form of the approximate wave function. In that case, the relative accuracy of two or more approximate wave functions, which give comparable energy values, could be determined by the ratios of the corresponding I_N values.

It should be noted that a trial wave function which minimizes the energy, with respect to some variational parameter, satisfies the Hellmann-Feynman theorem.[2,6]

[6] G. G. Hall, Phil. Mag. (London) **6**, 249 (1961).

One further point is in order. Equation (9) implies the phase relationship (8). Care must be taken not to make simultaneous use of formulas which presuppose another choice of phase.

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Singularities in Angular Momentum of the Scattering Amplitude for a Class of Soluble Potentials*

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The analyticity of the scattering amplitude in the variables, energy, and angular momentum is explicitly studied for square well and a class of continuous potentials having a $1/r^2$ type of core or tail. The trajectories of the poles in the l plane and their residues have been determined numerically.

I. INTRODUCTION

THE conjectures about the existence and properties of poles of the relativistic S matrix in complex angular momentum in the theory of strong interactions are based on the corresponding situation in potential scattering. The analyticity in the variables momentum k and angular momentum l in the case of potential scattering has been discussed by Regge[1] and by Froissart.[2]

The purpose of the present work is to study explicitly the trajectories of the poles and their residues for a class of soluble potentials. In Sec. II we discuss the analyticity of the S matrix in terms of the logarithmic derivative of the wave functions. The conclusions of this section are valid for all cutoff potentials. The contours in the l plane for the Watson-Sommerfeld transformation are discussed in Sec. III and the determination of the singularities in Sec. IV. In the following sections the potentials are considered explicitly and the numerical results and their interpretations are given.

II. ANALYTIC CONTINUATION OF THE S MATRIX IN l AND k

We consider the radial Schrödinger equation with complex k and complex $\lambda=l+\frac{1}{2}$ in units $\hbar^2=2m=1$,

$$\left(\frac{d^2}{dr^2}+k^2-\frac{\lambda^2-\frac{1}{4}}{r^2}-V(r)\right)\varphi(k,\lambda,r)=0. \tag{1}$$

The potential $V(r)$ is assumed to vanish outside a sphere of radius r_0, or behave as $1/r^2$ for $r>r_0$

$$V(r)=V_0(r)\theta(r_0-r)+\frac{\Lambda}{r^2}\theta(r-r_0).$$

The tail of the potential could be eliminated by writing

$$V(r)=\left(V_0(r)-\frac{\Lambda}{r^2}\right)\theta(r_0-r)+\frac{\Lambda}{r^2},$$

and absorbing the last part into the centrifugal term in Eq. (1). The effective λ in Eq. (1) would be then $(\lambda^2-\Lambda)^{\frac{1}{2}}$. However, it is more convenient to treat the tail of the potential separately. Inside the region $r<r_0$, the potential is arbitrary as long as the logarithmic derivative of φ exists at $r=r_0$. The solution of Eq. (1) in the region $r>r_0$ may be written as

$$\varphi(k,\nu,r)=B_1(k,\nu)(kr)^{1/2}J_\nu(kr)+B_2(k,\nu)(kr)^{1/2}J_{-\nu}(kr),\quad r>r_0, \tag{2}$$

where

$$\nu=\lambda=l+\tfrac{1}{2}\quad\text{if}\quad V=0\quad\text{for}\quad r>r_0, \tag{3a}$$

$$\nu=[(l+\tfrac{1}{2})^2-\Lambda]^{1/2}\quad\text{if}\quad V=\Lambda/r^2\quad\text{for}\quad r>r_0. \tag{3b}$$

Here $J_\nu(z)$ and $J_{-\nu}(z)$ are the usual Bessel functions which are linearly independent except when ν is an integer. They are entire functions of ν in the product domain of the whole ν plane and the z plane except for a possible branch point at $z=0$. The circuit relation at

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[1] T. Regge, Nuovo cimento **14**, 951 (1959); **18**, 947 (1960); A. Bottino, A. M. Longoni, and T. Regge, *ibid.* **23**, 954 (1962).
[2] M. Froissart (to be published).

$z=0$ is

$$J_\nu(ze^{im\pi})=e^{im\pi\nu}J_\nu(z), \quad m \text{ integer}. \tag{4}$$

The asymptotic form of Eq. (2) is

$$\varphi(k,\nu,r)\underset{r\to\infty}{\longrightarrow}B_1(k,\nu)\sin\left(kr-\frac{\pi\nu}{2}+\frac{\pi}{4}\right)+B_2(k,\nu)\sin\left(kr+\frac{\pi\nu}{2}+\frac{\pi}{4}\right). \tag{5}$$

To define the S matrix, we compare Eq. (5) with the asymptotic form of φ in terms of the phase shifts

$$\varphi(k,\nu,r)\underset{r\to\infty}{\longrightarrow}C(k,\nu)\sin\left(kr-\frac{\pi\lambda}{2}+\frac{\pi}{4}+\delta(k,\nu)\right), \tag{6}$$

where the phase shifts $\delta(k,\nu)$ have been defined as in the case of real k and integer l. Note that $\pi\lambda/2$ and not $\pi\nu/2$ appears in the argument of the sine function in Eq. (6), so that even if $\nu\neq\lambda$ the partial-wave amplitudes are given by

$$A(k,\nu)=\frac{1}{2ik}(e^{2i\delta(k,\nu)}-1)=\frac{1}{k}e^{i\delta(k,\nu)}\sin\delta(k,\nu). \tag{7}$$

Clearly, this definition of $\delta(k,\nu)$ and, therefore, of the S matrix, is not unique since δ can be multiplied by factors of the form

$$1+F(k,l)/\Gamma(-l),$$

which are equal to unity for physical values of l. Here $F(k,l)$ is an arbitrary function which does not have poles at positive integral values of l. We shall come back to this question in the next section. We note here that we have chosen a particular analytic continuation, Eq. (16), namely, the normal one which is also that used by Regge for the superposition of Yukawa potentials.[1]

From Eqs. (5) and (6) we obtain the reaction matrix K:

$$K\equiv\tan\delta(k,\nu)=\frac{B_1(k,\nu)\sin[\frac{1}{2}\pi(\lambda-\nu)]+B_2(k,\nu)\sin[\frac{1}{2}\pi(\lambda+\nu)]}{B_1(k,\nu)\cos[\frac{1}{2}\pi(\lambda-\nu)]+B_2(k,\nu)\cos[\frac{1}{2}\pi(\lambda+\nu)]}, \tag{8}$$

the S matrix

$$S(k,\nu)=\frac{1+iK}{1-iK}=e^{i\pi(\lambda-\nu)}\frac{1+e^{i\pi\nu}(B_2/B_1)}{1+e^{-i\pi\nu}(B_2/B_1)}, \tag{9}$$

and the partial wave amplitude

$$A(k,\nu)=\frac{1}{k}\frac{K}{1-iK}=\frac{1}{k}\frac{(B_2/B_1)\sin\pi\nu+e^{\frac{1}{2}i\pi(\lambda-\nu)}\sin[\frac{1}{2}\pi(\lambda-\nu)]}{1+e^{-i\pi\nu}(B_2/B_1)}. \tag{10}$$

For convenience, we also give another form of the S matrix which is useful for studying the threshold behavior of the amplitude and its singularities[3]

$$S(k,\nu)=e^{i\pi(\lambda-\nu)}\frac{Y(k,\nu)+k^{2\nu}e^{i\pi\nu}}{Y(k,\nu)+k^{2\nu}e^{-i\pi\nu}}, \tag{11}$$

$$Y(k,\nu)=(B_1/B_2)k^{2\nu}. \tag{12}$$

The unknown function (B_2/B_1) in Eqs. (8) to (12) can be related to the logarithmic derivative $L(k,\lambda)$ of the interior solution of Eq. (1) at $r=r_0$ as follows:

$$r_0\tilde{L}(k,\nu)\equiv r_0L(k,\lambda)=\tfrac{1}{2}+kr_0\frac{J_\nu'(kr_0)+(B_2/B_1)J_{-\nu}'(kr_0)}{J_\nu(kr_0)+(B_2/B_1)J_{-\nu}(kr_0)}. \tag{13}$$

