Structure of Unitary Scattering Amplitudes. II*

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A representation for the invariant scattering amplitude $A(s,t,u)$ is constructed that satisfies elastic unitarity in the s channel. The development is based upon the usual fixed-s dispersion relations and rigorous expansions of the partial-wave amplitudes. The amplitude has the singularity structure of the Mandelstam representation and leads to new expressions for the double spectral functions in the elastic region. The relationship of resonances to the Mandelstam representation is clarified. The continuation of the unitary scattering amplitude to the inelastic region and the physical regions of the crossed channels is discussed.

I. INTRODUCTION

In an earlier article[1] a representation for the invariant scattering amplitude $A(s,t,u)$ was constructed that satisfied elastic unitarity in the s channel. The representation had a structure like that of the Mandelstam representation,[2] yet it differed in some crucial details that led to difficulties with crossing symmetry and inelastic unitarity. The unitary scattering amplitude was developed by using the N/D method[3] to construct unitary partial-wave amplitudes and by using a class of Legendre sums to sum the Legendre expansion of the full amplitude. In the process it was necessary to make an *Ansatz* for the numerator function in the N/D representation and it was conjectured that the difficulties could be traced to a failure of the *Ansatz.*

The purpose of this work is to construct unitary scattering amplitudes without the necessity of questionable *Ansatzen.* The major assumption is the validity of the fixed-s dispersion relations, or equivalently, the validity of the Froissart-Gribov formula in the elastic region. Together with rigorous expansions for the partial-wave amplitudes that are consequences of elastic unitarity, this assumption leads to scattering amplitudes that are manifestly unitary. It is shown that the singularity structure in the momentum-transfer variables is precisely that of the Mandelstam representation for s in the elastic region. Also, the unitary scattering amplitude yields new expressions for the elastic-region double spectral functions. They are expressed as finite sums, for finite values of (s,t) or (s,u), and are determined by two real functions in contrast to the two complex functions of the usual Mandelstam representation.

It is shown that the construction procedure goes through for an arbitrary number of subtractions in the original fixed-s dispersion relation and that the formulas for the double spectral functions are unchanged in the presence of subtractions. The construction also sheds light on the relation of resonances to the Mandelstam representation. It is found that the existence of resonances requires the presence of subtractionlike terms in the representation. That is, the double spectral functions themselves cannot give rise to resonances in the elastic region. The possible connection of this fact to recent conjectures on the finite-energy sum rules is discussed.

The continuation of the unitary scattering amplitude to the inelastic region and to the physical regions of the crossed channels requires further assumptions related to the Froissart-Gribov formula. The inverse amplitude representation for the partial waves is used to extend the expansions out of the elastic region. In contrast to the results of I, a simple prescription is found that provides for the additional singularities of the Mandelstam representation in the inelastic region required by crossing symmetry. The continuation to the crossed-channel physical regions is a different story, however. Difficulties apparently related to the existence of essential singularities on the second sheet prevent the continuation of the present form of the amplitude to the crossed channels. Possible methods of surmounting these difficulties are outlined.

To simplify kinematical problems in the construction the scattering of distinguishable, spinless particles of equal mass is considered. This permits a general approach to questions of crossing relations. The unequal-mass situation will not give rise to serious complications, but the extension to spin will be somewhat more cumbersome. The expansions of the partial-wave amplitudes central to the development are given in Sec. II, while Sec. III contains the details of the construction procedure and the resulting unitary scattering amplitude. The elastic-region double spectral functions implied by the representation are determined in Sec. IV. The situations with subtractions and with resonances are presented in Secs. V and VI, respectively, and the questions of the continuation of the representation to the inelastic region and the crossed-channel physical regions are dealt with in Secs. VII and VIII. Section IX is devoted to conclusions, and some necessary mathematical details are presented in an Appendix.

