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Strip Approximation with Regge Poles and Bootstrap Equations for Pion-Pion Scattering.

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Summary. — In an earlier paper concerned with potential scattering a modified Regge representation had been developed that is distinguished from other Regge representations by the property that its analytic structure is of just the type that is required by Mandelstam's double dispersion relation. Here we discuss how this representation may be utilized within the framework of a relativistic S-matrix theory for pion-pion scattering. An approximation to the exact scattering amplitude, termed the strip approximation, is defined as the sum of all possible contributions to the scattering amplitude from Regge poles in the direct or one of the crossed channels. This strip approximation combines the following features: analyticity properties as described by the Mandelstam representation, exact crossing symmetry, « Regge behaviour » at high energies, description of bound states and resonances at low energies, and of forces due to the exchange of reggeized particles. Partial-wave amplitudes are discussed and the unitarity condition for complex angular momentum is introduced. A set of equations is then obtained that seems to be sufficient in principle for a self-consistent determination of the Regge trajectories, as far as they are located in the right half-plane, and of the background term of the scattering amplitude. Due to the complexity of the equations a numerical treatment is not yet in sight. Our work is related to other recent work on bootstrapping of Regge trajectories by Chew and Jones and by Frautschi, Kaus and Zachariasen.

1. – Introduction.

In an earlier paper (1) we have shown that in the theory of potential scattering a decomposition of the scattering amplitude into a Born term, a background term and a number of Regge terms can be obtained such that, apart

(1) M. KRETZSCHMAR: *Nuovo Cimento*, 32, 1405 (1964).

from the Born term, each term separately satisfies a Mandelstam representation with the correct boundary for the spectral function. All unphysical cuts had been eliminated. For the calculation of a Regge term only that part of the corresponding trajectory that was located in the right half of the angular momentum plane ($\mathrm{Re}\, l > -\frac{1}{2}$) had to be known, and as a consequence the spectral function associated with the Regge term was nonvanishing only within certain strips. The strip concept thus induced was closely related to the one of CHEW, FRAUTSCHI and MANDELSTAM [2].

In the present paper we wish to show how these results can be used in the framework of relativistic S-matrix theory if one is willing to accept a Regge formalism that is constructed in analogy to potential scattering theory. Although the basic considerations on which our formalism is built are more general, we discuss for reasons of simplicity only the amplitude for elastic pion-pion scattering. Complications due to spin and due to unequal masses are absent in that case. In the first few Sections we briefly review analyticity properties, crossing symmetry, and implications of elastic unitarity for the pion-pion scattering amplitude, mainly in order to introduce our notation and to cite some important equations which are refered to later on. In Sect. 5 the Regge formalism is introduced. The amplitude for elastic pion-pion scattering when expressed in terms of the generalized potential [3] is assumed to be analogous in structure to the scattering amplitude for potential scattering: The generalized potential and the elastic spectral function of S-matrix theory correspond, respectively, to the first Born approximation and the ordinary spectral function of potential scattering theory. In the latter case we know from ref. [1] how Regge trajectories in the right half of the angular momentum plane contribute to the spectral function within certain strips. For S-matrix theory we now postulate that in an analogous fashion Regge poles in the *direct* channel are associated only with the *elastic* spectral function, not with the generalized potential [*]. For a known set of Regge parameters $\alpha(s)$, $\beta(s)$ the corresponding contribution to the elastic spectral function can be calculated explicitly, using the methods of ref. [1]. This contribution is nonvanishing only within a strip, the width of which is determined by the energy interval over which the trajectory passes through the right half of the angular momentum plane. Carrying out a double dispersion integral over this strip and supplementing it, if necessary, by pole terms we obtain the contribution from the direct-channel Regge pole to the scattering amplitude. A simple appli-

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(2) G. F. CHEW, S. C. FRAUTSCHI and S. MANDELSTAM: *Phys. Rev.*, **126**, 1202 (1962).

(3) G. F. CHEW and S. C. FRAUTSCHI: *Phys. Rev.*, **124**, 264 (1961).

(*) For reasons of simplicity we are discussing here only a one-channel ($\pi\pi$) theory. For a many-channel ($\pi\pi$, $\pi\omega$, ...) theory this prescription has to be slightly modified so as to reflect the extent to which unitarity is taken into account.

cation of crossing symmetry yields immediately all contributions from Regge poles in the crossed channels, and in particular their contribution to the generalized potential.

The sum of all terms contributed by Regge poles, whether in the direct or in the crossed channels, is called the « strip approximation » to the scattering amplitude and combines the following features: analyticity properties as described by the Mandelstam representation, exact crossing symmetry, « Regge behaviour » at high energies, description of bound states and resonances at low energies and of forces due to the exchange of reggeized particles. The strip approximation is thus believed to describe most of the interesting features of the scattering amplitude. In Sect. **6** the contribution of the strip approximation to the partial-wave amplitude of complex index l ($\mathrm{Re}\, l > -\frac{1}{2}$) is discussed in detail. In Sect. **7** unitarity is imposed on the partial-wave amplitudes for all complex l in the right half-plane. The effects of inelastic processes are taken into account only to the extent that they are described already by the strip approximation, otherwise multichannel calculations would be unavoidable. We obtain a pair of equations along each Regge trajectory, and N/D equations elsewhere. In principle, the set of equations obtained seems to be sufficient to determine in a self-consistent manner the parameters of the Regge trajectories, as far as they are located in the right half-plane, as well as the background term of the scattering amplitude. A practical evaluation of our equations, however, will presumably be very difficult.

The work presented in this paper is closely related to recent work by CHEW and JONES (4) and by FRAUTSCHI, KAUS and ZACHARIASEN (5). Whereas the general philosophy adopted in this paper is the same as in the work of these authors our approach differs from theirs in several important points, as will be discussed later on.

2. – Definition of amplitudes and spectral functions.

We consider elastic pion-pion scattering. Each external particle is characterized by a four-momentum p_i ($i=1, 2, 3, 4$) and an isotopic index that assumes the values 1, 2, 3 and is denoted by α, β, γ or δ. In order to have energy and momentum conservation expressed by $p_1+p_2+p_3+p_4=0$ we choose for incoming (outgoing) particles the timelike component p_{i0} of the four-momentum p_i to be positive (negative), while the spacelike part $\boldsymbol{p}_i$ of p_i is parallel (antiparallel) to the actual physical momentum. Let us consider the

(4) G. F. CHEW and C. E. JONES: *Phys. Rev.*, **135**, B 208, 214 (1964).

(5) S. C. FRAUTSCHI, P. E. KAUS and F. ZACHARIASEN: *Phys. Rev.*, **133**, B 1607 (1964).

process where the particles 1 and 2 with isotopic indices α and β are incoming, *i.e.* $(p_1\alpha)+(p_2\beta)\rightarrow(-p_3\gamma)+(-p_4\delta)$. Then we can introduce the Lorentz-invariant Mandelstam variables s, t, u by

$$(2.1)\qquad \begin{cases} s=(p_1+p_2)^2=(p_3+p_4)^2=4(q_s^2+\mu^2)\,,\\ t=(p_3+p_1)^2=(p_2+p_4)^2=-2q_s^2(1-\cos\theta_s)\,,\\ u=(p_2+p_3)^2=(p_1+p_4)^2=-2q_s^2(1+\cos\theta_s)\,,\end{cases}$$

where μ denotes the pion mass, q_s the center-of-mass momentum and θ_s the scattering angle. Since s is the square of the center-of-mass energy we speak of « scattering in the s-channel ». The variables s, t, u are related by

$$(2.2)\qquad s+t+u=4\mu^2\,.$$

For definition of the « t-channel » we consider the reaction

$$(p_3\gamma)+(p_1\alpha)\rightarrow(-p_2\beta)+(-p_4\delta)\,,$$

for which center-of-mass momentum q_t and scattering angle θ_t are related to s, t, u by

$$(2.3)\qquad \begin{cases} t=(p_3+p_1)^2=(p_2+p_4)^2=4(q_t^2+\mu^2)\,,\\ u=(p_2+p_3)^2=(p_1+p_4)^2=-2q_t^2(1-\cos\theta_t)\,,\\ s=(p_1+p_2)^2=(p_3+p_4)^2=-2q_t^2(1+\cos\theta_t)\,,\end{cases}$$

Similarly the « u-channel » is defined on the basis of the reaction

$$(p_2\beta)+(p_3\gamma)\rightarrow(-p_1\alpha)+(-p_4\delta)$$

and in terms of the center-of-mass momentum q_u and scattering angle θ_u we have

$$(2.4)\qquad \begin{cases} u=(p_2+p_3)^2=(p_1+p_4)^2=4(q_u^2+\mu^2)\,,\\ s=(p_1+p_2)^2=(p_3+p_4)^2=-2q_u^2(1-\cos\theta_u)\,,\\ t=(p_3+p_1)^2=(p_2+p_4)^2=-2q_u^2(1+\cos\theta_u)\,.\end{cases}$$

Our formulas for the t-channel and the u-channel are derived from those for the s-channel by cyclic permutation of particles 1, 2, 3. The formulas, therefore, differ slightly from those of CHEW and MANDELSTAM [6], who have

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([6]) G. F. CHEW and S. MANDELSTAM: *Phys. Rev.*, **119**, 467 (1960).

based their definitions on the use of simple transpositions of pairs of particles. The advantage of the present scheme lies in the fact that any relation for an amplitude in the t-channel or u-channel can be derived from the corresponding relation in the s-channel by a cyclic permutation of s, t, u without having to observe additional rules.

Deferring the problem of subtractions to a later discussion we assume the amplitude for pion-pion scattering in the s-channel in the isotopic spin state I $(I=0, 1, 2)$ to satisfy a Mandelstam representation:

$$(2.5)\qquad T_s^I(s,t,u)=\frac{1}{\pi^2}\int\limits_{4\mu^2}^{\infty}\mathrm{d}s'\int\limits_{4\mu^2}^{\infty}\mathrm{d}t'\,\frac{v_s^I(s',t')}{(s'-s)(t'-t)}+\frac{1}{\pi^2}\int\limits_{4\mu^2}^{\infty}\mathrm{d}t'\int\limits_{4\mu^2}^{\infty}\mathrm{d}u'\,\frac{\sigma_s^I(t',u')}{(t'-t)(u'-u)}+\frac{1}{\pi^2}\int\limits_{4\mu^2}^{\infty}\mathrm{d}u'\int\limits_{4\mu^2}^{\infty}\mathrm{d}s'\,\frac{\tau_s^I(u',s')}{(u'-u)(s'-s)}\,.$$

The scattering amplitudes in the t-channel and u-channel will correspondingly be denoted by $T_t^I(t,u,s)$ and $T_u^I(u,s,t)$. In order to make clear in what sense the symbols v, σ, τ for the spectral functions are used we rewrite eq. (2.5) for the t-channel:

$$(2.6)\qquad T_t^I(t,u,s)=\frac{1}{\pi^2}\int\limits_{4\mu^2}^{\infty}\mathrm{d}t'\int\limits_{4\mu^2}^{\infty}\mathrm{d}u'\,\frac{v_t^I(t',u')}{(t'-t)(u'-u)}+\frac{1}{\pi^2}\int\limits_{4\mu^2}^{\infty}\mathrm{d}u'\int\limits_{4\mu^2}^{\infty}\mathrm{d}s'\,\frac{\sigma_t^I(u',s')}{(u'-u)(s'-s)}+\frac{1}{\pi^2}\int\limits_{4\mu^2}^{\infty}\mathrm{d}s'\int\limits_{4\mu^2}^{\infty}\mathrm{d}t'\,\frac{\tau_t^I(s',t')}{(s'-s)(t'-t)}\,.$$

Equation (2.6) is obtained from (2.5) by applying the cyclic permutation $s\to t$, $t\to u$, $u\to s$, as suggested by eq. (2.3). Similarly in the u-channel we have a representation with the spectral functions (*) $v_u^I(u',s')$, $\sigma_u^I(s',t')$ and $\tau_u^I(t',u')$.