Hence

$$B_2/B_1=-\frac{kr_0J_\nu'(kr_0)-[r_0L(k,\lambda)-\frac{1}{2}]J_\nu(kr_0)}{kr_0J_{-\nu}'(kr_0)-[r_0L(k,\lambda)-\frac{1}{2}]J_{-\nu}(kr_0)}=-\frac{[\nu+\frac{1}{2}-r_0L(k,\lambda)]J_\nu(kr_0)-kr_0J_{\nu+1}(kr_0)}{[\nu+\frac{1}{2}-r_0L(k,\lambda)]J_{-\nu}(kr_0)+kr_0J_{-(\nu+1)}(kr_0)}. \tag{14}$$

This expression can now be inserted into Eqs. (8) to (12). It may seem from Eq. (12) that the scattering amplitude vanishes for $\lambda=\nu=n=$ integer (nonphysical). However, from (14), $B_2/B_1=-(-1)^n$; hence $S(k,n)$ is indeterminate. A convenient formula for the S matrix, which eliminates this indeterminacy and which is useful for numerical calculations, is

$$S(k,\nu)=-e^{i\pi(\lambda-\nu)}\times\frac{[\nu+\frac{1}{2}-r_0L(k,\lambda)]H_\nu^{(2)}(kr_0)-kr_0H_{\nu+1}^{(2)}(kr_0)}{[\nu+\frac{1}{2}-r_0L(k,\lambda)]H_\nu^{(1)}(kr_0)-kr_0H_{\nu+1}^{(1)}(kr_0)}, \tag{15}$$

where the Hankel functions are given by

$$H_\nu^{(1)}(z)=(i/\sin\pi\nu)[e^{-i\pi\nu}J_\nu(z)-J_{-\nu}(z)],$$
$$H_\nu^{(2)}(z)=(-i/\sin\pi\nu)[e^{i\pi\nu}J_\nu(z)-J_{-\nu}(z)].$$

We now discuss, with the help of Eq. (15), the unitarity and the symmetry of the S matrix. If the logarithmic derivative $\tilde{L}(k,\nu)$ is a real analytic function of both arguments, we obtain the analytically continued unitarity condition

$$[S(k^*,\nu^*)]^*S(k,\nu)=1. \tag{16}$$

Furthermore, from the circuit relations for the Hankel functions and assuming that $L(k,\nu)$ is even in k, $L(-k,\nu)=L(k,\nu)$, we obtain

$$S(ke^{i\pi},\nu)=-e^{2i\pi\nu}S^{-1}(k,\nu)+2e^{i\pi\nu}\cos\pi\nu, \tag{17}$$

and the circuit relation for the S matrix around the

[3] A. O. Barut and D. E. Zwanziger, Phys. Rev. **127**, 974 (1962).

branch point at $k=0$:

$$S(ke^{2i\pi},\,\nu)=2e^{i\pi\nu}\cos\pi\nu+\frac{S(k,\nu)}{1-2e^{-i\pi\nu}\cos\pi\nu S(k,\nu)}. \quad (18)$$

Equations (17) and (18) have also been proved for Yukawa-type potentials[1,2] and in the relativistic case.[3] We note that the branch point at $k=0$ disappears for half-integer (physical) values of ν in which case Eq. (17) reduces to the familiar symmetry relation $S(-k,\nu)\times S(k,\nu)=1$. We also note the useful relation obtained by combining Eqs. (16) and (17)

$$S(ke^{i\pi},\nu)=2e^{i\pi\nu}\cos\pi\nu-e^{2i\pi\nu}[S(k^*,\nu^*)]^*, \quad (19)$$

which is the generalization of the relation $S(-k,l)=[S(k^*,l)]^*$ for integral values of l.

We also see that if $\lambda=\nu$, the so-called Mandelstam symmetry[4]

$$A(k,\lambda)=A(k,-\lambda),\quad \lambda=\text{integer (nonphysical)}, \quad (20)$$

holds, provided $L(k,\lambda)$ has this property, as is the case in the examples considered in Sec. V. If $\lambda\neq\nu$ (i.e., $1/r^2$ tail in the potential), this property, in general, is not true.

We are now in the position to discuss the analyticity of the S matrix in both variables k and λ. Except for the singularities of $L(k,\lambda)$ which, of course, depend on the potential in the interior region, the S matrix is meromorphic in the product of the whole λ plane and the whole k plane with a branch point at $k=0$ (except when ν is half-integer), and an essential singularity at infinity both in the k and λ planes.

It is seen from Eq. (3b) that for potentials with a $1/r^2$ tail the function $\nu(\lambda)$ has two fixed branch points independent of energy. However, in the S matrix the corresponding cut in the λ plane cancels; in fact, it is not even present in the radial wave function $\varphi(k,\lambda,r)$. To show this, we notice that going around one of the branch points changes ν into $-\nu$, which, by Eq. (14), exchanges B_2 into B_1. But the S matrix, Eq. (9), does not change when $\nu\leftrightarrow-\nu$ and $B_1\leftrightarrow B_2$ for fixed λ.

Finally, we remark that the S matrix may have, in general, the so-called nonessential singularities of the second kind (indeterminacy points),[5] where it is of the form 0/0. The indeterminacy points are a peculiarity of functions of two or more complex variables. They are isolated points in the case of two complex variables, in the neighborhood of which the S matrix takes any value. These indeterminacy points can be interpreted as "states" for which the wave function is completely confined inside the region $r<r_0$. For, if we write Eq. (2) in the form

$$\varphi(k,\nu,r)=A_1(k,\nu)(kr)^{\frac{1}{2}}H_\nu^{(1)}(kr)+A_2(k,\nu)(kr)^{\frac{1}{2}}H_\nu^{(2)}(kr),\quad r>r_0,$$

the S matrix is given by the ratio of A_1 and A_2, and the indeterminacy points correspond to both A_1 and A_2 being zero. Of course, these points do not occur for physical values of k and l. In principle, one expects to find the indeterminacy points for any potential (e.g., for an attractive Coulomb potential they occur at $l=-[(n+m)/2]-1$, $E=-e^4/(m-n)^2$, where m and n are nonnegative integers].

III. WATSON-SOMMERFELD TRANSFORMATION

The analytic continuation in angular momentum l is used to separate from the scattering amplitude singular terms which correspond to bound states and resonances on the one hand, and, on the other hand, control the asymptotic behavior of the scattering amplitude for large momentum transfer. For this purpose the partial-wave expansion of the scattering amplitude

$$A(k^2,\cos\theta)=\sum\nolimits_l(2l+1)A(k^2,l)P_l(\cos\theta) \quad (21)$$

is transformed into a contour integral in the right-hand l plane. For the contour C shown in Fig. 1, we have[1]

$$A(k_2,\cos\theta)=\frac{1}{2i}\int_C\frac{(2l+1)dl\,A(k^2,l)P_l(-\cos\theta)}{\sin\pi l}, \quad (22)$$

if $A(k^2,l)$ is holomorphic inside the contour and approaches zero exponentially for $l\rightarrow\infty$ (see Appendix I). If $A(k^2,l)$ is meromorphic, to be precise, holomorphic except for a finite number of single poles, in the region $\mathrm{Re}l>-\frac{1}{2}$, k fixed, and the integral (22) exists for the contour C' (Fig. 1), then Eq. (22) can be transformed into the following representation;

$$A(k^2,\cos\theta)=\frac{1}{2i}\int_{\mathrm{Re}l=-1/2}\frac{(2l+1)dl\,A(k^2,l)P_l(-\cos\theta)}{\sin\ l}+\sum_n\frac{(2\alpha_n+1)\beta_n(k)P_\alpha(-\cos\theta)}{\sin\pi\alpha_n(k)}, \quad (23)$$

where we have also assumed that the integral over the

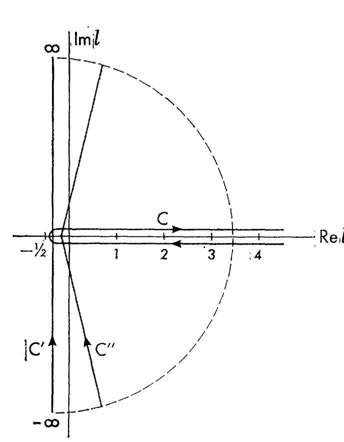

FIG. 1. Contours in the l plane for Watson-Sommerfeld transformation.