II. PARTIAL-WAVE AMPLITUDES

The scattering of distinguishable, spinless particles of equal mass is described by an invariant scattering amplitude $A(s,t,u)$, where s, t, and u are the usual Mandel-

* Research supported in part by the National Science Foundation.
[1] A. W. Martin, Phys. Rev. **161**, 1528 (1967); hereafter referred to as I.
[2] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[3] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

stam variables satisfying[4] $s+t+u=1$. The amplitude is normalized so that the elastic unitarity condition in the s channel reads

$$\mathrm{Im}A(s,z)=(4\pi)^{-1}[(s-1)/s]^{1/2}\times\int d\Omega_n A^*(s,z_{fn})A(s,z_{ni}), \quad (1)$$

where z is the cosine of the center-of-mass scattering angle and the integration runs over all angles of the intermediate meson pair. The partial-wave amplitudes are related to the scattering amplitude in the usual way,

$$A(s,z)=\sum(2l+1)P_l(z)A_l(s),$$
$$A_l(s)=\frac{1}{2}\int_{-1}^{1}dz\,P_l(z)A(s,z), \quad (2)$$

and by virtue of (1) satisfy the elastic unitarity condition

$$\mathrm{Im}A_l(s)=[(s-1)/s]^{1/2}|A_l(s)|^2. \quad (3)$$

Two well-known consequences of (3) are, first, the phase shift parametrization

$$A_l(s)=[s/(s-1)]^{1/2}\sin\delta_l\, e^{i\delta_l}, \quad (4)$$

where δ_l is real in the elastic region, and second, the inverse amplitude representation

$$A_l(s)=s^{1/2}[\phi_l(s)-i(s-1)^{1/2}]^{-1}, \quad (5)$$

where we have adopted the form used by Martin.[5] In (5) the branch cut of the function $(s-1)^{1/2}$ is taken to be $[1,\infty)$ and the function is real on the cut with a non-negative imaginary part on the first sheet. Similarly, the function $s^{1/2}$ is given the cut $(-\infty,0]$ and is positive for $0<s$.

It is convenient for our purposes to express the inverse amplitude representation in terms of the inverse of $\phi_l(s)$,

$$A_l(s)=s^{1/2}\psi_l(s)[1-i(s-1)^{1/2}\psi_l(s)]^{-1}, \quad (6)$$

with $\psi_l(s)=\phi_l^{-1}(s)$. $\psi_l(s)$ is real analytic in the s plane cut along $(-\infty,0]$ and $[s_i,\infty)$, where s_i is the first inelastic threshold. In the elastic region $\psi_l(s)$ is real, related to the phase shift by

$$\psi_l(s)=(s-1)^{-1/2}\tan\delta_l,$$

and an important feature of (6) is that $A_l(s)$ satisfies elastic unitarity for arbitrary real $\psi_l(s)$.

The present method for constructing unitary scattering amplitudes utilizes the expansion of $\mathrm{Im}A_l(s)$ in powers of $\mathrm{Re}A_l(s)$. Equation (3) provides a quadratic equation in $\mathrm{Im}A_l(s)$ and the two roots of this equation hold for different ranges of $\mathrm{Im}A_l(s)$, or equivalently, for different ranges of the phase shift. The two roots can be expanded in powers of $\mathrm{Re}A_l(s)$ with the results (suppressing the angular-momentum subscript for the moment)

$$\mathrm{Im}A(s)=\sum\frac{\Gamma(k+\frac{1}{2})4^k(s-1)^{k+1/2}}{\Gamma(k+2)\Gamma(\frac{1}{2})s^{k+1/2}}[\mathrm{Re}A(s)]^{2k+2}, \quad (7)$$

and

$$\mathrm{Im}A(s)=[s/(s-1)]^{1/2}-\sum\frac{\Gamma(k+\frac{1}{2})4^k(s-1)^{k+1/2}}{\Gamma(k+2)\Gamma(\frac{1}{2})s^{k+1/2}}[\mathrm{Re}A(s)]^{2k+2}. \quad (8)$$

The first of these expansions [Eq. (7)] is valid for

$$0\leqslant[(s-1)/s]^{1/2}\,\mathrm{Im}A(s)\leqslant\tfrac{1}{2},$$

or in terms of the phase shift δ,

$$-\tfrac{1}{4}\pi+n\pi\leqslant\delta\leqslant\tfrac{1}{4}\pi+n\pi,$$

where n is a suitable integer. The second expansion [Eq. (8)] holds for

$$\tfrac{1}{2}\leqslant[(s-1)/s]^{1/2}\,\mathrm{Im}A(s)\leqslant 1$$

or

$$\tfrac{1}{4}\pi+n\pi\leqslant\delta\leqslant\tfrac{3}{4}\pi+n\pi.$$

We note that the hypergeometric expansions in (7) and (8) are absolutely convergent within and on the unit circle, or for

$$|\mathrm{Re}A(s)|\leqslant\tfrac{1}{2}[s/(s-1)]^{1/2},$$

so there is no question of the convergence of the sums in the elastic region. Instead it is a question of choosing the appropriate expansion depending upon the magnitude of the phase shift. This will have important consequences in the construction of unitary scattering amplitudes.