The Mandelstam representation, eq. (2.5), implies for the scattering amplitude $T_s^I(s,t,u)$ three representations by single dispersion relations, namely for fixed s, for fixet t, and for fixed u. We denote the discontinuity of the scattering amplitude across the cut along the positive real t-axis (from $t=4\mu^2$ to $t=+\infty$) by $M_{s(t)}^I(s,t)$, similarly we denote the discontinuity across the cut along the positive real u-axis (from $u=4\mu^2$ to $u=+\infty$) by $M_{s(u)}^I(s,u)$.

(*) Note that we have $v_s^I(x,y)\equiv v_t^I(x,y)\equiv v_u^I(x,y)$ and similar identities for σ and τ. The subscripts s, t, u merely indicate which combination of variables appears in the argument, but are superfluous otherwise.

Equation (2.5) then requires

$$M^I_{s(t)}(s,t)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}s'\,\frac{\tau^I_s(s',t)}{s'-s}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}u'\,\frac{\sigma^I_s(t,u')}{u'-(4\mu^2-s-t)}\,, \tag{2.7}$$

$$M^I_{s(u)}(s,u)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}s'\,\frac{\tau^I_s(u,s')}{s'-s}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}t'\,\frac{\sigma^I_s(t',u)}{t'-(4\mu^2-s-u)}\,, \tag{2.8}$$

and the dispersion relation for fixed s for the scattering amplitude $T^I_s(s,t,u)$ reads

$$T^I_s(s,t,u)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}t'\,\frac{M^I_{s(t)}(s,t')}{t'-t}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}u'\,\frac{M^I_{s(u)}(s,u')}{u'-u}\,. \tag{2.9}$$

Because the pion is a boson of isotopic spin 1 the scattering amplitude T^I_s is symmetrical in t and u for $I=0$ and $I=2$, and antisymmetrical for $I=1$, thus we have

$$T^I_s(s,t,u)=(-1)^I\,T^I_s(s,u,t)\,. \tag{2.10}$$

To satisfy this symmetry we have to require (x, y being real variables)

$$M^I_{s(t)}(s,x)=(-1)^I\,M^I_{s(u)}(s,x) \tag{2.11}$$

and

$$\tau^I_s(y,x)=(-1)^I\,\tau^I_s(x,y)\,,\qquad \sigma^I_s(x,y)=(-1)^I\,\sigma^I_s(y,x)\,. \tag{2.12}$$

It is convenient to introduce the amplitude

$$T^{I(+)}_s(s,t)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}t'\,\frac{M^I_{s(t)}(s,t')}{t'-t}\,, \tag{2.13}$$

which allows us to write

$$T^I_s(s,t,u)=T^{I(+)}_s(s,t)+(-1)^I\,T^{I(+)}_s(s,u)\,. \tag{2.14}$$

Denoting the discontinuity of $T^{I(+)}_s$ across the positive real s-axis by M^I_s we have (*)

$$M^I_s(s,t)=\theta(s-4\mu^2)\lim_{\varepsilon\to 0}\frac{1}{2i}\left(T^{I(+)}_s(s+i\varepsilon,t)-T^{I(+)}_s(s-i\varepsilon,t)\right)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}t'\,\frac{\tau^I_s(s,t')}{t'-t} \tag{2.15}$$

(*) We use $M^I_s(s,t)$ as an abbreviation instead of writing $M^{I(+)}_{s(s)}(s,t)$, which would be suggested by consistent use of our notation.

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Now the discontinuity of the full scattering amplitude T^I_s can be written as

$$M^I_{s(s)}(s,t,u)=\theta(s-4\mu^2)\lim_{\varepsilon\to 0}\frac{1}{2i}\big(T^I_s(s+i\varepsilon,t,u)-T^I_s(s-i\varepsilon,t,u)\big)= \\ = M^I_s(s,t)+(-1)^I M^I_s(s,u)\,. \tag{2.16}$$

3. – Crossing symmetry.

In the last Section we have exploited already the symmetry conditions which follow from Bose-Einstein statistics. These are, however, only special cases of the more general crossing symmetry that has been described in more detail in the paper of CHEW and MANDELSTAM (6). As these authors have shown, the scattering amplitudes T^I_s in the s-channel can be written as linear combinations of the scattering amplitudes T^I_t in the t-channel or of the amplitudes T^I_u in the u-channel. More specifically, the following relations are supposed to hold (*):

$$T^I_s(s,t,u)=\sum_{I'=0}^{2}(-1)^{I'}\beta_{II'}T^{I'}_t(t,u,s)=(-1)^I\sum_{I'=0}^{2}\beta_{II'}T^{I'}_u(u,s,t)\,, \tag{3.1}$$

where the coefficients $\beta_{II'}$ are the elements of the so-called crossing matrix

$$(\beta_{II'})=\begin{pmatrix}\frac{1}{3} & 1 & \frac{5}{3}\\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6}\\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{pmatrix}. \tag{3.2}$$

The crossing matrix satisfies the following two relations

$$\sum_{I'=0}\beta_{II'}\beta_{I'I''}=\delta_{II''}\,, \tag{3.3}$$

$$\sum_{I'=0}\beta_{II'}(-1)^{I'}\beta_{I'I''}=(-1)^{I+I''}\beta_{II''}\,. \tag{3.4}$$

Equation (3.1) together with the Mandelstam representation for the amplitudes T^I_s, T^I_t and T^I_u [cf. eq. (2.5) and (2.6)] implies immediately crossing

(*) Equation (3.1) is identical in content to the relations of Chew and Mandelstam, formally, however, it differs by additional sign factors $(-1)^{I'}$ and $(-1)^I$. These arise from the fact that in going from one channel to the other we permute the three variables (s, t, u) cyclically, while CHEW and MANDELSTAM perform only a transposition of two variables [cf. eqs. (2.1), (2.3), (2.4)].

relations for the spectral functions:

$$(3.5)\qquad v_s^I(s,t)=\sum_{I'=0}^{2}(-1)^{I'}\beta_{II'}\tau_t^{I'}(s,t)=(-1)^I\sum_{I'=0}^{2}\beta_{II'}\sigma_u^{I'}(s,t)\,,$$

$$(3.6)\qquad \sigma_s^I(t,u)=\sum_{I'=0}^{2}(-1)^{I'}\beta_{II'}v_t^{I'}(t,u)=(-1)^I\sum_{I'=0}^{2}\beta_{II'}\tau_u^{I'}(t,u)\,,$$

$$(3.7)\qquad \tau_s^I(u,s)=\sum_{I'=0}^{2}(-1)^{I'}\beta_{II'}\sigma_t^{I'}(u,s)=(-1)^I\sum_{I'=0}^{2}\beta_{II'}v_u^{I'}(u,s)\,.$$

Equation (2.12) allows us to eliminate the spectral functions τ^I and σ^I completely and to express everything in terms of the v^I. We obtain

$$(3.8)\qquad \sigma_s^I(t,u)=\sum_{I'=0}^{2}(-1)^{I'}\beta_{II'}\cdot\tfrac{1}{2}\left(v_t^{I'}(t,u)+(-1)^I v_u^{I'}(u,t)\right)\,.$$

The functions v^I must moreover satisfy the symmetry relation

$$(3.9)\qquad v_s^I(s,t)=\sum_{I'=0}^{2}\beta_{II'}v_t^{I'}(t,s)$$

as follows from eq. (3.5). Thus there are only two independent spectral functions (*).

The crossing relations for the absorptive amplitudes $M^I_{s(s)}$, $M^I_{s(t)}$ and $M^I_{s(u)}$ can also be easily obtained from eq. (3.1), but we are not going to need these relations in what follows.

4. – Some implications of unitarity.

The unitarity condition tells us that the absorptive amplitude $M^I_{s(s)}(s,t,u)$ can be written as a sum of contributions from all states that are accessible from the given initial state. We thus have a decomposition

$$(4.1)\qquad M^I_{s(s)}(s,t,u)=M^I_{s(s)\,\mathrm{el}}(s,t,u)+M^I_{s(s)\,\mathrm{in}}(s,t,u)\,,$$

where the first term is the contribution from intermediate states representing elastic pion-pion scattering, and where the second term represents the contribution from inelastic intermediate states. The elastic part can be calculated explicitly in terms of the scattering amplitude T_s^I, and from the expression

(*) We remind the reader of the footnote following eq. (2.6).

so obtained [cf. eqs. (4.5) and (4.6) below] it follows that $M^I_{s(s)\,\mathrm{el}}$ has a decomposition analogous to eq. (2.16), namely

$$M^I_{s(s)\,\mathrm{el}}(s, t, u) = M^I_{s,\,\mathrm{el}}(s, t) + (-1)^I M^I_{s,\,\mathrm{el}}(s, u)\,, \tag{4.2}$$

where $M^I_{s,\,\mathrm{el}}(s, t)$ as a function of t has a cut along the positive t-axis only, from $t = 16\mu^2$ to $t = +\infty$. Combining eqs. (4.2) and (2.16) we find a similar decomposition for $M^I_{s(s)\,\mathrm{in}}(s, t, u)$:

$$M^I_{s(s)\,\mathrm{in}}(s, t, u) = M^I_{s,\,\mathrm{in}}(s, t) + (-1)^I M^I_{s,\,\mathrm{in}}(s, u)\,. \tag{4.3}$$

Thus we can conclude that $M^I_s(s, t)$ can be divided, just as the full absorptive amplitude, into an elastic and an inelastic part

$$M^I_s(s, t) = M^I_{s,\,\mathrm{el}}(s, t) + M^I_{s,\,\mathrm{in}}(s, t)\,. \tag{4.4}$$

As was shown by MANDELSTAM (7) the absorptive amplitude $M^I_{s,\mathrm{el}}(s, t)$ can be expressed as follows in terms of the scattering amplitude (*) $T^{I(+)}_s$:

$$M^I_{s,\,\mathrm{el}}(s, t) = \frac{1}{4\pi}\sqrt{\frac{s-4\mu^2}{s}}\int\limits_0^{+1}\mathrm{d}z'\int\limits_0^{2\pi}\mathrm{d}\varphi'\cdot \left(T^{I(+)*}_s(s, t')\,T^{I(+)}_s(s, t'') + [(-1)^I T^{I(+)}_s(s, u')]^*\cdot(-1)^I T^{I(+)}_s(s, u'')\right) \tag{4.5}$$

where t, t', t''; u, u', u'' and z, z', z'' are related by

$$\left\{\begin{aligned} t &= -2q_s^2(1-z)\,, & u &= -2q_s^2(1+z)\,,\\ t' &= -2q_s^2(1-z')\,, & u' &= -2q_s^2(1+z')\,,\\ t'' &= -2q_s^2(1-z'')\,, & u'' &= -2q_s^2(1+z'')\,,\\ q_s^2 &= \tfrac{1}{4}(s-4\mu^2)\,, & z'' &= zz' + \cos\varphi'\cdot[(1-z^2)(1-z')]^{\frac{1}{2}}. \end{aligned}\right. \tag{4.6}$$

(*) In writing eq. (4.5) we took into account that two-pion states that are eigenstates of total isotopic spin are either symmetrical or antisymmetrical. In the center-of-mass system, when one reverses for a given eigenstate the momenta of both mesons the state remains unchanged up to a sign factor $(-1)^I$. [Examples: $|\pi^+(+\boldsymbol{q})\pi^+(-\boldsymbol{q})\rangle$ or $(1/\sqrt{2})\left(|\pi^+(+\boldsymbol{q})\pi^0(-\boldsymbol{q})\rangle - |\pi^+(-\boldsymbol{q})\pi^0(+\boldsymbol{q})\rangle\right)$]. Therefore, summing over all states of the two-particle Hilbert space, as, *e.g.*, in the evaluation of the unitarity relation, the angular integration has to be restricted to one *half* of the unit sphere otherwise double-counting results. What is defined by most authors as their scattering amplitude is actually only $\frac{1}{2}$ of the true scattering amplitude. These authors integrate over the whole unit sphere and carry an additional factor 2 in eq. (4.7), wherefore their formalism is identical in content to ours. The present formalism, however, conforms more strictly to the usual prescriptions of quantum mechanics (see also footnote (**) on p. 846).