[4] S. Mandelstam, Ann. Phys. (New York) **19**, 254 (1962).

[5] H. Behnke and P. Thullen, *Theorie der Funktionen Mehrerer Komplexer Veranderlichen* (Verlag Julius Springer, Berlin, 1934).

large half-circle of C' vanishes. In Eq. (23) $\alpha_n(k)$ are the poles of the scattering amplitude $A(k^2,l)$ in the l plane ($\mathrm{Re}l>-\frac{1}{2}$) for fixed k, and $\beta_n(k)$ are its residues at the poles.

Assuming now that the representation (23) holds, we obtain the asymptotic form of the amplitude for large momentum transfer t, where $t=-2k^2(1-\cos\theta)$, from the position $\alpha_{\max}(k)$ of the pole in the l plane with the largest real part:

$$A(k^2,\cos\theta) \underset{t\to\infty}{\longrightarrow} t^{\alpha_{\max}(k)}. \tag{24}$$

Note that the first integral in Eq. (23) behaves as $t^{-\frac{1}{2}}$ for $t\to\infty$. Furthermore, we see from the representation (23) that the scattering amplitude is analytic in the whole $z=\cos\theta$ plane cut from $z=1$ to ∞, namely, the cut of $P_\alpha(-z)$. This cut, however, may be reduced by cancellations as is the case, for example, for Yukawa-type potentials where the cut runs from $z=1+\mu^2/2k^2$ to $z=\infty$, μ being the lowest mass of the Yukawa potential. It follows from Eq. (23) that the physical partial-wave amplitudes have the form

$$A(k^2,l)=A_0(k^2,l)+\frac{1}{\pi}\sum_n \frac{[2\alpha_n(k)+1]\beta_n(k)}{[\alpha_n(k)-l][\alpha_n(k)+l+1]}, \tag{25}$$

where $A_0(k^2,l)$ is holomorphic in l. When l becomes equal to $\mathrm{Re}\alpha_n(k)$, as k is varied, the corresponding partial wave has a bound-state pole or a resonance depending on whether $\alpha_n(k^2)$ is real or complex, the two cases corresponding to negative or positive k^2, respectively. In fact, it can be proved quite generally from Eq. (1) that for $\mathrm{Re}l>-\frac{1}{2}$, $\alpha_n(k^2)$ is real for $k^2<0$.[1] ($\mathrm{Re}l=-\frac{1}{2}$ is excluded, see Sec. IV).

Finally, we remark that when the representation (23) is true the analytic continuation of the amplitude to complex l is unique. For a function that is zero at integral values of l and behaves as $e^{-\rho|l|}$, $\rho>0$ for $\mathrm{Re}\geqq$const is identically zero.[6] This is just the behavior we must impose on $A(k,l)$ in order that the Watson-Sommerfeld transformation can be done in some region of the z plane (see Appendix I).

Now we study the validity of the representation (23) for the class of potentials considered here. We start with the asymptotic behavior of $A(k,\nu)$ in the ν plane. For fixed k, we have

$$J_\nu(kr_0) \underset{|\nu|\to\infty}{\longrightarrow} \frac{(kr_0/2)^\nu}{\Gamma(\nu+1)}[1+O(k^2/\nu)], \quad |\arg\nu|<\pi;$$

hence, from Eq. (14),

$$B_2/B_1 \underset{|\nu|\to\infty}{\longrightarrow} \frac{\pi}{2\nu^2}\frac{(kr_0/2)^{2\nu}}{\sin\pi\nu}[\Gamma(\nu)]^{-2} \times\frac{\nu(2\nu-2r_0L+1)-k^2r_0{}^2}{2\nu+2r_0L-1}+[1+O(k^2/\nu)]. \tag{26}$$

Furthermore, in the right-hand ν plane

$$\Gamma(\nu) \underset{|\nu|\to\infty}{\longrightarrow} (2\pi)^{\frac{1}{2}}e^{-\nu}e^{(\nu-\frac{1}{2})\ln\nu}[1+O(1/\nu)], \quad |\arg\nu|<2\pi.$$

Thus, unless $L(k,\nu)$ is such that the last factor in Eq. (26) becomes important, (see below), (B_2/B_1) approaches zero faster than any exponential along any ray in the right-hand ν plane. Therefore, by Eq. (10), the amplitude goes also to zero faster than any exponential. However, along any vertical, $\mathrm{Re}\nu=\mathrm{const}>0$, the Γ function and $\sin\pi\nu$ compensate each other, (B_2/B_1) behaves as a power and the amplitude goes to a constant for $\mathrm{Im}\nu\to+\infty$, and to infinity for $\mathrm{Im}\nu\to-\infty$. The factor containing the unknown logarithmic derivative $L(k,\nu)$,

$$[\nu(2\nu-2r_0L(k,\nu)+1)-k^2r_0{}^2]/(2\nu+2r_0L-1), \tag{27}$$

can change the results only if $L(k,\nu)$ has such a special dependence on k and ν as to make this factor vanish exponentially. We assume in the following that this does not happen, as it will be shown to be true in the next section.

Therefore we see that the integrand in Eq. (23) will converge along any ray in the l plane, but not along a vertical direction. We cannot modify the contour C into C', and therefore must choose a path C'' (Fig. 1) to separate the pole terms as in Eq. (23). In this new representation, the angle between C' and C'' may be as small as one pleases and interpretation of the poles as representing the bound states and resonances remains unchanged. But the asymptotic behavior of the amplitude for large momentum transfer t is no longer given by Eq. (24), because it will not be dominated by the pole farthest to the right in the l plane. This result can also be obtained as follows, which makes its physical meaning clearer. The partial-wave expansion (21) converges, by Faber's theorem,[7] uniformly and absolutely, in an ellipse with semimajor axes $\frac{1}{2}(h\pm h^{-1})$, where

$$h=\lim_{l\to\infty}|1/A(l)|^{1/l}.$$

In our case $A(l)\to e^{-2l\ln l}$, hence the expansion converges in the whole $\cos\theta$ plane, except at infinity. This argument shows that the cut of $P_l(-\cos\theta)$ from $\cos\theta=1$ to ∞ in Eq. (23) actually cancels in the sum of all terms for the potentials we are considering. Therefore, the amplitude is an entire function for all finite values of

[6] G. Prosperi, University of California Radiation Laboratory Report UCRL-10116 (unpublished). See also E. J. Squires, University of California Radiation Laboratory Report UCRL-10033 (unpublished).

[7] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, New York, 1952), p. 95.

$\cos\theta$ and, because it is not a polynomial, as all partial-wave amplitudes, Eq. (15), do not vanish, it must have an essential singularity at infinity.

With the new contour C'', the uniqueness of the analytic continuation of $A(k^2,l)$ in l is no longer true without further restrictions, such as unitarity, etc. Because the threshold behavior of the amplitude is the same for all short-range potentials, we have chosen the same analytic continuation as in the case of Yukawa potentials.