Finally, in anticipation of questions of inelastic unitarity, we observe that (7) and (8), with suitable modifications, enjoy a greater range of validity. Using the inverse amplitude representation [Eq. (6)], we define

$$A_R(s)=s^{1/2}\psi(s)[1+(s-1)\psi^2(s)]^{-1}$$

and (9)

$$A_I(s)=[s(s-1)]^{1/2}\psi^2(s)[1+(s-1)\psi^2(s)]^{-1}.$$

In the elastic region, where $\psi(s)$ is real, A_R and A_I are simply the real and imaginary parts, respectively, of the partial-wave amplitude. In the inelastic region A_R and A_I become complex but the partial-wave amplitude is still given by

$$A(s)=A_R(s)+iA_I(s).$$

The useful aspect of this separation is that the expansions (7) and (8) hold with $\mathrm{Re}A(s)\to A_R(s)$ and $\mathrm{Im}A(s)\to A_I(s)$ in complex domains discussed in detail in the Appendix. As a consequence we will be able to extend the unitary scattering amplitude into the inelastic region in a natural way.

[4] The units $\hbar=c=2m=1$ are used.
[5] A. Martin, Nuovo Cimento **47**, 265 (1967).

III. CONSTRUCTION OF THE UNITARY SCATTERING AMPLITUDE

Our purpose in this section is to sum the Legendre series [Eq. (2)] to obtain a scattering amplitude that automatically satisfies elastic unitarity. The first requirement is a statement of the l dependence of the partial-wave amplitudes. This is provided by the Froissart-Gribov formula:

$$A_l(s)=2[(s-1)\pi]^{-1}\int_1^\infty dt\, A_t(t,s)Q_l[1+2t/(s-1)]$$
$$+2(-1)^l[(s-1)\pi]^{-1}\int_1^\infty du\, A_u(u,s)\times Q_l[1+2u/(s-1)], \quad (10)$$

where A_t and A_u are the absorptive parts in the crossed channels. The necessary subtractions in (10) have been suppressed. We will discuss the question of subtractions later on, and it will be shown that they are readily incorporated into the final representation.

The Froissart-Gribov formula expresses the l dependence of $\mathrm{Re}A_l(s)$ as weighted integrals over Legendre functions of the second kind. At the same time, elastic unitarity provides rapidly convergent expansions [Eqs. (7) and (8)] for $\mathrm{Im}A_l(s)$ in powers of $\mathrm{Re}A_l(s)$. We can therefore express the unitary partialwave amplitudes in terms of two unknown real functions (apart from subtractions) that we introduce as follows:

$$\mathrm{Re}A_l(s)=2[(s-1)\pi]^{-1}\int_1^\infty dx[f_t(x,s)+(-1)^l f_u(x,s)]\,Q_l[1+2x/(s-1)]. \quad (11)$$

In the elastic region we have the identification

$$f_t(x,s)=\mathrm{Re}A_t(x,s)\,,\quad f_u(x,s)=\mathrm{Re}A_u(x,s)\,,$$

so that with reference to the Mandelstam representation we would expect f_t and f_u to be given by principal-value integrals over double spectral functions (dsf) together with contributions from the "third" dsf. At the first inelastic threshold, however, it will be useful to have f_t and f_u develop imaginary parts in keeping with the discussion of the previous section. In the suggestive framework of the Mandelstam representation this can be viewed as applying the principal-part integrations only to the "first wings" of the spectral functions, while the "second wings" representing inelastic effects contribute the imaginary parts.