(7) S. MANDELSTAM: *Phys. Rev.*, **112**, 1344 (1958).

Both terms of the integrand give equal contributions to $M^I_{s,\,\mathrm{el}}(s,t)$. The discontinuity of $M^I_{s,\,\mathrm{el}}(s,t)$ across the t-axis is given by

$$(4.7)\qquad v^I_{s,\,\mathrm{el}}(s,t)=\frac{2}{\pi}\cdot\frac{\theta(s-4\mu^2)}{\sqrt{s(s-4\mu^2)}}\int\limits_{4\mu^2}^{\infty}\mathrm{d}t'\int\limits_{4\mu^2}^{\infty}\mathrm{d}t''\,\frac{\theta\big(K(s;t,t',t'')\big)}{\sqrt{K(s;t,t',t'')}}\,M^{I*}_{s(t)}(s,t')\,M^I_{s(t)}(s,t'')$$

with $K(s;t,t',t'')$ defined by

$$(4.8)\qquad K(s;t,t',t'')=t^2+t'^2+t''^2-2(tt'+tt''+t't'')-\frac{4tt't''}{s-4\mu^2}\,.$$

Equation (4.7) serves as the definition of the « elastic spectral function » $v^I_{s,\,\mathrm{el}}(s,t)$. From the fact that the integral is actually extended over only those values of t' and t'' for which $K\ (s;t,t',t'')$ is positive, one can conclude that the elastic spectral function $v^I_{s,\,\mathrm{el}}$ is nonvanishing only in the region (6)

$$(4.9)\qquad s\geqslant 4\mu^2\,,\qquad t\geqslant\frac{16\mu^2 s}{s-4\mu^2}\,.$$

An inelastic spectral function $v^I_{s,\,\mathrm{in}}$ can be defined by the equation

$$(4.10)\qquad v^I_{s,\,\mathrm{in}}(s,t)=v^I_s(s,t)-v^I_{s,\,\mathrm{el}}(s,t)$$

and can be shown to be nonvanishing only in the region (6)

$$(4.11)\qquad s\geqslant 16\mu^2\,,\qquad t\geqslant\frac{4\mu^2 s}{s-16\mu^2}\,.$$

Whereas $v^I_{s,\,\mathrm{el}}$ can be calculated explicitly in terms of the scattering amplitude itself [eq. (4.7)], there is no similar expression available for $v^I_{s,\,\mathrm{in}}$; and even if we would know how to calculate $v^I_{s,\,\mathrm{in}}$ the calculation would involve all possible inelastic amplitudes, which are even less known than the elastic amplitude. Thus we shall introduce an approximate spectral function $\hat{v}^I_s(s,t)$, which comprises all of the elastic spectral function, and of the inelastic spectral function only so much as is necessary to maintain crossing symmetry in connection with $v^I_{s,\,\mathrm{el}}$. To be more specific we define

$$(4.12)\qquad \hat{v}^I_s(s,t)=v^I_{s,\,\mathrm{el}}(s,t)+\sum_{I'=0}^{2}\beta_{II'}\,v^I_{t,\,\mathrm{el}}(t,s)\,.$$

The second term on the right-hand side represents inelastic processes, specifically the contribution from all inelastic collisions that are mediated by one-pion exchange. We shall refer to this approximation as the « elastic unitarity » approximation.

It is easy to show that $\hat{v}_s^I$ satisfies the crossing relation

$$\hat{v}_s^I(s, t) = \sum_{I'=0}^{2} \beta_{II'} \hat{v}_t^{I'}(t, s) . \tag{4.13}$$

The remaining approximate spectral functions can be defined by

$$\begin{aligned}\hat{\sigma}_s^I(t, u) &= \sum_{I'=0}^{2} (-1)^{I'} \beta_{II'} \cdot \tfrac{1}{2}(\hat{v}_t^{I'}(t, u) + (-1)^I \hat{v}_u^{I'}(u, t)) \\ &= \sum_{I'=0}^{2} (-1)^{I'} \beta_{II'} \cdot (v_{t,\,\mathrm{el}}^{I'}(t, u) + (-1)^I v_{u,\,\mathrm{el}}^{I'}(u, t)) ,\end{aligned} \tag{4.14}$$

$$\hat{v}_s^I(u, s) = (-1)^I \hat{v}_s^I(s, u) . \tag{4.15}$$

5. – Introduction of Regge hypothesis and strip-approximation.

The Regge hypothesis will now be introduced, as is customary, by tracing analogies to potential scattering. Such analogies are most clearly exhibited when we make use of the generalized potential due to CHEW and FRAUTSCHI [8], defined by

$$V_{s(t)}^I(s, t) = M_{s(t)}^I(s, t) - \frac{1}{\pi} \int_{4\mu^2}^{\infty} \mathrm{d}s' \, \frac{v_{s,\,\mathrm{el}}^I(s', t)}{s' - s} \tag{5.1}$$

and

$$V_{s(u)}^I(s, u) = (-1)^I V_{s(t)}^I(s, u) . \tag{5.2}$$

With its help the scattering amplitude $T_s^{I(+)}(s, t)$ can be written as

$$T_s^{I(+)}(s, t) = \frac{1}{\pi} \int_{4\mu^2}^{\infty} \mathrm{d}t' \, \frac{V_{s(t)}^I(s, t')}{t' - t} + \frac{1}{\pi^2} \int_{4\mu^2}^{\infty} \mathrm{d}s' \int_{4\mu^2}^{\infty} \mathrm{d}t' \, \frac{v_{s,\,\mathrm{el}}^I(s', t')}{(s' - s)(t' - t)} , \tag{5.3}$$

which is to be compared to the potential scattering amplitude [1]

$$f(s, t) = -\int_{m^2}^{\infty} \mathrm{d}\mu^2 \, \frac{\sigma(\mu^2)}{\mu^2 - t} + \frac{1}{\pi^2} \int_{0}^{\infty} \mathrm{d}s' \int_{4m^2}^{\infty} \mathrm{d}t' \, \frac{\varrho(s', t')}{(s' - s)(t' - t)} \tag{5.4}$$

([8]) G. F. CHEW and S. C. FRAUTSCHI: *Phys. Rev*, **124**, 264 (1961) ignoring for the moment bound states and subtraction terms. In potential scattering, when we calculate the analytically continued partial-wave amplitude we find that the contribution to it from the first term, the potential term, is a holomorphic function of l in the right half-plane ($\operatorname{Re} l > -\frac{1}{2}$), and that all singular terms, in particular the Regge poles, stem from the second, the spectral function term. We shall assume now that the structure of the relativistic amplitude, eq. (5.3), is quite similar: All singularities in the right half l-plane of the s-channel, in particular all s-channel Regge poles, will be supposed to be due to the double integral term in eq. (5.3), and consequently the term due to the generalized potential will be assumed to depend only on the Regge poles in the crossed channels (*). It is well known that the Regge poles in the crossed channels are responsible for generating the forces acting in the direct channel and for the high-energy behaviour in the direct channel. Possible branch cuts in the l-plane must be treated in an analogous fashion.

The partial-wave expansion of the scattering amplitude is given by (**)

$$T_s^I(s, t, u) = \sum_{l=0}^{\infty} (2l+1)\, T_s^{I,l}(s) \cdot \left[P_l\left(1 + \frac{2t}{s-4\mu^2}\right) + (-1)^I P_l\left(1 + \frac{2u}{s-4\mu^2}\right)\right] \tag{5.5}$$

whence we conclude for $T_s^{I(+)}(s, t)$

$$T_s^{I(+)}(s, t) = \sum_{l=0}^{\infty} (2l+1)\, T_s^{I,l}(s)\, P_l\left(1 + \frac{2t}{s-4\mu^2}\right). \tag{5.6}$$

(*) We wish to emphasize that this assumption is an approximation only, appropriate for one-channel calculations as in the present paper, where essentially only the $\pi\pi$-channel is considered. More generally, inelastic amplitudes too may involve Regge poles in the direct channel (*e.g.*, $\pi\pi \to \rho \to \pi\omega$), and via unitarity these may contribute to the inelastic spectral function $v_{s,\text{in}}^I$ of the elastic scattering amplitude and thus to the generalized potential. However, a self-consistent determination of the associated residues can be achieved only within the framework of a multichannel calculation. Our approximation, therefore, can be understood as resulting from a decomposition of the residue function according to the intermediate states in the unitarity relation, of which only one term is retained: $\beta_{ii}(s) = \sum_j \varphi_{ij}(s)\,\beta_{ji}(s) \approx \varphi_{ii}(s)\,\beta_{ii}(s)$ [whence $\varphi_{ii}(s) = 1$; cf. eq. (7.10)]. This decomposition has been discussed by CHENG and SHARP (9), and its connection with the factorization theorem for the residues has been pointed out.

(**) This definition of the partial-wave expansion already takes into account the symmetrical or antisymmetrical character of the initial and final states. The partial-wave amplitude so defined satisfies the usual unitarity relation. The total elastic scattering cross-section is obtained by integrating the amplitude squared times $4/s$ over *half* a unit sphere (because of the symmetry or antisymmetry of the final states):

$$\sigma_{\text{el}}^I(s) = (4/s)\cdot \pi 2 \sum_{l=0}^{\infty} (2l+1)(1+(-1)^{I+l})^2\, |T_s^{I,l}(s)|^2 = (8\pi/q_s^2) \sum_{\substack{l=0,2,4,\dots \text{ for } I=0 \text{ or } 2\\ l=1,3,5,\dots \text{ for } I=1}} (2l+1)\sin^2\delta_l^I.$$

(9) H. CHENG and D. SHARP: *Ann. Phys.*, 22, 481 (1963).