IV. SINGULARITIES IN THE l PLANE

We have seen in Sec. II that for the class of potentials we are considering the S matrix is a meromorphic function in the product of the entire k and l planes except for (1) the branch point at $k=0$, (2) the essential singularities at infinity, both in k and l planes, and (3) possible singularities of $\tilde{L}(k,\nu)$ which will be discussed for specific cases in the next sections. The poles in k and l are given by the zeros of the denominator in Eq. (15) and satisfy the equation

$$F(k,\nu)\equiv\nu+\tfrac{1}{2}-r_0\tilde{L}(k,\nu)-xH_{\nu+1}^{(1)}(x)/H_\nu^{(1)}(x)=0, \quad (28)$$

where

$$x=kr_0.$$

This equation takes a simple form at $x=0$:

$$F(0,\nu_0)=-\nu_0+\tfrac{1}{2}-r_0\tilde{L}(0,\nu_0)=0;\quad \mathrm{Re}\nu_0>0,$$
$$F(0,\nu_0)=\nu_0+\tfrac{1}{2}-r_0\tilde{L}(0,\nu_0)=0;\quad \mathrm{Re}\nu_0<0, \quad (29)$$

where ν_0 is the position of the poles at zero energy. Equation (29) will be used to locate and "count" the poles in the l plane in the numerical calculations (Sec. V). For $\mathrm{Re}\nu_0=0$, one does not get a simple limit as $x\to 0$ for the ratio of the Hankel functions in Eq. (28). As a matter of fact, poles along the imaginary ν axis for $k^2\leqq 0$ cannot be excluded in general.[8]

The symmetry of the zeros and poles of the S matrix for complex angular momentum is easily discussed. It follows from Eq. (16) that if (k,ν) is a pole of the S matrix, then (k^*,ν^*) is a zero. Clearly, then, if a pole occurs at a point where both ν and k are real, its residue must vanish. This is generally the case for the poles at $(0,\nu_0)$ (where ν_0 is real). Note that the points where the residues are zero form a line in the four-dimensional (k,ν) space. This is based on unitarity alone and is true for all potentials.

[8] The proof that for $k^2\leqq 0$ the poles in the l plane are along the real ν axis is based on the equation (obtained simply from the Schrödinger equation as in reference 1 where, however, the first term is missing):

$$\lim_{R\to\infty}\left[\mathrm{Re}k\,|\varphi(R)|^2+\mathrm{Re}k\,\mathrm{Im}k\int_0^R|\varphi|^2dr\right.$$
$$\left.-2\,\mathrm{Re}\lambda\,\mathrm{Im}\lambda\int_0^R(|\varphi|^2/r^2)dr\right]=0,$$

so that if $\mathrm{Re}\lambda\neq 0$, we have $\mathrm{Im}\lambda=0$ for $\mathrm{Re}k=0$. Thus, if $\mathrm{Re}\lambda=0$, $\mathrm{Im}\lambda$ can be different from zero. The trajectories of these poles (if they exist) will leave the imaginary axis for $k^2>0$.

Furthermore, we see from the circuit relation (17) that a zero at (k,ν) implies a pole at $(ke^{+i\pi},\nu)$, but not *vice versa*, and from (19) that if there exists a pole at (k^*,ν^*), then there is also one at $(ke^{i\pi},\nu)$. This last property is a generalization of the symmetry of the poles in the lower half k plane with respect to the imaginary axis for physical values of l.

We denote the solutions of Eq. (17) in the l plane for fixed k by $\alpha_n(k)$. As solutions of an analytic equation of the form $F(k,l)=0$, these functions $\alpha_n(k)$ are themselves analytic everywhere where $F(k,l)$ is analytic in both variables and the derivatives $\partial F/\partial l$ do not vanish. Clearly, $F(k,\nu)$ is analytic in both variables except for singularities of $L(k,\nu)$; the branch points of the Hankel functions at $x=0$ cancel, as can be seen from Eq. (29).

We shall concentrate in the next section on the determination of the poles in the l plane as a function of the energy k^2 (k^2 real, negative, and positive) and refer to them as Regge trajectories. In the k^2 plane the functions $\alpha_n(k^2)$ have the usual right-hand kinematical cut from 0 to ∞. It follows quite generally from Eq. (1) that $\alpha_n(k)$ has a positive imaginary part for $k^2>0$ and is real for $k^2<0$, both for $\mathrm{Re}\alpha_n>-\frac{1}{2}$.[1]

V. SQUARE-WELL POTENTIAL

For the three-dimensional "square-well" potential

$$V(r)=-V_0\quad\text{for}\quad r\leqq r_0,$$
$$V(r)=0\quad\text{for}\quad r>r_0. \quad (30)$$

(V_0 positive corresponds to an attractive potential, V_0 negative to a repulsive potential) the logarithmic derivative function $r_0L(k,\lambda)$ of Eq. (13) is easily found to be

$$r_0L(k,\nu)=y\frac{J_\lambda'(y)}{J_\lambda(y)}+\tfrac{1}{2}=\lambda+\tfrac{1}{2}-y\frac{J_{\lambda+1}(y)}{J_\lambda(y)}, \quad (31)$$

where

$$y=(x^2+V)^{\frac{1}{2}},\quad V=V_0r_0^2,\quad x=kr_0.$$

Although both y and $J_\lambda(y)$ have branch points in the k^2 plane at $k^2=k_0^2=-V_0(y=0)$, the function $r_0L(k,\lambda)$ is a holomorphic function of k at this point, for

$$y\frac{J_{\lambda+1}(y)}{J_\lambda(y)}\underset{|y|\to 0}{\longrightarrow}\frac{y^2}{2(\lambda+1)}\to 0,\quad \lambda\neq -n.$$

Hence $r_0L(k,\lambda)$ is a meromorphic function of both variables k and λ. Note, however, that at the points $k^2=k_0^2$, $\lambda=-n$ the function is undetermined. In fact we get, for instance,

$$\lim_{k\to k0}L(k,-n)=n+\tfrac{1}{2},$$

but

$$\lim_{\lambda\to -n}L(k_0,\lambda)=-n+\tfrac{1}{2}.$$

since

$$y\frac{J_{-n+1}(y)}{J_{-n}(y)}\underset{|y|\to 0}{\longrightarrow}-2n,\quad n=1,2,3,\cdots. \quad (32)$$

These indeterminacy points are, by definition, inside the domain of meromorphy. Functions of several complex variables cannot have isolated singularities, and the indeterminacy points lie, in fact, at the intersection of the "pole surfaces" with the "zero surfaces." It follows then that the trajectories of the poles in the l plane as a function of k^2 originate or pass through these points $(k_0^2, \lambda=-n)$.

We also note the simple expression

$$r_0 L(k_0,\lambda)=\lambda+\tfrac{1}{2}, \quad \lambda\neq -n,$$

and the properties

$$L(k,\lambda)=L(-k,\lambda),$$

so that the generalized unitarity relation (16) holds, and

$$L(k,n)=L(k,-n), \quad n \text{ integer},$$

(also for $k=k_0$) so that, because $\lambda\equiv\nu$ in this case, the amplitude has the symmetry of Eq. (20).

These properties of L completely specify the analytic behavior of the S matrix, as discussed in Sec. II, which is, we summarize, a meromorphic function in the product domain of the finite k and λ planes except for a branch "line" at $k=0$ (all λ except half-integer). The essential singularities at infinity in k and λ have been discussed in Sec. II.

For large λ we have

$$r_0 L(k,\lambda)\to\lambda+\tfrac{1}{2}-y^2/2\lambda,$$

and the factor of Eq. (27) vanishes only as a power. Therefore the conclusions of Sec. III about the Watson-Sommerfeld transformation hold.