To simplify the analysis of the construction procedure, we begin with a restricted class of amplitudes. We suppose that in the elastic region none of the partial-wave phase shifts exceed $\frac{1}{4}\pi$ in magnitude. It follows from the results of the previous section that each partial-wave amplitude can be written in the unitary form

$$A_l(s)=\mathrm{Re}A_l(s)+i\sum_k \frac{\Gamma(k+\frac{1}{2})4^k(s-1)^{k+1/2}}{\Gamma(k+2)\Gamma(\frac{1}{2})s^{k+1/2}}\times[\mathrm{Re}A_l(s)]^{2k+2}, \quad (12)$$

with $\mathrm{Re}A_l(s)$ given by (11). The l dependence of (12) occurs as multiple products of Legendre functions, and the Legendre sums over such products were evaluated in I. We define

$$I_n(s,t;x_1,\cdots,x_n)=2(s-1)^{-1}\sum_l(2l+1)P_l(1+2t/(s-1))\times Q_l(1+2x_1/(s-1))\times\cdots\times Q_l(1+2x_n/(s-1)) \quad (13)$$

and obtain from I,

$$I_1(s,t;x_1)=(x_1-t)^{-1},$$
$$I_n(s,t;x_1,\cdots,x_n)=\int_{t_+(x_1,x_2)}^\infty \frac{dt_1}{K(x_1,x_2,t_1)}\int_{t_+(x_3,t_1)}^\infty \frac{dt_2}{K(x_3,t_1,t_2)}\cdots\times\int_{t_+(x_n,t_{n-2})}^\infty \frac{dt_{n-1}}{(t_{n-1}-t)K(x_n,t_{n-2},t_{n-1})}, \quad (14)$$

where the latter form holds for $n\geqslant 2$. The functions introduced in (14) are given by

$$t_\pm(a,b)=(s-1)^{-1}[a^{1/2}(b+s-1)^{1/2}\pm b^{1/2}(a+s-1)^{1/2}]^2,$$
$$K(a,b,c)=[c-t_+(a,b)]^{1/2}[c-t_-(a,b)]^{1/2} \quad (15)$$
$$=[a^2+b^2+c^2-2ab-2ac-2bc-4abc/(s-1)]^{1/2},$$

and $K(a,b,c)$ is the Mandelstam kernel.

A further detail must be taken care of before we proceed to write down the unitary scattering amplitude. It concerns the complications due to the two different crossed channels. We define two sets of functions, $F_t^{(n)}$ and $F_u^{(n)}$, by

$$F_t^{(n)}(x_1,\cdots,x_n,s)+(-1)^l F_u^{(n)}(x_1,\cdots,x_n,s)=[f_t(x_1,s)+(-1)^l f_u(x_1,s)]\times[f_t(x_2,s)+(-1)^l f_u(x_2,s)]\times\cdots\times[f_t(x_n,s)+(-1)^l f_u(x_n,s)], \quad (16)$$

meaning that the terms must be multiplied out and collected into two groups, one with and the other without the factor $(-1)^l$. The terms without the factor will appear with t denominators in the scattering amplitude; the terms with the factor will appear with u denominators.

The Legendre sum of the unitary partial-wave amplitudes [Eq. (12)] yields a unitary scattering amplitude that can be represented as

$$A(s,t,u)=A^{(1)}(s,t,u)+\sum_{\substack{n=2\\ \text{(even)}}}^{\infty} A^{(n)}(s,t,u), \quad (17)$$

where

$$A^{(1)}(s,t,u)=\frac{1}{\pi}\int_1^\infty \frac{dx\, f_t(x,s)}{x-t}+\frac{1}{\pi}\int_1^\infty \frac{dx\, f_u(x,s)}{x-u}, \quad (18)$$

and for $n=2, 4, 6, \cdots$,

$$A^{(n)}(s,t,u)=\frac{i2^{2n-3}\Gamma(\frac{1}{2}n-\frac{1}{2})}{\pi^n\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}n+1)[s(s-1)]^{n/2-1/2}}\int_1^\infty\cdots\int_1^\infty dx_1,\cdots,dx_n$$
$$\times[F_t^{(n)}(x_1,\cdots,x_n,s)I_n(s,t;x_1,\cdots,x_n)+F_u^{(n)}(x_1,\cdots,x_n,s)I_n(s,u;x_1,\cdots,x_n)]. \quad (19)$$