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We can now introduce the Regge hypothesis and the strip-approximation. The Regge hypothesis consists in the following assumption: A Sommerfeld-Watson transformation be applicable to eq. (5.6), so that for physical $s\geqslant 4\mu^2$ the amplitude $T_s^{I(+)}$ can be written in the usual fashion as a sum of terms contributed by Regge poles, plus a background integral that can be taken along the line $\operatorname{Re} l=-\frac{1}{2}$. This includes the assumption that all physical partial waves including S- and P-waves can be obtained from higher partial waves by analytic continuation, in other words that there are no « elementary » particles in the theory. Regge trajectories for isospin I in the s-channel are described by functions $\alpha_n^I(s)$ with $n=1, 2, \ldots$, the associated reduced residues are $\gamma_n^I(s)$. Suppose that for $n=1, 2, \ldots, N_I$ (and for no other n) a part of the n-th trajectory, corresponding to the energy interval $s_{n0}^I\leqslant s\leqslant s_{n1}^I$, lies in the right half of the angular-momentum plane ($\operatorname{Re} l\geqslant-\frac{1}{2}$), then, using the methods of ref. [1], we can write the contribution to the amplitude $T_s^{I(+)}$ due to the presence of the n-th Regge pole in the right half-plane in the following form:

$$\overline{T}_{s,n}^{I(+)}(s,t)=\frac{1}{\pi}\int\limits_{4\mu^2}^{\infty}\mathrm{d}s'\,\frac{g_n^I(s',t)}{s'-s}\,, \tag{5.7}$$

where the absorptive amplitude $g_n^I(s,t)$ is defined by

$$\begin{aligned} g_n^I(s,t) = {} & -\Theta\left(s-\operatorname{Max}[s_{n0}^I, 4\mu^2]\right)\Theta(s_{n1}^I-s)\cdot\frac{1}{2i}\cdot \\ & \cdot\left\{(2\alpha_n^I(s)+1)\gamma_n^I(s)\left(\frac{s}{4}-\mu^2\right)^{\alpha_n^I(s)}\left[\frac{\pi P_{\alpha_n^I(s)}(-z)}{\sin\pi\alpha_n^I(s)}+\frac{1}{\sqrt{2}}\int\limits_{-\infty}^{2\xi}\mathrm{d}x\,\frac{\exp\left[\left(\alpha_n^I(s)+\frac{1}{2}\right)x\right]}{(\cosh x-z)^{\frac{1}{2}}}\right]-\right. \\ & \left.-(2\alpha_n^{I*}(s)+1)\gamma_n^{I*}(s)\left(\frac{s}{4}-\mu^2\right)^{\alpha_n^{I*}(s)}\left[\frac{\pi P_{\alpha_n^{I*}(s)}(-z)}{\sin\pi\alpha_n^{I*}(s)}+\frac{1}{\sqrt{2}}\int\limits_{-\infty}^{2\xi}\mathrm{d}x\,\frac{\exp\left[\left(\alpha_n^{I*}(s)+\frac{1}{2}\right)x\right]}{(\cosh x-z)^{\frac{1}{2}}}\right]\right\}. \end{aligned} \tag{5.8}$$

Here the abbreviation $z=1+2t\cdot(s-4\mu^2)^{-1}$ has been used, and ξ is determined by the fact that the smallest mass, that can be exchanged between two pions, is 2μ. Thus (cf. ref. [1])

$$\cosh\xi=1+\frac{2\cdot 4\mu^2}{s-4\mu^2}=\frac{s+4\mu^2}{s-4\mu^2}\,, \tag{5.9}$$

which is equivalent to

$$\cosh 2\xi=1+\frac{2}{s-4\mu^2}\cdot\frac{16\mu^2 s}{s-4\mu^2}=1+\frac{2t_0}{s-4\mu^2}\,. \tag{5.10}$$

The absorptive amplitude $g_n^I(s, t)$ has a branch cut in the t-plane, from $t = t_0 = 16\mu^2 s\cdot(s-4\mu^2)^{-1}$ to $t = +\infty$. The discontinuity across the branch cut is easily determined to be [1]

$$(5.11)\qquad \varrho_n^I(s, t) = \Theta(s-\mathrm{Max}[s_{n0}^I, 4\mu^2])\,\Theta(s_{n1}^I - s)\,\Theta(t-t_0)\,\frac{1}{\sqrt{2}}\int\limits_{t_0}^{t}\frac{d\tau}{\sqrt{t-\tau}}\cdot$$

$$\cdot\,\mathrm{disc}_s\left\{\frac{(2\alpha_n^I(s)+1)\gamma_n^I(s)2^{-\alpha_n^I(s)}\cdot[\frac{1}{2}(s-4\mu^2)+\tau+\sqrt{(s-4\mu^2)\tau+\tau^2}]^{\alpha_n^I(s)+\frac{1}{2}}}{\sqrt{(s-4\mu^2)\tau+\tau^2}}\right\}.$$

This spectral function may be used to write a subtracted dispersion relation in t for the absorptive amplitude $g_n^I(s, t)$. The function $\varrho_n^I(s, t)$ is nonvanishing only in a strip parallel to the t-axis, $\mathrm{Max}[s_{n0}^I, 4\mu^2]\leqslant s\leqslant s_{n1}^I$, which strip is contained in the larger region where $v_{s,\,\mathrm{el}}^I(s, t)$ is nonvanishing as defined in (4.9). We shall consider $\varrho_n^I(s, t)$ as the contribution to the elastic spectral function $v_{s,\,\mathrm{el}}^I(s, t)$ due to the presence in the right half angular-momentum plane of the n-th Regge pole in the s-channel. The total contribution from s-channel Regge poles (in the right half-plane) to the elastic spectral function can be written as

$$(5.12)\qquad \bar{v}_{s,\,\mathrm{el}}^I(s, t) = \sum_{n=1}^{N_I}\varrho_n^I(s, t)\,.$$

The contribution from a possible branch cut in the angular-momentum plane should also be separated from the background term and be included in $\bar{v}_{s,\,\mathrm{el}}^I(s, t)$.

Crossing symmetry now tells us how to obtain the general contribution from Regge poles, in the direct channel and in the crossed channels, to the three spectral functions v_s^I, σ_s^I, and τ_s^I. Recalling eqs. (4.12)–(4.15) we define

$$(5.13)\qquad \bar{v}_s^I(s, t) = \bar{v}_{s,\,\mathrm{el}}^I(s, t) + \sum_{I'=0}^{2}\beta_{II'}\bar{v}_{t,\,\mathrm{el}}^{I'}(t, s)\,,$$

$$(5.14)\qquad \bar{\sigma}_s^I(t, u) = \sum_{I'=0}^{2}(-1)^{I'}\beta_{II'}\left(\bar{v}_{t,\,\mathrm{el}}^{I'}(t, u) + (-1)^I\bar{v}_{u,\,\mathrm{el}}^{I'}(u, t)\right)\,,$$

$$(5.15)\qquad \bar{\tau}_s^I(u, s) = (-1)^I\bar{v}_s^I(s, u)\,.$$

The spectral functions so defined satisfy all conditions of crossing symmetry, and they are nonvanishing only within certain « strips ».

Thus the prescription to replace in the Mandelstam representation, eq. (2.5), the exact spectral functions v_s^I, σ_s^I, τ_s^I by the corresponding barred spectral functions $\bar{v}_s^I$, $\bar{\sigma}_s^I$, $\bar{\tau}_s^I$ defines an approximation $\bar{T}_s^I$ to the exact scattering amplitude $T_s^I(s, t, u)$ that we may term appropriately the « strip approximation » (*).

(*) For some remarks concerning the validity of this approximation see the end of this Section.

In essentially the same spirit the term has been used in the more recent work of CHEW, FRAUTSCHI and MANDELSTAM [2]. From the above it is obvious how to define the « strip approximation » for any other quantity that can be deduced from the Mandelstam representation, like absorptive amplitudes, generalized potentials etc.

In order to analyse the structure of the strip approximation it is useful to define

$$G^I(s,t)=\sum_{n=1}^{N_I} g_n^I(s,t)\,, \tag{5.16}$$

$$H^I(s,t)=\sum_{n=1}^{N_I} h_n^I(s,t)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}s'\,\frac{\bar{v}^I_{s,\,\mathrm{el}}(s',t)}{s'-s}\,, \tag{5.17}$$

where $g_n^I(s,t)$ is given by eq. (5.8) and $h_n^I(s,t)$ by

$$h_n^I(s,t)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}s'\,\frac{\varrho_n^I(s',t)}{s'-s}\,. \tag{5.18}$$

The integrals in these equations are convergent as the spectral functions are nonvanishing only across the finite width of the strip. The absorptive amplitude $M^I_{s(t)}(s,t)$ in the strip approximation can be expressed [cf. eq. (2.7)] in terms of these functions:

$$\bar{M}^I_{s(t)}(s,t)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}s'\,\frac{\bar{v}^I_s(s',t)}{s'-s}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}u'\,\frac{\bar{\sigma}^s(t,u')}{u'-(4\mu^2-s-t)} \tag{5.19}$$

and using eqs. (5.13), (5.14), (5.16) and (5.17) we obtain

$$\begin{aligned}\bar{M}^I_{s(t)}(s,t)=H^I(s,t)+\sum_{I'=0}^{2}\beta_{II'}G^{I'}(t,s)+{}\\ +\sum_{I'=0}^{2}(-1)^{I'}\beta_{II'}\big(G^{I'}(t,4\mu^2-s-t)+(-1)^I H^{I'}(4\mu^2-s-t,t)\big)\,.\end{aligned} \tag{5.20}$$

From eqs. (5.1) and (5.17) we obtain the generalized potential in the strip approximation:

$$\begin{aligned}\bar{V}^I_{s(t)}(s,t)=\sum_{I'=0}^{2}\beta_{II'}G^{I'}(t,s)+{}\\ +\sum_{I'=0}^{2}(-1)^{I'}\beta_{II'}\big(G^{I'}(t,4\mu^2-s-t)+(-1)^I H^{I'}(4\mu^2-s-t,t)\big)\,.\end{aligned} \tag{5.21}$$

The first term on the right-hand side represents the effect of (peripheral) inelastic processes in the s-channel, and has in the s-plane a right-hand cut only, starting above $s=16\mu^2$. The second term, due to the third spectral function, has only a left-hand cut in the s-plane and describes the forces due to the crossed channels. The absorptive amplitude $M_s^I(s, t)$, defined by eq. (2.15), can similarly be approximated by

$$\overline{M}_s^I(s, t) = G^I(s, t) + \sum_{I'=0}^{2} \beta_{II'} H^{I'}(t, s) . \tag{5.22}$$

Equations such as (2.15) or (5.19) are in general not valid as they stand, they should properly be written with the necessary subtractions applied. In the strip approximation, however, when the corresponding absorptive amplitudes are written in the form of (5.20) or (5.22), all subtraction problems are automatically taken care of. One can also see that the integral representing the contribution to the scattering amplitude from the generalized potential

$$\frac{1}{\pi}\int_{4\mu^2}^{\infty} \mathrm{d}t' \, \frac{\overline{V}^I_{s(t)}(s, t')}{t'-t} \tag{5.23}$$

is convergent, as follows from the finite width of the strip and from the results of the Appendix of ref. (1).

Finally we can find a decomposition of the strip-approximated scattering amplitude $\overline{T}_s^{I(+)}(s, t)$ similar to the decomposition obtained in ref. (1) for the case of potential scattering [eq. (2.28) of ref. (1)]. We only have to observe that

$$\frac{1}{\pi}\int_{16\mu^2}^{\infty} \mathrm{d}t' \, \frac{h_n^I(s, t')}{t'-t} = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \mathrm{d}s' \, \frac{g_n^I(s', t)}{s'-s} , \tag{5.24}$$

where the left-hand side should actually be written in a properly subtracted form, while the right-hand side does not need any subtraction. We can then write

$$\overline{T}_s^{I(+)}(s, t) = \overline{T}_{s,\mathrm{B}}^{I(+)}(s, t) + \overline{T}_{s,\,\mathrm{in}}^{I(+)}(s, t) + \sum_{n=1}^{N_I} \overline{T}_{s,n}^{I(+)}(s, t) \tag{5.25}$$

with the right hand terms defined by

$$\overline{T}_{s,\,\mathrm{B}}^{I(+)}(s, t) = \tag{5.26}$$

$$= \sum_{I'=0}^{2} (-1)^{I'} \beta_{II'} \left(\frac{1}{\pi}\int_{4\mu^2}^{\infty} \mathrm{d}t' \, \frac{G^{I'}(t', 4\mu^2 - s - t')}{t'-t} + \frac{(-1)^{I}}{\pi}\int_{16\mu^2}^{\infty} \mathrm{d}t' \, \frac{H^{I'}(4\mu^2 - s - t', t')}{t'-t} \right),$$

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$$\bar{T}^{I(+)}_{s,\,\mathrm{in}}(s,t)=\sum_{I'=0}^{2}\beta_{II'}\cdot\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}t'\,\frac{G^{I'}(t',s)}{t'-t}\,, \tag{5.27}$$

$$\bar{T}^{I(+)}_{s,n}(s,t)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}s'\,\frac{g^I_n(s',t)}{s'-s} \qquad (n=1,2,\dots,N_I). \tag{5.28}$$

Equation (5.26) represents the analogon of the nonrelativistic Born term. One therefore associates the amplitude $\bar{T}^{I(+)}_{s,\mathrm{B}}$ with the forces acting in pion-pion states in the s-channel. In the following we will refer to it as the « force term ». The amplitude $\bar{T}^{I(+)}_{s,\mathrm{B}}$ has a cut in the s-plane along the negative real axis from $s=-16\mu^2$ to $s=-\infty$. Equation (5.27) describes diffraction scattering due to peripheral inelastic processes, and dominates the high-energy behaviour of the amplitude. It has a cut in the s-plane from $s=16\mu^2$ to $s=+\infty$. Finally the amplitude $\bar{T}^{I(+)}_{s,n}$ describes the effect of the n-th s-channel Regge pole for isotopic spin I. This amplitude dominates the low-energy resonance region. It has a finite right-hand cut in the s-plane, the length of which is determined by the width of the corresponding strip of the spectral function.