Equations (28) and (29), which determine the poles of the scattering amplitude, become

$$y\frac{J_{\lambda+1}(y)}{J_\lambda(y)}-x\frac{H_{\lambda+1}^{(1)}(x)}{H_\lambda^{(1)}(x)}=0, \tag{33}$$

and for $k=0$

$$V^{\frac{1}{2}}\frac{J_{\lambda_0+1}(V^{\frac{1}{2}})}{J_{\lambda_0}(V^{\frac{1}{2}})}\begin{aligned}&=2\lambda_0 \quad \text{for} \quad \mathrm{Re}\lambda_0>0,\\ &=0 \quad\ \ \text{for} \quad \mathrm{Re}\lambda_0<0,\end{aligned}$$

or, using the recursion relations of J_λ and the fact that $J_\lambda(z)$ and $J_{\lambda+n}(z)$ $(n=1, 2, 3, \cdots)$ have no common zeros other than at $z=0$,[9]

$$\begin{aligned}J_{\lambda_0-1}(V^{\frac{1}{2}})&=0 \quad \text{for} \quad \mathrm{Re}\lambda_0>0,\\ J_{\lambda_0+1}(V^{\frac{1}{2}})&=0 \quad \text{for} \quad \mathrm{Re}\lambda_0<0.\end{aligned} \tag{34}$$

Thus, for $k^2=0$, the poles of the S matrix are obtained from the zeros of the Bessel functions with respect to the order. The number of trajectories and their positions at $k^2=0$ are then completely determined. There are a denumerable infinity of poles along the negative real λ axis approaching asymptotically the negative integers as $n\to\infty$. If the potential is strong enough and attractive, some of the poles occur along the positive real λ axis, as shown in Fig. 2. Furthermore, as the potential goes to zero, the position of the poles at $k=0$ also approaches the negative integers.

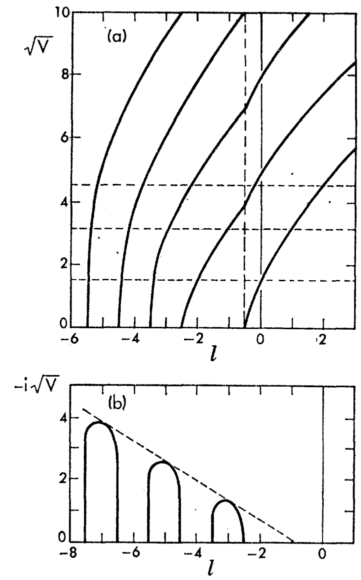

FIG. 2. Position of the poles in the l plane at $k^2=0$ as a function of potential strength. $V=V_0 r_0^2$. (a) Attractive square well. (b) Repulsive square well.

We may also study the effect of a $1/r^2$ core and tail to the square-well potential:

$$V=-V_0\{[1+\rho-\rho(r_0/r)^2]\theta(r_0-r) +\epsilon(r_0/r)^2\theta(r-r_0)\}. \tag{35}$$

The discussion of the effect of the core in Eq. (35) will be reduced to that of a tail as follows. The S matrix for the potential (35), $S(\lambda,V_0,\epsilon,\rho)$ is the same as that for the potential

$$\bar{V}=-V_0[(1+\rho)\theta(r_0-r)+(\rho+\epsilon)(r_0/r)^2\theta(r-r_0)],$$

which has no core, *and* λ replaced by

$$\bar{\lambda}=(\lambda^2+\rho V)^{1/2}.$$

Or,

$$S(\lambda,V_0,\epsilon,\rho) =S[(\lambda^2+\rho V)^{1/2},\ V_0(1+\rho),\ (\rho+\epsilon)/(1+\rho),\ 0]. \tag{36}$$

It is thus sufficient to study potentials with a tail only.

There is one interesting feature of the core and that is the cut introduced in the λ plane due to the term $(\lambda^2+\rho V)^{1/2}$. Different analytic continuations are therefore possible according to the choice of the branch cuts. The branch points are on the imaginary axis at $\lambda=\pm i(\rho V)^{1/2}$. If we choose the branch cut from $-i(\rho V)^{1/2}$ to $+i(\rho V)^{1/2}$, we obtain a continuation that goes continuously over to the case $\rho=0$. If, however, we choose the cuts running from $-i\infty$ to $-i(\rho V)^{1/2}$ and from $i(\rho V)^{1/2}$ to $+i\infty$, then S in (36) is an even

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[9] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, New York, 1953), Vol. 2, p. 60; M. J. Coulomb, Bull. Sci. Math. **60**, 297 (1936).

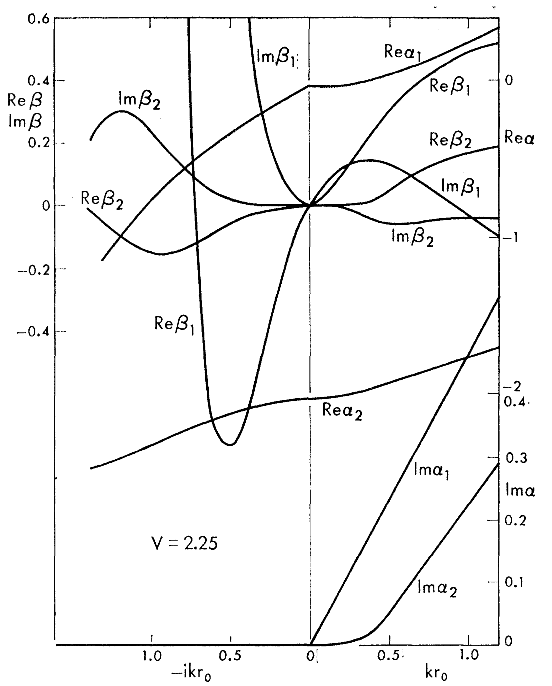

FIG. 3. Threshold behavior of the positions and residues of the two highest poles α_1 and α_2. Square well, $V=2.25$.

function of λ. For this case we have the curious result that there are a finite number of poles symmetric with respect to the imaginary axis, namely, the "physical poles" and their images.

Numerical calculations, using IBM 704, have been performed to determine explicitly the trajectories of the poles in the l plane and their residues. Two different subroutines have been used for the Bessel functions of complex order and argument to verify the results.

The threshold behavior of the parameters of the poles is shown in Figs. 3, 4, and 5 for three different strengths of the pure square-well potential. One expects on general grounds that the threshold behavior is the same for all potentials, even for the relativistic case (if α_0 is real at $k=0$), and the numerical result agrees with this general theoretical behavior.[3] Near the indeterminacy points in the left-hand l plane, the residues of the pole become very large and oscillate, and we have not been able to continue the trajectories beyond these points. Below threshold $\beta(k^2)/k^{2\alpha}$ is real and checks again with the general result.[3]

Figure 6 shows the trajectories for larger values of $(kr_0)^2$, and some typical trajectories in the l plane are shown in Fig. 7. The potentials discussed here take a place intermediate between Coulomb and Yukawa potentials: At threshold they behave more like a Yukawa potential, but the trajectories extend to infinity instead of being bounded. Although the Reα curve cuts the integers infinitely many times, Imα becomes so large that these points can not be interpreted

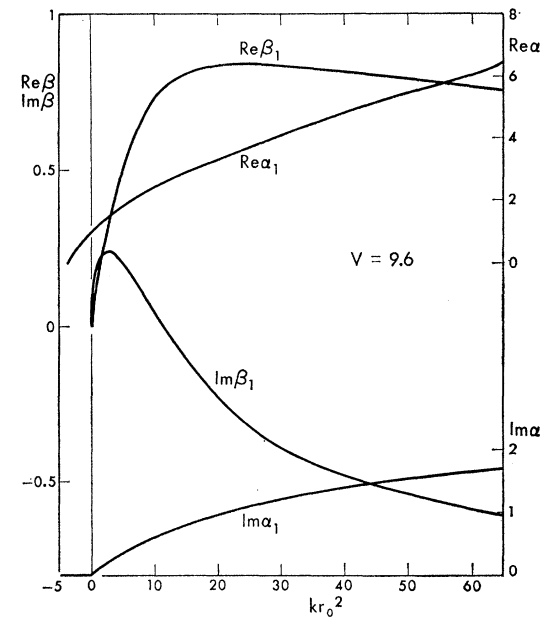

FIG. 4. Threshold behavior of the position and residue of the highest pole. Square well, $V=9.6$. For negative energies both Reβ and Imβ oscillate very rapidly and have not been shown.

as true or sharp resonances. The residues for large values of energy are shown in Figs. 8 and 9. The asymptotic form of the trajectories as $k^2\rightarrow\infty$ is discussed in Appendix II.