It is the appropriate place here to add a few remarks about contributions from possible pole terms which have been disregarded up to now. Pole terms always occur when a trajectory $\alpha^I_n(s)$ passes through a nonnegative integer l as s is increased from $-\infty$ to $4\mu^2$. They are associated with physical bound states of the $\pi\pi$-system when l is a physical angular momentum for isotopic spin I. Experimentally such bound states are believed to be nonexisting, but the framework of the theory must be general enough to allow to investigate the possibility of such bound states theoretically. On the other hand, even in the absence of bound states, we must ask whether or not pole terms associated with nonphysical angular-momentum values l might have any effects, *e.g.*, from an $I=1$ trajectory passing through $l=0$ or from a Pomeranchuk-type trajectory passing through $l=1$.

Since we have constructed the Regge formalism of our relativistic S-matrix theory so as to be closely analogous to that of potential scattering theory we can take over the results of ref. [1], and upon application of crossing symmetry we obtain the desired crossing-symmetrical set of pole terms. Suppose that the Regge trajectory $\alpha^I_n(s)$ passes through $l=0$ at $s=\sigma^I_{n0}$, through $l=1$ at $s=\sigma^I_{n1}$, ..., through $l=l^I_n$ at $s=\sigma^I_{nl^I_n}$, and then turns away from the real axis into the upper half of the complex l-plane, and denote the first derivative of $\alpha^I_n(s)$ by $\alpha^{I\prime}_n(s)$, then the results of ref. [1] imply that eq. (5.28) should be replaced by

$$\bar{T}^{I(+)}_{s,n}(s,t)=R^I_n(s,t)+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\mathrm{d}s'\,\frac{g^I_n(s',t)}{s'-s}\,, \tag{5.29}$$

where

$$(5.30)\qquad R_n^I(s,t)=\sum_{l=0}^{l_n^I}\frac{1}{\sigma_{nl}^I-s}\cdot\left[\frac{(2l+1)\beta_n^I(\sigma_{nl}^I)}{\alpha_n^{I\prime}(\sigma_{nl}^I)}P_l\left(1+\frac{2t}{\sigma_{nl}^I-4\mu^2}\right)\right].$$

This result can also be expressed by saying that the double-dispersion integral over the spectral function $\varrho_n^I(s,t)$ is associated with the set of pole terms $R_n^I(s,t)$. Crossing symmetry, however, requires that each such dispersion integral in the representation of the complete scattering amplitude must be associated with a corresponding set of pole terms. Considering now the complete scattering amplitude in the strip approximation

$$(5.31)\qquad \overline{T}_s^I(s,t,u)=\overline{T}_s^{I(+)}(s,t)+(-1)^I\,\overline{T}_s^{I(+)}(s,u)$$

and defining

$$(5.32)\qquad R^I(s,t)=\sum_{n=1}^{N_I}R_n^I(s,t)\,,$$

we obtain the following pole terms. From Regge poles in the direct channel

$$(5.33)\qquad R^I(s,t)+(-1)^I R^I(s,u)\,,$$

secondly, associated with the amplitude $\overline{T}^I_{s,\,\text{in}}(s,t,u)$ that describes the high-energy diffraction scattering

$$(5.34)\qquad \sum_{I'=0}^{2}\beta_{II'}\left(R^{I'}(t,s)+(-1)^I R^{I'}(u,s)\right)$$

and finally, I associated with the force term $\overline{T}^I_{s,\text{B}}(s,t,u)$ that originates from the third spectral function [cf. eq. (5.14)]

$$(5.35)\qquad \sum_{I'=0}^{2}(-1)^{I'}\beta_{II'}\left(R^{I'}(t,u)+(-1)^I R^{I'}(u,t)\right).$$

It is now easy to see that the $l=0$ pole of an $I=1$ trajectory (*e.g.*, ρ-meson) contributes nothing at all [expression (5.33) vanishes and (5.34) cancels (5.35)]. On the other hand, a Pomeranchuk-type ($I=0$) trajectory $\alpha_{\text{P}}(s)$ with residue (*) $\beta_{\text{P}}(s)$ passing through $l=1$ at $s=\sigma$ (*e.g.*, $\sigma=0$) gives rise to a set of pole terms that add up to a constant:

$$5\cdot\frac{2\beta_{\text{P}}(\sigma)}{\alpha'_{\text{P}}(\sigma)(\sigma-4\mu^2)}\ \text{for}\ I=0,\qquad 0\ \text{for}\ I=1,\qquad 2\cdot\frac{2\beta_{\text{P}}(\sigma)}{\alpha'_{\text{P}}(\sigma)(\sigma-4\mu^2)}\ \text{for}\ I=2.$$

(*) As usual $\beta_{\text{P}}(s)$ is assumed to vanish when $\alpha_{\text{P}}(s)$ passes through $l=0$.

In a theory taking the Pomeranchuk trajectory into consideration eq. (2.5) must be supplemented by these constants.

In closing this Section we add a few remarks on the range of validity of the strip approximation. We remember that the approximation has actually been obtained in a two-step process: At first we made the replacement $v_s^I \approx \hat{v}_s^I$, $\sigma_s^I \approx \hat{\sigma}_s^I$, $\tau_s^I \approx \hat{\tau}_s^I$, based on the crossing-symmetrical « elastic unitarity » approximation. Secondly we assumed $\hat{v}_s^I \approx \bar{v}_s^I$, $\hat{\sigma}_s^I \approx \bar{\sigma}_s^I$, $\hat{\tau}_s^I \approx \bar{\tau}_s^I$, thereby dropping the background integrals along the line $\operatorname{Re} l = -\frac{1}{2}$.

Let us introduce an amplitude $\widetilde{T}_s^{I(+)}(s, t)$ representing the difference between the *exact* scattering amplitude and the strip approximation, that is

$$T_s^{I(+)}(s, t) = \widetilde{T}_s^{I(+)}(s, t) + \overline{T}_s^{I(+)}(s, t)\ (^*)\ . \tag{5.36}$$

When we denote by $\hat{T}_s^{I(+)}(s, t)$ the amplitude obtained from $T_s^{I(+)}(s, t)$ by the elastic unitarity approximation of Sect. **4**, then we can write $\widetilde{T}_s^{I(+)}(s, t)$ as being composed of two parts:

$$\widetilde{T}_s^{I(+)}(s, t) = \left(T_s^{I(+)}(s, t) - \hat{T}_s^{I(+)}(s, t)\right) + \left(\hat{T}_s^{I(+)}(s, t) - \overline{T}_s^{I(+)}(s, t)\right)\ . \tag{5.38}$$

The second part obviously consists of the various contributions due to background integrals associated with the line $\operatorname{Re} \bar{l} = -\frac{1}{2}$. It is thus expected to be a rather smooth function and to possess no particularly interesting structure. Its asymptotic properties will not give rise to any troubles. In the low-energy *resonance* region it will certainly be negligible.

About the first part much less is known. It describes the influence of all nonperipheral inelastic channels in $\pi\pi$-scattering (*e.g.*, $\pi\pi \to \pi\omega$). These have been neglected in the crossing-symmetrical « elastic unitarity » approximation of Sect. **4**, where only inelastic diagrams with one-pion exchange were taken into account. The first term in eq. (5.38) thus describes the majority of inelastic effects, and it may possibly be of considerable importance in any quantitative calculation, as is indicated, *e.g.* by the work of ZACHARIASEN and ZEMACH [10]. This term, therefore, needs further consideration. In general one will be forced into a many-channel calculation which is, however, beyond the scope of the present paper.

The main advantage of the strip approximation derives from the fact that the approximated scattering amplitude and all derived quantities are mathe-

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(*) Taking into account eqs. (2.13) and (5.20) we can write (for later use) a similar equation for the discontinuity of $T_s^{I(+)}(s, t)$ across the t-axis:

$$M_{s(t)}^I(s, t) = \widetilde{M}_{s(t)}^I(s, t) + \overline{M}_{s(t)}^I(s, t)\ . \tag{5.37}$$

([10]) F. ZACHARIASEN and C. ZEMACH: *Phys. Rev.*, **128**, 849 (1962).

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matically well-defined functions of the Regge parameters $\alpha_n^I(s)$ and $\gamma_n^I(s)$, containing no further unknown parameters. They can be calculated in a straightforward way once a set of Regge parameters is given. At least for resonating amplitudes in the low-energy region, where inelastic effects are still small, a strip approximation alone should give a valid description of the actual scattering amplitude.

6. – Partial-wave amplitudes.

In this Section we shall discuss properties of partial-wave amplitudes for complex l, in particular we are interested in the contribution to the partial-wave amplitude from the various components that make up the strip-approximated scattering amplitude. We shall see that the poles in the angular momentum plane are produced only by the corresponding Regge terms $T_{s,n}^{I(+)}(s, t)$ in the scattering amplitude.

Partial-wave amplitudes for integer l have already been defined by eq. (5.6). To extend the definition to complex l we make use of eq. (2.13) and of Heine's expansion for $(t'-t)^{-1}$. Thus we obtain

$$T_s^{I,l}(s) = \frac{2}{\pi(s-4\mu^2)} \int_{4\mu^2}^{\infty} \mathrm{d}t' M_{s(t)}^I(s, t') Q_l\left(1 + \frac{2t'}{s-4\mu^2}\right), \tag{6.1}$$

where Q_l is a Legendre function of the second kind. This equation is valid for all complex l with $\operatorname{Re} l > A^I$, where by definition the number A^I is chosen so that no Regge singularities are found to the right of the line $\operatorname{Re} l = A^I$:

$$A^I = \operatorname{Max}\{\operatorname{Re}\alpha_n^I(s):\ s_{n0}^I \leqslant s \leqslant s_{n1}^I,\ n = 1, 2, \ldots, N_I\}. \tag{6.2}$$

As is well known it is advantageous with regard to analytic properties to introduce a reduced partial-wave amplitude

$$t_s^{I,l}(s) = \left(\frac{1}{4}s - \mu^2\right)^{-l} \cdot T_s^{I,l}(s). \tag{6.3}$$

Using eqs. (5.20) and (5.37) we can immediately write down a decomposition of the partial-wave amplitude, which is analogous to eqs. (5.25) and (5.36),

$$t_s^{I,l}(s) = \tilde{t}_s^{I,l}(s) + \bar{t}_{s,\mathrm{B}}^{I,l}(s) + \bar{t}_{s,\mathrm{in}}^{I,l}(s) + \sum_{n=0}^{N_I} \bar{t}_{s,n}^{I,l}(s). \tag{6.4}$$