For potentials with a tail the quantity ν is imaginary for $|\lambda|^2 < \epsilon V$, λ real $[\nu=(\lambda^2-\epsilon V)^{\frac{1}{2}}]$. Returning to Eq. (29) and the discussion following it, we see that we

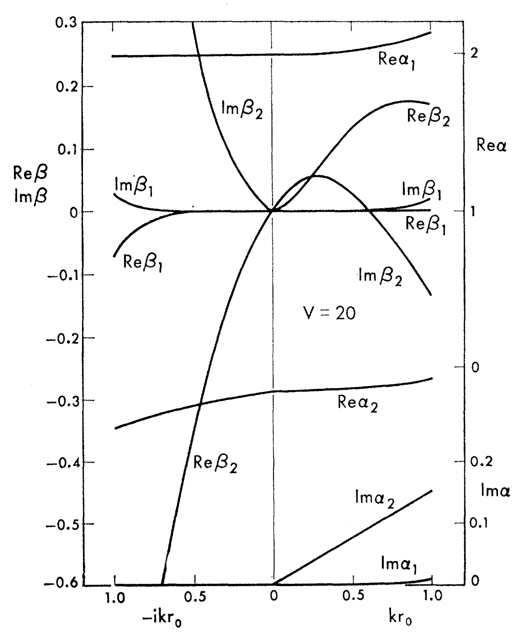

FIG. 5. Threshold behavior of the positions and residues of the two highest poles. Square well, $V=20.0$.

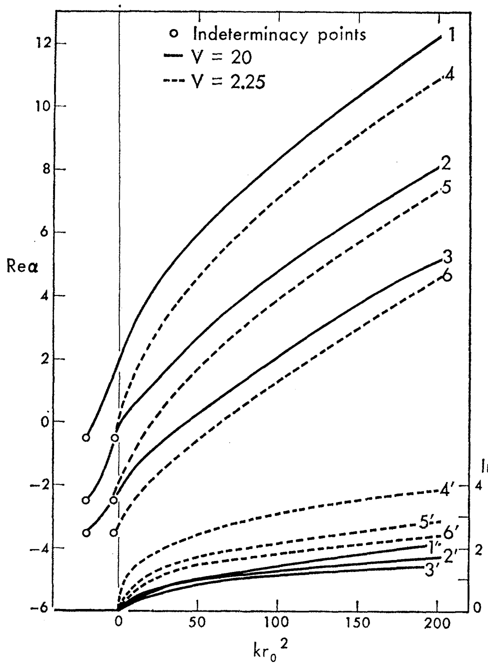

FIG. 6. Positions of the poles for larger values of energy. Curves 1, 2, 3, and 1′, 2′, 3′ refer to real and imaginary parts of the first three poles for $V=20$. Curves 4, 5, 6, and 4′, 5′, 6′ to those for $V=2.25$. The case $V=9.6$ is intermediate and similar.

can have poles with pure imaginary ν_0 (but still real l) not given in the previous analysis. In fact the "missing pole" ending at $-\frac{3}{2}$ in Fig. 2 is found in this case. These poles have also quite a different threshold behavior, because ν is not real at threshold as shown in the two examples (Figs. 10 and 11). For $\lambda^2>\epsilon V$ the trajectories are again "normal," as shown by the upper curve in Fig. 8. The asymptotic behavior of the trajectories is again as before.

VI. CONTINUOUS SOLUBLE POTENTIALS

The S matrix for the following class of continuous potentials can also be explicitly evaluated:

$$\begin{aligned} V(r)&=-V_0[1+\epsilon-(1+\epsilon\rho-\rho)(r/r_0)^2-\rho(r_0/r)^2] && \text{for } r<r_0, \qquad (37)\\ &=-V_0\epsilon(1-\rho)(r_0/r)^2 && \text{for } r>r_0. \end{aligned}$$

The parameters V_0 and r_0 determine again the strength and the range of the potential, whereas ϵ and ρ characterize its shape. The potential is continuous for all values of ϵ and ρ. For $\epsilon=1$, its first derivative is also continuous. For $\epsilon\neq 0$ the potential has a $1/r^2$ tail outisde r_0. The behavior of $V(r)$ at $r=0$ is controlled by the parameter ρ; when $\rho=0$ the potential attains its extremum value $-V_0(1+\epsilon)$, whereas for $\rho\neq 0$ it has in the origin a core of the form $V_0\rho(r_0/r)^2$. In the latter case the scattering amplitude is defined only if $V_0\rho$ is positive, i.e., a repulsive core.

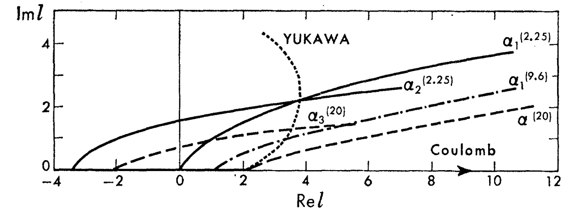

FIG. 7. Typical trajectories in the l plane for square-well potential. Superscripts on α refer to the potential, subscripts to the number of pole. For comparison a Coulomb trajectory and (schematically) a Yukawa trajectory is shown [A. Ahmedzadeh, P. G. Burke, and C. Tate, Lawrence Radiation Laboratory Report UCRL-10140 (unpublished)]. The scale of this figure is such that one does not see the correct threshold behavior which is shown in detail in Figs. 3, 4, and 5.

In the S-matrix expression of Eq. (15) we now have

$$\nu=[\lambda^2-\epsilon(1-\rho)V]^{1/2},\quad V=r_0^2V_0,\quad \lambda=l+\tfrac{1}{2},$$

and the logarithmic derivative function

$$r_0L(k,\lambda)=c-\tfrac{1}{2}-z+2z\Phi'(a,c;z)/\Phi(a,c;z)$$

obtained from the interior solution of the Schrödinger equation

$$\varphi=(r/r_0)^{c-\frac{1}{2}}e^{-\frac{1}{2}z(r/r_0)^2}\Phi[a,c;z(r/r_0)^2]. \qquad (38)$$

Here $\Phi(a,c;z)$ is the confluent hypergeometric function defined by the series

$$\Phi(a,c;z)=1+\sum_{n=1}^{\infty}\frac{\Gamma(a+n)\Gamma(c)}{\Gamma(a)\Gamma(c+n)}\frac{z^n}{n!}, \qquad (39)$$

and a, c, and z are given by

$$\begin{aligned} z&=-[V(1+\epsilon\rho-\rho)]^{1/2},\\ c&=1+(\lambda^2+\rho V)^{1/2}, \qquad (40)\\ a&=\frac{1}{2}\left[c-\frac{x^2+V(1+\epsilon)}{2z}\right],\quad x=kr_0. \end{aligned}$$

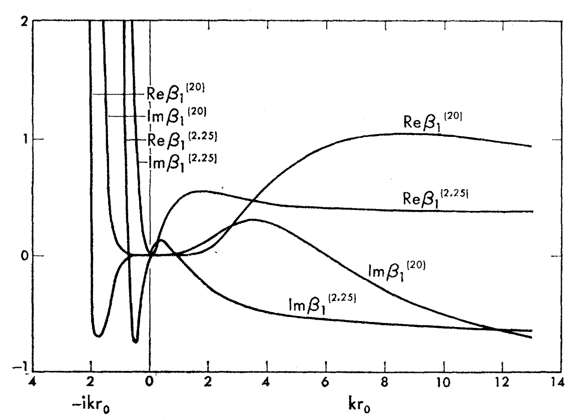

FIG. 8. Residues for the highest pole. Square well, $V=20$ and $V=2.25$.

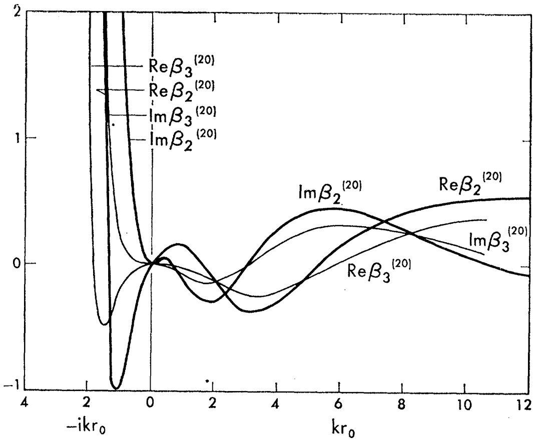

FIG. 9. Residues for the second and the third pole. Square well, $V=20$.