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The various terms on the right-hand side are given by

$$(6.5)\qquad \tilde{t}_s^{I,l}(s) = \left(\frac{s}{4}-\mu^2\right)^{-(l+1)} \cdot \frac{1}{2\pi}\int\limits_{4\mu^2}^{\infty} \mathrm{d}t'\, \tilde{M}^{I}_{s(t)}(s,t')\, Q_l\left(1+\frac{2t'}{s-4\mu^2}\right),$$

$$(6.6)\qquad \bar{t}^{I,l}_{s,\mathrm{B}}(s) = \left(\frac{s}{4}-\mu^2\right)^{-(l+1)} \cdot \frac{1}{2\pi}\sum_{I'=0}^{2}(-1)^{I'}\beta_{II'}\cdot$$

$$\cdot\left[\int\limits_{4\mu^2}^{\infty}\mathrm{d}t'\, G^{I'}(t', 4\mu^2-s-t')\, Q_l\left(1+\frac{2t'}{s-4\mu^2}\right)+\right.$$

$$\left. +(-1)^{I}\int\limits_{16\mu^2}^{\infty}\mathrm{d}t'\, H^{I'}(4\mu^2-s-t', t')\, Q_l\left(1+\frac{2t'}{s-4\mu^2}\right)\right],$$

$$(6.7)\qquad \bar{t}^{I,l}_{s,\mathrm{in}}(s) = \left(\frac{s}{4}-\mu^2\right)^{-(l+1)} \cdot \frac{1}{2\pi}\sum_{I'=1}^{2}\beta_{II'}\int\limits_{4\mu^2}^{\infty}\mathrm{d}t'\, G^{I'}(t', s)\, Q_l\left(1+\frac{2t'}{s-4\mu^2}\right),$$

$$(6.8)\qquad \bar{t}^{I,l}_{s,n}(s) = \left(\frac{s}{4}-\mu^2\right)^{-(l+1)} \cdot \frac{1}{2\pi}\int\limits_{16\mu^2}^{\infty}\mathrm{d}t'\, h^{I}_{n}(s,t')\, Q_l\left(1+\frac{2t'}{s-4\mu^2}\right).$$

All these amplitudes are analytic in the complex s-plane, apart from branch cuts along the real axis. On the real axis for each amplitude we have a left-hand branch cut from $s=-\infty$ to $s=0$, a region free from singularities from $s=0$ to $s=4\mu^2$, and, with the exception of the «force term» $\bar{t}^{I,l}_{s,\mathrm{B}}(s)$, a right-hand branch cut from $s=4\mu^2$ to $s=+\infty$. The force term has a left-hand branch cut only. Thus for each term in eq. (6.4) we have a dispersion relation of the type

$$(6.9)\qquad t_s^{I,l}(s) = \frac{1}{\pi}\int\limits_{4\mu^2}^{\infty}\mathrm{d}s'\,\frac{\mathrm{disc}\, t_s^{I,l}(s')}{s'-s} + \frac{1}{\pi}\int\limits_{-\infty}^{0}\mathrm{d}s'\,\frac{\mathrm{disc}\, t_s^{I,l}(s')}{s'-s}$$

properly subtracted if necessary. As discussed in detail in ref. ([1]), these dispersion relations may be utilized for the analytic continuation of the partial-wave amplitudes into the remaining part of the right half of the l-plane $(-\frac{1}{2}<\mathrm{Re}\, l\leqslant A^I)$.

Next we will give the explicit expressions for the discontinuities across the branch cuts. Using the same tricks as in ref. ([1]) we obtain the following result for the Regge terms $\bar{t}^{I,l}_{s,n}(s)$:

For $4\mu^2 \leqslant s < +\infty$

$$\text{disc}\,\bar{t}^{I,l}_{s,n}(s) = \Theta(s - \text{Max}\,[s^I_{n0}, 4\mu^2])\,\Theta(s^I_{n1} - s)\cdot \cdot\,\text{disc}\left\{\gamma^I_n(s)\cdot\left(\frac{s}{4} - \mu^2\right)^{\alpha^I_n(s)-l}\cdot\frac{2\alpha^I_n(s)+1}{2l+1}\cdot\frac{\exp\left[2\xi\left(\alpha^I_n(s)-l\right)\right]}{l-\alpha^I_n(s)}\right\}, \tag{6.10}$$

and for $-\infty < s \leqslant -12\mu^2$

$$\text{disc}\,\bar{t}^{I,l}_{s,n}(s) = -\frac{1}{4}\left(\mu^2 - \frac{s}{4}\right)^{-(l+1)}\cdot\int\limits_{16\mu^2}^{4\mu^2-s}\text{d}t'\,h^I_n(s,t')\,P_l\left(-1-\frac{2t'}{s-4\mu^2}\right). \tag{6.11}$$

There are no singularities in the interval $-12\mu^2 \leqslant s \leqslant 0$, so that we have a larger gap free from singularities $(-12\mu^2 \leqslant s \leqslant +4\mu^2)$ than for the complete partial-wave amplitude, eq. (6.4).

For the contribution describing the diffraction scattering due to inelastic processes in the s-channel, eq. (6.7), we obtain for $16\mu^2 \leqslant s \leqslant +\infty$

$$\text{disc}\,\bar{t}^{I,l}_{s,\text{in}}(s) = \sum_{I'=0}^{2}\beta_{II'}\left(\frac{s}{4} - \mu^2\right)^{-(l+1)}\cdot\frac{1}{2\pi}\int\limits_{4\mu^2}^{\infty}\text{d}t'\,\bar{v}^{I'}_{t,\text{el}}(t', s)\,Q_l\left(1+\frac{2t'}{s-4\mu^2}\right) \tag{6.12}$$

and for $-\infty \leqslant s \leqslant 0$

$$\text{disc}\,\bar{t}^{I,l}_{s,\text{in}}(s) = -\frac{1}{4}\left(\mu^2 - \frac{s}{4}\right)^{-(l+1)}\cdot\sum_{I'=0}^{2}\beta_{II'}\int\limits_{4\mu^2}^{4\mu^2-s}\text{d}t'\,G^{I'}(t', s)\,P_l\left(-1-\frac{2t'}{s-4\mu^2}\right). \tag{6.13}$$

We notice that the discontinuity vanishes in the physical region below the inelastic threshold, as it should be. Thus we have again a singularity-free gap of length $16\mu^2$. In eq. (6.11) as well as in (6.13) the discontinuity across the left-hand branch cut is merely a reflection of the properties of the Legendre function Q_l and has no dynamical origin.

This is different in the case of the « force term », eq. (6.6). As we mentioned already, there is no right-hand branch cut whatsoever, and the left-hand branch cut is entirely of dynamical origin, describing the forces due to the crossed channels. To give the explicit expression for the discontinuity, we first must define the Legendre function of the second kind on the interval $-1 \leqslant x \leqslant +1$ of the real axis. This is done by [11]

$$\widetilde{Q}_l(x) = \tfrac{1}{2}\left(Q_l(x+i0) + Q_l(x-i0)\right) \qquad (-1 \leqslant x \leqslant +1). \tag{6.14}$$

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[11] W. MAGNUS and F. OBERHETTINGER: *Formeln und Sätze für die Speziellen Funktionen der Mathematischen Physik* (Berlin, Göttingen, Heidelberg, 1948), p. 76.

The discontinuity is then obtained as

$$\text{disc}\,\bar{t}^{I,l}_{s,\text{B}}(s) = -\frac{1}{4}\left(\mu^2 - \frac{s}{4}\right)^{-(l+1)} \cdot \left(\Delta_1^{I,l}(s) + \Delta_2^{I,l}(s) + \Delta_3^{I,l}(s)\right) \tag{6.15}$$

with

$$\Delta_1^{I,l}(s) = \sum_{I'=0}^{2}(-1)^{I'}\beta_{II'}\Theta(-s)\cdot \cdot \int_{4\mu^2}^{4\mu^2-s} \text{d}t' \cdot \text{Re}\, G^{I'}(t', 4\mu^2 - s - t') \cdot P_l\left(-1 - \frac{2t'}{s-4\mu^2}\right), \tag{6.16}$$

$$\Delta_2^{I,l}(s) = \sum_{I'=0}^{2}(-1)^{I+I'}\beta_{II'}\Theta(-12\mu^2 - s)\cdot \cdot \int_{16\mu^2}^{4\mu^2-s} \text{d}t' \cdot \text{Re}\, H^{I'}(4\mu^2 - s - t', t') \cdot P_l\left(-1 - \frac{2t'}{s-4\mu^2}\right), \tag{6.17}$$

$$\Delta_3^{I,l}(s) = \sum_{I'=0}^{2}(-1)^{I'}\beta_{II'}\Theta(-16\mu^2 - s)\cdot \frac{2}{\pi}\int_{16\mu^2}^{-s} \text{d}u'\, \bar{v}^{I'}_{t,\text{el}}(4\mu^2 - s - u', u')\cdot \cdot\left\{\widetilde{Q}_l\left(1 + \frac{2u'}{s-4\mu^2}\right) + (-1)^I\,\widetilde{Q}_l\left(-1 - \frac{2u'}{s-4\mu^2}\right)\right\}. \tag{6.18}$$

In the last equation the expression in curly brackets vanishes for all integer odd l when $I=1$, and for all integer even l when $I=0$ or 2. Thus $\Delta_3^{I,l}(s)$ vanishes for all physical values of l, but it is in general nonvanishing for unphysical l.

Having obtained now all the discontinuities we can perform the analytic continuation in l for the various amplitudes, given in the form of eq. (6.9), from the region $\text{Re}\, l > A^I$ into the remaining part of the right half-plane, *i.e.* $-\frac{1}{2} < \text{Re}\, l \leqslant A^I$. The problems connected with the analytic continuation of amplitudes such as $\bar{t}^{I,l}_{s,n}(s)$ have been discussed in detail in ref. (1). The essential point is the fact that $\text{disc}\,\bar{t}^{I,l}_{s,n}(s)$ is singular for $l = \alpha^I_n(s)$ and $l = \alpha^{I*}_n(s)$ [see eq. (6.10)], so that a pole moves up from the unphysical sheet into the physical sheet whenever l is varied along a path in the l-plane that crosses the Regge trajectory or the complex conjugate trajectory. The dispersion relation for $\bar{t}^{I,l}_{s,n}(s)$ then acquires in addition to the integrals a pole term

$$\frac{\gamma^I_n(\sigma^I_{nl})}{\alpha^{I'}_n(\sigma^I_{nl})(\sigma^I_{nl} - s)}, \tag{6.19}$$

where σ^I_{nl} is the root of the equation $l = \alpha^{I'}_n(s)$ and where $\alpha^I_n(\sigma^I_{nl})$ denotes the first derivative of $\alpha_n^I(s)$ at $s=\sigma_{nl}^I$. No particular problems are encountered in the analytic continuation of $\bar{t}_{s,\mathrm{B}}^{I,l}(s)$ and $\bar{t}_{s,\mathrm{in}}^{I,l}(s)$. Both amplitudes depend on l only over the index of Legendre functions (cf. eqs. (4.12)–(4.18)), and throughout the half-plane $\mathrm{Re}\, l > -\frac{1}{2}$ Legendre functions of either kind are regular functions of l.

Our explicit expressions for the discontinuities across the cuts enable us also to discuss the subtraction problem for the strip-approximated amplitude. For lack of space we shall not give any details in the present paper, but mention only that a method described by Omnès ([12]) can be adapted to our case. Answers to all relevant questions can thus be obtained, and the results are generally analogous to those of Omnès.

7. – Unitarity and bootstrap equations.

Our work so far has resulted in the formulation of a strip approximation $\bar{T}_s^I(s, t, u)$ to the exact scattering amplitude $T_s^I(s, t, u)$. Put in different words, we have obtained a prescription for separating from the scattering amplitude $T_s^I(s, t, u)$ the contribution from all Regge poles in the direct and the crossed channels, provided the Regge parameters $\alpha_n^I(s)$ and $\gamma_n^I(s)$ are known, thereby preserving crossing symmetry and the correct analyticity properties. The motivation behind this rewriting of the scattering amplitude is, of course, the belief that in a number of problems this strip approximation constitutes a useful approximation.

In order to give dynamical content to our equations we have to impose on them the unitarity condition and to show that we are led to a number of equations which, in principle, may be sufficient to determine all unknown functions in our problem including the Regge parameters $\alpha_n^I(s)$ and $\gamma_n^I(s)$ in a self-consistent manner, in other words that we are led to a bootstrap situation.

The unitarity condition for a physical partial wave in the presence of inelastic processes takes the form ([13])

$$\text{(7.1)} \qquad \mathrm{disc}\, T_s^{I,l}(s) = \sqrt{\frac{s-4\mu^2}{s}} \cdot |T_s^{I,l}(s)|^2 + \sqrt{\frac{s}{s-4\mu^2}} \cdot \frac{1}{4}[1-(\eta_l^I(s))^2]\,.$$

The absorption parameter η_l^I gives a summary description of the inelastic processes and is connected with the imaginary part ε_l of the scattering phase shift δ_l^I by $\eta_l^I = \exp[-2\varepsilon_l^I]$. The variation of η_l^I is thus restricted to the interval $0 \leqslant \eta_l^I \leqslant 1$. The equation essentially says that $\mathrm{disc}\, T_s^{I,l}(s)$ can be decom-

([12]) R. Omnès: *Phys. Rev.*, **133**, B 1543 (1964).

([13]) See *e.g.*, G. F. Chew: *S-Matrix Theory of Strong Interactions* (New York, 1961) posed, just like the absorptive amplitude $M^I_{s(s)}(s, t, u)$ in eq. (4.1), into an elastic and an inelastic part. On the other hand, from eq. (6.4) we know that

$$\text{disc}\, T_s^{I,l}(s) = \text{disc}\, \widetilde{T}_s^{I,l}(s) + \sum_{n=1}^{N_I} \text{disc}\, \overline{T}_{s,n}^{I,l}(s) + \text{disc}\, \overline{T}_{s,\text{in}}^{I,l}(s)\,, \tag{7.2}$$

whence we conclude that $\text{disc}\, \widetilde{T}_s^{I,l}(s)$ must also be decomposable into elastic and inelastic contributions.

We now introduce an approximation which consists in neglecting the inelastic contribution to $\text{disc}\, \widetilde{T}_s^{I,l}(s)$. In other words, only those inelastic processes are being considered that can be described by Regge poles in the crossed channels. It almost goes without saying that this approximation may not really be well justified from the point of view of physics, but in the present paper we are forced to introduce this approximation since we do not wish to go beyond the framework of a one-channel ($\pi\pi$) theory. As is well known from the work of ZACHARIASEN and ZEMACH [10] any theory of the $\pi\pi$ phenomena aiming at quantitative predictions must necessarily be a multichannel theory, including, *e.g.*, the $\pi\omega$ and similar channels.

Within our approximation we can rewrite eq. (7.1) as follows:

$$\text{disc}\, \widetilde{T}_s^{I,l}(s) + \sum_{n=1}^{N_I} \text{disc}\, \overline{T}_{s,n}^{I,l}(s) = \sqrt{\frac{s-4\mu^2}{s}}\; T_s^{I,l}(s)\cdot\left(T_s^{I,l^*}(s)\right)^*, \tag{7.3}$$

$$\text{disc}\, \overline{T}_{s,\text{in}}^{I,l}(s) = \sqrt{\frac{s}{s-4\mu^2}}\cdot\frac{1}{4}\left[1-\left(\eta_l^I(s)\right)^2\right]. \tag{7.4}$$

Combining eq. (7.4) with eq. (6.12) we obtain the following three conditions ($I=0, 1, 2$) on the Regge parameters:

$$0 \leqslant \frac{8}{\sqrt{s(s-4\mu^2)}}\cdot\frac{1}{\pi}\int_{4\mu^2}^{\infty} \text{d}t'\, Q_l\left(1+\frac{2t'}{s-4\mu^2}\right)\sum_{I'=0}^{2}\beta_{II'}\,\overline{v}_{t,\text{el}}^{I'}(t', s) \leqslant 1\,. \tag{7.5}$$

These conditions must be valid for all physical l and real $s \geqslant 16\mu^2$. From the crossing matrix we see immediately that a theory taking only one $I=1$ Regge trajectory into account leads to inconsistencies: eq. (7.5) will be violated either for $I=1$ or for $I=2$. Looking at the crossing matrix we conclude that eq. (7.5) requires a dominant contribution from $I=0$ Regge trajectories (*) (*e.g.*, Pomeranchuk trajectory).

(*) Considering eq. (7.5) in the asymptotic region one also is forced to the conclusion $\alpha(t) \leqslant 1$ for real $t \leqslant 0$ for all Regge trajectories.

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As is well known the unitarity condition can be analytically continued into the complex l-plane. Equation (7.3) has already been written in a form appropriate for complex angular momentum. As is evident from eq. (6.12) the function $\operatorname{disc} \overline{T}^{I,l}_{s,\text{in}}(s)$ is holomorphic in the half-plane $\operatorname{Re} l > -\frac{1}{2}$, and thus eq. (7.4) defines η^I_l everywhere in this half-plane

$$\eta^I_l(s) = \left(1 - 4\sqrt{\frac{s-4\mu^2}{s}} \operatorname{disc} \overline{T}^{I,l}_{s,\text{in}}(s)\right)^{\frac{1}{2}}, \tag{7.6}$$

for all real $s \geqslant 4\mu^2$. Equation (7.5) hlds for all physical l. It is reasonable to postulate the validity of the condition (7.5) not only for physical l, but on the whole real semi-axis $l > -\frac{1}{2}$. Then η^I_l is seen to be a nonnegative real function of l on this semi-axis. The analytic continuation into the l-plane goes according to eq. (7.6) through the known l-dependence of $\operatorname{disc} \overline{T}^{I,l}_{s,\text{in}}(s)$. This l-dependence, in turn, is only through the Legendre function Q_l, so that we have

$$\operatorname{disc} \overline{T}^{I,l^*}_{s,\text{in}}(s) = \left(\operatorname{disc} \overline{T}^{I,l}_{s,\text{in}}(s)\right)^* \tag{7.7}$$

$$\eta^I_{l^*}(s) = \left(\eta^I_l(s)\right)^* . \tag{7.8}$$

We now wish to write down what form the unitarity condition takes along a Regge trajectory, *i.e.*, for values of l and s such that $l = \alpha^I_n(s)$ where $n = 1, 2, \ldots$ or N_I. From the results obtained in Sect. **6** we see that for a fixed value of s and for l in the vicinity of the n-th trajectory the scattering amplitude $T^{I,l}_s(s)$ may be decomposed as follows

$$T^{I,l}_s(s) = \frac{\beta^I_n(s)}{l - \alpha^I_n(s)} + B^{I,l}_{s,n}(s) , \tag{7.9}$$

where $B^{I,l}_{s,n}(s)$ as a function of l is regular at $l = \alpha^I_n(s)$. Inserting this into eq. (7.3) and comparing on both sides the pole term at $l = \alpha^I_n(s)$ and the part that is regular at $l = \alpha^I_n(s)$ we obtain the two equations (*)

$$1 + 2i\sqrt{\frac{s-4\mu^2}{s}} \cdot T^{I,(\alpha^I_n(s))^*}_s(s) = 0 , \tag{7.10}$$

$$\beta^I_n(s) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}l} T^{I,l}_s(s)\right)^*_{l=(\alpha^I_n(s))^*} = \frac{1}{4} [\eta^I_l(s)]^2 \cdot \frac{s}{s-4\mu^2} . \tag{7.11}$$

(*) These equations are valid only *above* the threshold, *i.e.* away from the singularities of the partial-wave amplitude associated with $l = \alpha^I_n(4\mu^2)$ and $s = 4\mu^2$. If the second term in eq. (7.9) is neglected and $\eta^I_l = 1$ is assumed, both equations reduce to $\beta^I_n(s) = s^{\frac{1}{2}}(s-4\mu^2)^{-\frac{1}{2}} \operatorname{Im} \alpha^I_n(s)$, an approximation discussed in ref. (5).

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In the derivation of eq. (7.11) we have repeatedly made use of eq. (7.10). When we consider in eq. (7.3) the pole at $l=(\alpha_n^I(s))^*$ we obtain equivalent results.

For each Regge trajectory we have thus obtained one pair of equations (7.10), (7.11). The total number of equations equals the total number of Regge parameters $\alpha_n^I(s)$, $\gamma_n^I(s)$. Even when the Regge parameters are given, the expression for the scattering amplitude (eq. (6.4)) contains still another unknown function, namely $\tilde{t}_s^{I,l}(s)$. To determine this function too we must obtain an equation that contains l as one parameter. This can obviously be achieved by exploiting the unitarity condition also for values of l that do not lie on a Regge trajectory, for example by using the N/D method. FROISSART [14] has described the general procedure to be followed in the presence of inelastic processes. We shall, however, postpone the explicit formulation of equations until we have discussed the asymptotic properties of the various functions that occur in our approach. In principle, the N/D equations complete our set of dynamical equations, which now numbers as many equations as we have unknown functions. One may hope that these equations determine all unknown functions of the problem uniquely.

One could imagine that our dynamical equations allow an iterative procedure to be followed. The first step would consist of two parts: Assuming $\tilde{t}_s^{I,l}(s)\equiv 0$ one tries to find a set of Regge parameters that satisfy eqs. (7.10), (7.11). In other words, one imposes eqs. (7.10) and (7.11) on $\bar{T}_s^{I,l}(s)$, the strip approximation to the full scattering amplitude. Secondly one uses the Regge parameters so obtained as an input for the N/D calculation. The r-th step of the iterative procedure follows the same scheme, using the approximation to $\tilde{t}_s^{I,l}(s)$ obtained in the $(r-1)$-th step as an input. Unfortunately, our equations are extremely complicated and highly nonlinear, so that a rigorous numerical treatment, even of the first step of the iteration procedure, seems to be out of question. But still our system of equations might be quite valuable as a starting point for developing useful calculational schemes that incorporate further approximations.

Equation (7.10) has also been derived by FRAUTSCHI, KAUS, and ZACHARIASEN [5]. These authors use as a second equation a dispersion relation for $\alpha_n^I(s)$. This approach suffers from the difficulty that dispersion relations of a simple structure hold only for the leading trajectories. For the trajectories following the leading one $(n=1)$ additional branch points, due to crossing of two or more trajectories, are known to occur which considerably complicate the structure of the dispersion relations. Within the framework of the present paper dispersion relations seem to be unsuitable from still another point of view. Inherent in our whole approach is a restriction to the right half of the angular-momentum plane, $\operatorname{Re} l > -\frac{1}{2}$, so that eqs. (7.10) and (7.11) must

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([14]) M. FROISSART: *Nuovo Cimento*, 22, 191 (1961).

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be exploited only in this half-plane. Thus we need to calculate only those parts of the Regge trajectories which lie in the right half-plane, but not the whole trajectories as would be necessary when using dispersion relations.

8. – Connection with the bootstrap equations of Zachariasen and Zemach.

In connection with the remarks concerning the solution of our dynamical equations it might be not quite without interest to see how the bootstrap equation of Zachariasen and Zemach ([10]) for the one-channel case is contained in our approach. We shall list below a number of assumptions and « approximations » that will result, when taken all together, in the input function for the N/D equations as assumed by these authors.

1) All Regge trajectories except the leading one for $I=1$ (ρ-meson) are neglected. As pointed out in the last Section, this assumption in a theory assuming correct analytic behaviour already leads to contradiction with unitarity, which can be removed only by considering at least one additional $I=0$ trajectory.