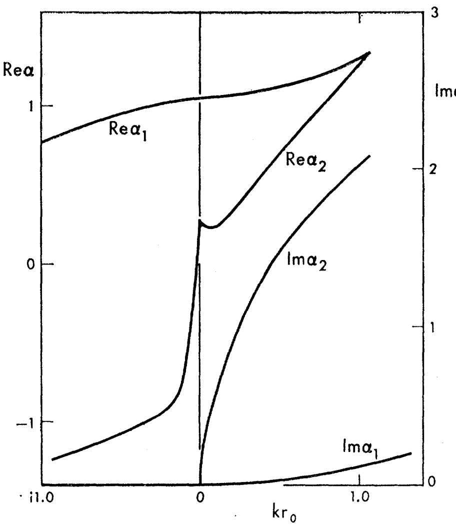

FIG. 10. Effect of a tail in the potential. Square well, $V=9.6$, $\epsilon=0.1$, $\rho=0$ [Eq. (35)]. A new trajectory appears at $\alpha(0)\cong 0.28$ with distinct threshold behavior (α_2). The behavior of "normal" poles is the same as before (α_1).

It is seen from Eq. (39) that the confluent hypergeometric function $\Phi(a,c;z)$ is an entire function of z and a; as a function of c it has simple poles at $c=0, -1, -2, -3, \cdots$. In fact $\Phi(a,c;z)/\Gamma(c)$ is an entire function of all three arguments. The ratio $\Phi'(a,c;z)/\Phi(a,c;z)$ is therefore a meromorphic function of the two variables a and c everywhere. (In our case the argument z is a constant independent of k and λ.) At the points $a=c=-n$, $n=0, 1, 2, 3, \cdots$ the ratio is indeterminate. (It can be seen that Φ'/Φ has different limits as $a=c\to -n$ and $c=-n$, $a\to c$.) These are again nonessential singularities of the second kind, as we have found them also in the square-well potential, i.e., fixed intersection points of the two zero surfaces of φ' and φ.

Now we have to pass from c and a to the variables k and λ. If $\rho=0$ (potentials without core), we have $c=1+\lambda$ and hence $L(k,\lambda)$ is meromorphic in the product domain of both variables. The indeterminacy points are at

$$\begin{aligned} &\lambda=-n, \\ &x^2=(kr_0)^2=(1-n)V^{1/2}-V(1+\epsilon), \qquad (41)\\ &n=0, 1, 2, 3, \cdots. \end{aligned}$$

For attractive potentials and $\epsilon>0$ these points occur always at real negative energies, for repulsive potentials at complex energies.

The case $\rho\neq 0$ introduces a cut in the λ plane (due to the square root in c) which has been discussed in the case of the square-well potential with core.

We recall that the cut in λ due to the $1/r^2$ tail in Eq. (3b) always cancels in the S matrix. The unitarity condition (16) holds again because L is an even function of k. For large λ and fixed k, $r_0L(k,\lambda)$ approaches λ, as in the case of square-well potential.

The poles of the S matrix are again given by Eq. (28) with $r_0L(k,\lambda)$ given above.

The numerical results are qualitatively the same as for the square-well potential. A typical example is given in Fig. 12.

VII. OTHER POTENTIALS

We consider two more cases of soluble potentials which are of some interest from the point of view of complex angular momenta. One is a three-dimensional

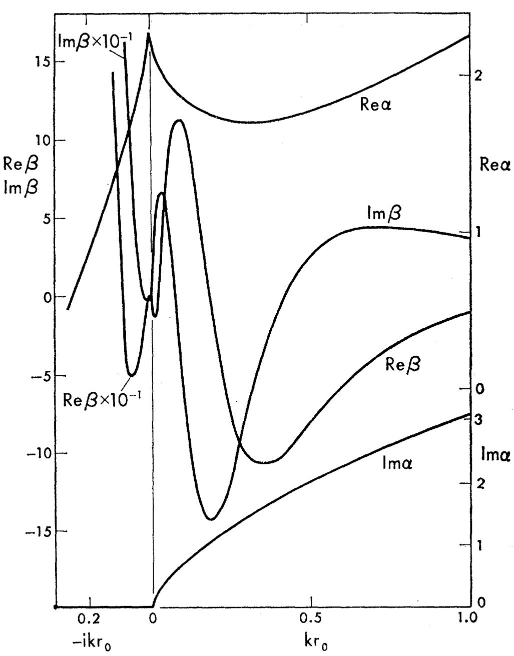

FIG. 11. Square well with a continuous tail, $V=9.6$, $\epsilon=1.0$, $\rho=0$. Note again the unusual threshold behavoir and the oscillations in Reβ and Imβ. (Note the change of scale in β for negative k^2.)

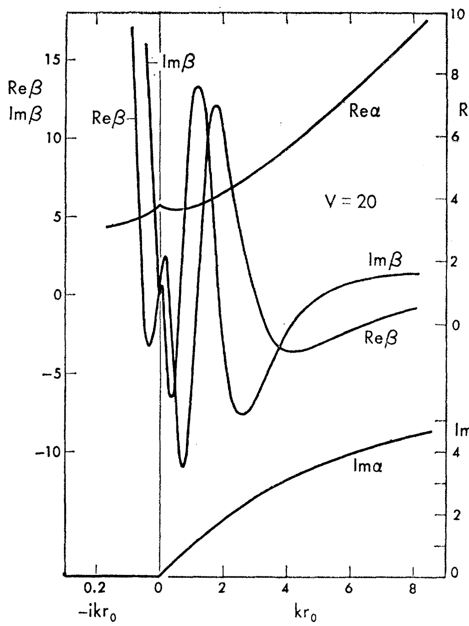

FIG. 12. A typical example of a trajectory for a continuous smooth potential. $V=20$, $\epsilon=1.0$, $\rho=0$ [Eq. (37)]. This is an "abnormal" trajectory. (Note change of the scale of the abscissa for negative k^2.)

finite-range potential

$$V=V_0\delta(r-r_0), \tag{41}$$

for which the S matrix is given by

$$S(k,\lambda)=\frac{1-\frac{1}{2}i\pi VJ_\lambda(x)H_\lambda^{(2)}(x)}{1+\frac{1}{2}i\pi VJ_\lambda(x)H_\lambda^{(1)}(x)}, \tag{42}$$

$$x=kr_0, \quad V=V_0r_0.$$

The poles are given by the solutions of the equation

$$J_\lambda(x)H_\lambda^{(1)}(x)=i2/\pi V.$$

At threshold and $\mathrm{Re}\lambda>0$, we get simply

$$\lambda_0=-V/2, \tag{43}$$

i.e., for a given attractive potential ($V<0$) there is a single trajectory $\alpha_0=-(V+1)/2$ at $k^2=0$. The points $(k^2=0, \lambda=-n)$ are again the indeterminacy points through which the trajectories with $\alpha_0<-\frac{1}{2}$ must pass.[10] Thus for all potentials of this type there is an infinite number of trajectories crossing the negative integers in the λ plane at $k=0$. Only for attractive potentials, there will be one extra trajectory that crosses the positive real value λ_0 given by Eq. (43) at $k=0$. For $\mathrm{Re}\lambda=0$ there may be again, at $k=0$, poles on the imaginary λ axis.

[10] Note that the functions $J_\lambda(x)$ considered as a function of two complex variables x and λ have indeterminacy points $\lambda=-n$, $x=0$.

The other interesting case is the infinite repulsive core at $r=r_0$ for which

$$S=-H_\lambda^{(2)}(x)/H_\lambda^{(1)}(x).$$

Clearly there are no poles for $\mathrm{Re}\lambda>0$ because $H_\lambda^{(1)}(x)$ has no zeros for $\lambda\geqq 0$ and $0\leqq\arg x\leqq\pi$. For $\mathrm{Re}\lambda<0$ the poles at threshold will again only be at the indeterminacy points $\lambda=-n$, $x=0$.