2) Assume $\gamma_\rho(s)=\gamma=\text{const}$ and $\alpha_\rho(m_\rho^2-i\Delta)=1$, where m_ρ denotes the mass of the ρ-meson. Then the expansion

$$\alpha_\rho(s) = 1+(s-m_\rho^2+i\Delta)\cdot\alpha'_\rho(m_\rho^2-i\Delta)+\dots \tag{8.1}$$

is used, assuming the imaginary part of $\alpha'_\rho(m_\rho^2-i\Delta)$ to be negligible compared with the real part. Assuming furthermore

$$\Delta\cdot\alpha'_\rho(m_\rho^2-i\Delta)\ll 1\,, \tag{8.2}$$

we insert this expansion into eq. (5.8). In the neighbourhood of $s=m_\rho^2$ we then have $\alpha_\rho(s)\approx 1$ and the leading term in eq. (5.8) becomes

$$g_\rho(s,t)\approx\frac{3}{4}\cdot\frac{\gamma}{\alpha'_\rho(m_\rho^2-i\Delta)}\cdot\frac{\Delta}{(s-m_\rho^2)^2+\Delta^2}\cdot(2t+s-4\mu^2)\,. \tag{8.3}$$

We emphasize that this approximation is *justified only in the immediate neighborhood of $s=m_\rho^2$*, and is already so crude that the originally correct analytic behaviour of $g_\rho(s,t)$ is completely destroyed.

3) In spite of all that we now postulate the validity of the approximation (8.3) on the whole real s-axis ($-\infty<s<+\infty$). Then we can calcu-

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late the amplitude $T_\rho^{(+)}(s, t)$ by

$$(8.4)\qquad T_\rho^{(+)}(s,t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} ds' \frac{g_\rho(s',t)}{s'-s-i0} = \frac{3}{4}\cdot\frac{\gamma}{\alpha'_\rho(m_\rho^2 - i\Delta)}\cdot\frac{2t+m_\rho^2-4\mu^2-i\Delta}{m_\rho^2-i\Delta-s} =$$

$$= \frac{3}{4}\cdot\frac{\gamma}{\alpha'_\rho(m_\rho^2-i\Delta)}\cdot\left[1+\frac{s-4\mu^2}{m_\rho^2-i\Delta-s}\left(1+\frac{2t}{s-4\mu^2}\right)\right].$$

We see that formally this amplitude contains a constant S-wave term and a resonant P-wave term, the latter one being described by a pole below the real axis on the unphysical sheet. Upon forming the complete s-channel contribution, the S-wave part, of course, drops out

$$(8.5)\qquad T_\rho(s,t,u) = T_\rho^{(+)}(s,t) - T_\rho^{(+)}(s,u) =$$

$$= \frac{3}{2}\cdot\frac{\gamma}{\alpha'_\rho(m_\rho^2-i\Delta)}\cdot\frac{t-u}{m_\rho^2-i\Delta-s} = \frac{c^2}{4\pi}\cdot\frac{q_s^2\cdot 2\cos\theta_s}{m_\rho^2-i\Delta-s}.$$

We have introduced here a $\pi\pi\rho$-coupling constant c by

$$(8.6)\qquad \frac{c^2}{4\pi} = \frac{3\gamma}{\alpha'_\rho(m_\rho^2-i\Delta)}.$$

4) For comparison with the N/D calculation of Zemach and Zachariasen we have now to calculate the contribution from the crossed channels. One introduces a zero-width approximation, $\Delta\to 0$, thus replacing eq. (8.3) by

$$(8.7)\qquad g_\rho(s,t) \approx \frac{1}{4}\cdot\frac{c^2}{4\pi}(2t+s-4\mu^2)\cdot\pi\delta(s-m_\rho^2).$$

Inserting this equation into eq. (6.6) we obtain as P-wave contribution from the third spectral function

$$(8.8)\qquad T_B(s) = \frac{1}{4}\cdot\frac{c^2}{4\pi}\cdot\frac{2s+m_\rho^2-4\mu^2}{s-4\mu^2}\,Q_1\left(1+\frac{2m_\rho^2}{s-4\mu^2}\right).$$

This term represents the forces due to the crossed channels. The integral containing $H^{I'}$ gives no contribution as $H^{I'}$ vanishes identically because of the faulty analytic properties of our approximation. Another P-wave contribution, however, equal in magnitude to $T_B(s)$, comes from eq. (6.7), *i.e.*, from the term that we have associated with inelastic processes. Thanks to the wrong analytic properties of the approximation this contribution is not associated with an imaginary part, as it should be in the inelastic region; and this circumstance removes the contradiction against unitarity, mentioned under 1), due to the neglect of the Pomeranchuk trajectory. The total input function for the N/D calculation is thus obtained as

$$T_L(s) = T_B(s) + T_{in}(s) = \frac{1}{2}\cdot\frac{c^2}{4\pi}\cdot\frac{2s + m_\rho^2 - 4\mu^2}{s - 4\mu^2}\, Q_1\left(1 + \frac{2m_\rho^2}{s - 4\mu^2}\right), \tag{8.9}$$

which is exactly what one expects from eq. (8.5) on the basis of crossing symmetry, and what has been used in the work of Zemach and Zachariasen.

It is clear from the above derivation that the primitive bootstrap calculation based on the exchange of nonreggeized one-particle states can be only a poor approximation to a fully reggeized bootstrap calculation, and that it can be expected to give only qualitatively correct results.

9. – Conclusions.

In the preceding Sections we have outlined, using pion-pion scattering as an example, a systematic procedure for introducing Regge poles into a relativistic S-matrix theory in such a way that exact crossing symmetry and Mandelstam analyticity are preserved. Contributions to the spectral functions due to Regge terms are nonvanishing only within the usual boundary curves. Summing over all contributions to the scattering amplitude that are due to Regge poles in the direct or one of the crossed channels we obtain an approximation to the scattering amplitude that we have termed the strip approximation. This approximated amplitude describes bound states, resonances at low energy, forces due to exchange of Reggeized particles, and high-energy « Regge behaviour ». The total contribution to the scattering amplitude from the n-th direct channel Regge pole for isospin I (cf. eqs. (5.29) and (2.14)).

$$\overline{T}^I_{s,n}(s, t, u) = \overline{T}^{I(+)}_{s,n}(s, t) + (-1)^I\,\overline{T}^{I(+)}_{s,n}(s, u) \tag{9.1}$$

may be associated with a graph such as the one in Fig. 1a, symbolizing a single intermediate *Regge* particle in the direct channel propagating from the initial vertex to the final vertex. The other graphs analogously represent pion-pion scattering by exchange of single intermediate *Regge* particles. The strip approximation may then be characterized as the total contribution to the scattering amplitude from all graphs with a single intermediate Regge particle. This last remark connects our formalism to those simple bootstrap models [15]

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[15] For a survey see F. ZACHARIASEN: *Bootstraps* [contained in *Strong Interaction and High-Energy Physics*, ed. by R. G. MOORHOUSE (Edinburgh, London, 1964)].

where the input forces are approximated by contributions from graphs that represent the exchange of a single meson of fixed spin. Our strip approximation and more generally all other developments in this paper too can be understood as being the result of a consistent Reggeization of one of those simple (one-channel) bootstrap models, subject only to the condition that crossing symmetry and Mandelstam analyticity must be correctly obtained.

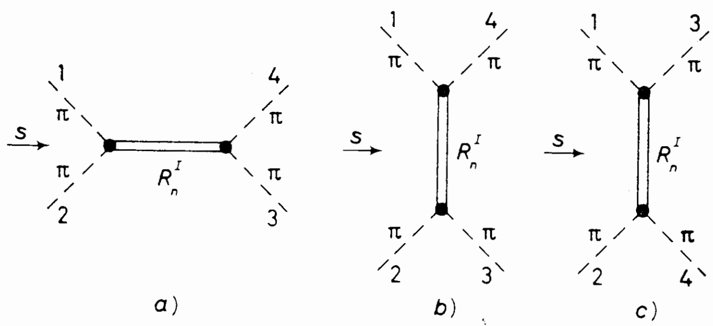

Fig. 1. – Graphs symbolizing pion-pion scattering through one intermediate Regge particle R^I_n. *a*) The Regge particle is in the direct channel, the graph symbolizes the bound states and resonances due ot R^I_n. *b*) and *c*) With R^I_n in a crossed channel the graph symbolizes exchange of a Regge particle.

When the unitarity condition is imposed one obtains bootstrap equations. This part of our work is related to the paper by FRAUTSCHI, KAUS, and ZACHARIASEN (5) inasmuch as we too exploit the unitarity condition for complex angular momentum, on the other hand, however, we would like to circumvent the use of dispersion relations for the Regge parameters, because in our approach only those parts of the Regge trajectories are used that are located in the right half-plane. An exploration of the left half of the angular-momentum plane is nowhere called for, which we consider a fortunate circumstance because analyticity properties in the left half-plane are presumably very complicated in general. Our work is more closely related to that of CHEW and JONES (4). The main difference is our use of another representation of the Regge terms. In the present formulation no need arises to distinguish the «asymptotic» regions from low-energy regions, and consequently the parameters s_1, t_1, u_1 of Chew and Jones, marking the beginning of the asymptotic region, are absent in our approach. One consequence of this is that the conventional N/D-method is presumably as well suitable in connection with our formulation as is the «modified» N/D-method of Chew (16). Furthermore we feel that a better description of the low-energy, long-range part of the generalized potential is achieved, because in our approach also those parts of the third spectral function are nonvanishing that are close to the physical region of the direct channel ($-s < u_1$ or t_1 in the notation of ref. (4)).

Our work could be generalized in various ways. For example, the present one-channel formulation might be extended to a many-channel formulation.

(16) G. F. CHEW: *Phys. Rev.*, **129**, 2363 (1963); **130**, 1264 (1963).

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This should not be too difficult as long as we restrict ourselves to only two-body final states. The inclusion of three- and more body final states presents a major difficulty because the analyticity properties of the corresponding production amplitudes are not well understood. Another generalization of our formalism would be the addition of terms associated with cuts in the angular-momentum plane. This can be done, but it complicates the bootstrap problem enormously. We believe, therefore, that it is not very useful to elaborate on this point as long as one has not learned how to do reliable numerical calculations for a model with Regge *poles* only.

RIASSUNTO (*)

In un precedente articolo sullo scattering di potenziale si è sviluppata una rappresentazione di Regge modificata che è distinta dalle altre rappresentazioni di Regge dalla proprietà che la sua struttura analitica è proprio del tipo richiesto dalla relazione di doppia dispersione di Mandelstam. Qui si discute come si possa utilizzare questa rappresentazione nello schema di una teoria relativistica della matrice S per lo scattering pione-pione. Si definisce un'approssimazione all'ampiezza di scattering esatta, chiamata approssimazione a strisce, come somma di tutti i possibili contributi all'ampiezza di scattering dati dai poli di Regge nel canale diretto od in uno di quelli incrociati. Questa approssimazione a strisce unisce le seguenti caratteristiche: proprietà di analiticità quali sono descritte dalla rappresentazione di Mandelstam, esatta simmetria incrociata, « comportamento di Regge » alle alte energie, descrizione degli stati legati e delle risonanze alle basse energie, e delle forze dovute allo scambio di particelle reggeizzate. Si discutono le ampiezze dell'onda parziale e si introduce la condizione di unitarietà per il momento angolare complesso. Si ottiene allora un gruppo di equazioni che sembra sufficiente in linea di principio per una determinazione autocoerente delle traiettorie di Regge, in quanto esse siano collocate nel semipiano destro, e del termine di fondo dell'ampiezza di scattering. Data la complessità delle equazioni, non è ancora prevista la possibilità di un trattamento numerico. Il nostro articolo si collega con altri articoli recenti di Chew e Jones, di Frautschi, e di Kaus e Zachariasen sul bootstrap delle traiettorie di Regge.

(*) *Traduzione a cura della Redazione.*