VIII. CONCLUSIONS

We conclude here with a brief summary of the results. For potentials which vanish or behave as $1/r^2$ outside a range r_0, but arbitrary inside, we have shown that:

(1) The S matrix is a meromorphic function of l and k except for a branch point at $k=0$, essential singularties at infinity in both l and k, and possible singularities of the "logarithmic derivative function."

(2) The Sommerfeld-Watson transformation cannot, in general, be written for the original contour but for a modified one so that one can still define Regge poles. However, they do not control the asymptotic behavior in $\cos\theta$ of the amplitude because the amplitude has an essential singularity at infinity in $\cos\theta$. The analytic continuation in l is not unique. We have chosen the one which corresponds to that of the Yukawa-type potentials. Note that the threshold behavior of the poles is the same for all short-range potentials and even for the relativistic case.

For a class of exactly soluble potentials we have explicitly studied the properties of the poles in l, their number, their trajectories and residues as a function of energy (in particular their threshold and asymptotic behavior) as shown in the figures. The trajectories go to infinity with energy in contrast to what happens for Yukawa-type potentials.

We have pointed out the existence of indeterminacy points of the S matrix and located them explicitly.

ACKNOWLEDGMENTS

We thank many members of the Theoretical Group at Berkeley for discussions during the course of this work, in particular C. Zemach, and D. Zwanziger. One of us (F. C.) wishes to thank the Commonwealth Fund for financial support.

APPENDIX I. ASYMPTOTIC BEHAVIOR OF LEGENDRE FUNCTIONS

We give here a more complete discussion than one can find in the literature of the asymptotic behavior of $P_l(z)/\sin\pi l$ for $|l|\to\infty$ and the existence of the Watson-Sommerfeld transform.

The Legendre functions can be transformed into suitable hypergeometric functions and one obtains[11]

$$P_l(z) \underset{|l|\to\infty}{\longrightarrow} \frac{e^{-\zeta/2}}{(\pi l)^{1/2}(1-e^{-2\zeta})^{1/2}}(e^{(l+\frac{1}{2})\zeta}+e^{\pm i(\pi/2)}e^{-(l+\frac{1}{2})\zeta}), \quad \text{(A1)}$$

$$-\tfrac{1}{2}\pi-\epsilon_1\leqq \arg l \leqq \tfrac{1}{2}\pi+\epsilon_2; \quad \epsilon_1\ \epsilon_2>0,$$

and

$$Q_l(z) \underset{|l|\to\infty}{\longrightarrow} (\pi/l)^{1/2}\frac{e^{-(l+1)\zeta}}{(1-e^{-2\zeta})^{1/2}}=(\pi/2l)^{1/2}\frac{e^{-(l+\frac{1}{2})\zeta}}{(\sinh\zeta)^{1/2}}, \quad \text{(A2)}$$

$$|\arg(z\pm 1)|\leqq\pi, \quad |\arg l|\leqq \pi-\delta.$$

Here

$$z=\cosh\zeta, \quad \zeta=\xi+i\eta.$$

We choose $\xi\geqq 0$ and $-\pi<\eta\leqq\pi$ so that ξ is unique when z is given except when z is real and <1. The $\pm$ sign in (A1) is according to Imz being greater than or less than zero, respectively. In z-plane curves with $\xi=$const are confocal ellipses, and curves with $\eta=$const are confocal hyperbolas with foci ± 1.

Let $\lambda=l+\frac{1}{2}=\lambda_1+i\lambda_2$; then

$$P_l(z)/\sin\pi l \underset{|l|\to\infty}{\longrightarrow} l^{-1/2}[Ae^{\lambda_1\xi-\lambda_2\eta-\pi|\lambda_2|}+Be^{-\lambda_1\xi+\lambda_2\eta-\pi|\lambda_2|}].$$

Depending on z and l, one of these exponential dominates, namely, according to $\lambda_1\xi-\lambda_2\eta$ being greater than or less than zero. We find then

$$P_l(z)/\sin\pi l\to l^{-1/2}e^{+\lambda_1\xi-\lambda_2\eta-\pi|\lambda_2|}, \quad \text{(A3)}$$

for z in the shaded regions (Fig. 13), and

$$P_l(z)/\sin\pi l\to l^{-1/2}e^{-\lambda_1\xi+\lambda_2\eta-\pi|\lambda_2|}, \quad \text{(A4)}$$

for z in the blank regions. For $\lambda_2=0$ the blank region reduces to zero, and we obtain the most stringent condition that

$$P_l(z)/\sin\pi l \underset{l_{\text{real}}\to\infty}{\longrightarrow} l^{-1/2}e^{\lambda_1\xi} \quad \text{for all } z. \quad \text{(A5)}$$

For $\epsilon=\epsilon/2$ $(\lambda_1=0)$ we obtain, again for all z,

$$P_l(z)/\sin\pi_l\to l^{-1/2}e^{|\lambda_2||\eta|-\pi|\lambda_2|}, \quad \text{(A6)}$$

where η greater than or less than zero corresponds to upper- and lower-half z planes. Because $|\eta|\leqq\pi$, in order to make the integral in Eq. (23) convergent, the amplitude along the imaginary axis need only vanish as a power in l, $A(l)<O(1/l^{\frac{1}{2}})$, as $|l|\to\infty$, and not as an exponential. [As a matter of fact, if $|\eta|\to\infty$, i.e., z not on negative real axis, $A(l)$ can even increase as a power.]

[11] G. N. Watson, Trans. Cambridge Phil. Soc. 22, 277 (1918).

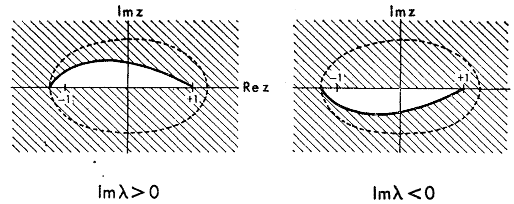

FIG. 13. Regions in z plane for the asymptotic expressions of $P_l(z)/\sin\pi l$.

APPENDIX II. ASYMPTOTIC BEHAVIOR OF TRAJECTORIES

To investigate the asymptotic behavior of the trajectories in the square-well case, we need the asymptotic form of the Bessel functions $J_\nu(x)$. This form is different depending whether x or ν approaches infinity faster. We first look if the trajectories are such that $|x|/|\nu|\to\infty$ as $x\to\infty$, Under these conditions we have the limits

$$H_\nu^{(1)}(x)\to(\pi x/2)^{-1/2}e^{i(x-\pi\nu/2-\pi/4)},$$
$$J_\nu(x)\to(\pi x/2)^{-1/2}\cos(x-\pi\nu/2-\pi/4),$$

which, when inserted into Eq. (33), give

$$\exp[-i(x-\tfrac{1}{2}\pi\nu-\tfrac{1}{2}\pi)]=0.$$

Clearly, we have no solutions unless $x\to -i\infty$. For real x, therefore, $|\nu|$ must go to infinity at least as fast as x. Also for $x\to +i\infty$, the trajectory does not approach a finite limit in the ν plane either. {For $x\to -i\infty$, we find from Eq. (33), $\nu_\infty^2+\nu_\infty+[(2V^2-7)/4]+O(1/x)=0$. Thus, two trajectories seem to have a finite limit if the energy is in the second sheet.}

Secondly, if $|\nu|/|x^2|\to\infty$, we can use the limit

$$J_\nu(x)\to(x/2)^\nu/\Gamma(\nu+1),$$

which shows that there is no solution of (33) for Re$\nu>0$ and Im$\nu>0$. We conclude, therefore, that the trajectories in the l plane go to infinity at infinite energies, in such a way that

$$|\nu|\sim x^\beta, \quad 1\leqq\beta\leqq 2.$$

This is also indicated by the numerical results (Fig. 7).