In particular, the first term in the expansion of the imaginary part of the amplitude takes the form

$$A^{(2)}(s,t,u)=\frac{2i}{\pi^2}\int_1^\infty\int_1^\infty\int_{t_+(x,y)}^\infty\frac{dxdydt'[f_t(x,s)f_t(y,s)+f_u(x,s)f_u(y,s)]}{[s(s-1)]^{1/2}(t'-t)K(x,y,t')}$$
$$+\frac{2i}{\pi^2}\int_1^\infty\int_1^\infty\int_{t_+(x,y)}^\infty\frac{dxdydu'[f_t(x,s)f_u(y,s)+f_u(x,s)f_t(y,s)]}{[s(s-1)]^{1/2}(u'-u)K(x,y,u')}. \quad (20)$$

It is not difficult at this point to verify that $A(s,t,u)$ as given by (17), (18), and (19) satisfies the elastic unitarity condition [Eq. (1)], provided f_t and f_u are real. This is done by means of the identity

$$(4\pi)^{-1}\int d\Omega_n I_k(s,t_{fn};x_1,\cdots,x_k)I_m(s,t_{ni};x_{k+1},\cdots,x_{k+m})$$
$$=2(s-1)^{-1}I_{k+m}(s,t;x_1,\cdots,x_{k+m}), \quad (21)$$

and the three related identities obtained by replacing the triplet of variables (t_{fn},t_{ni},t) in (21) by the triplets (u_{fn},u_{ni},t), (t_{fn},u_{ni},u), and (u_{fn},t_{ni},u). These identities follow directly from (13) and the properties of the Legendre polynomials.[6] It is also helpful to note the identities

$$F_t^{(k+m)}=F_t^{(k)}F_t^{(m)}+F_u^{(k)}F_u^{(m)},$$
$$F_u^{(k+m)}=F_t^{(k)}F_u^{(m)}+F_u^{(k)}F_t^{(m)},$$

which follow from the definition (16).

The unitary scattering amplitude constructed above is the central result of this paper. There remain however the following points to be discussed: (i) the determination of the double spectral functions in the elastic region implied by the representation, (ii) the modifications introduced by subtractions in the Froissart-Gribov formula, (iii) the lifting of the restriction $|\delta_l(s)|\leqslant\frac{1}{4}\pi$ and, in particular, the relationship of resonances to the Mandelstam representation, (iv) the continuation of the amplitude into the inelastic region, and (v) the continuation of the amplitude to the physical regions of the crossed channels and the implications of crossing symmetry. These points are treated in the subsequent sections.

IV. ELASTIC-REGION DOUBLE SPECTRAL FUNCTIONS

The unitary scattering amplitude developed above has the structure of the Mandelstam representation for s in the elastic region, so it is straightforward to determine the double spectral functions. The s-channel absorptive part in the elastic region is simply

$$A_s(s,t,u)=-i\sum_{\substack{n=2\\(\text{even})}}^\infty A^{(n)}(s,t,u),$$

with $A^{(n)}(s,t,u)$ given by (19), and the double spectral functions $\rho_{st}(s,t)$ and $\rho_{su}(s,u)$ are defined to be the discontinuities for positive t and u, respectively, of $A_s(s,t,u)$. These discontinuities come only from the functions I_n in (19), and it is convenient to define, for $n\geqslant 2$,

$$J_n(s,t;x_1,\cdots,x_n)$$
$$=(2\pi i)^{-1}[I_n(s,t+i\epsilon;x_1,\cdots,x_n)-I_n(s,t-i\epsilon;x_1,\cdots,x_n)]$$
$$=\int_{t_+(x_1,x_2)}^\infty\frac{dt_1}{K(x_1,x_2,t_1)}\int_{t_+(x_3,t_1)}^\infty\frac{dt_2}{K(x_3,t_1,t_2)}\cdots$$
$$\times\int_{t_+(x_{n-1},t_{n-3})}^\infty\frac{dt_{n-2}\theta[t-t_+(x_n,t_{n-2})]}{K(x_{n-1},t_{n-3},t_{n-2})K(x_n,t_{n-2},t)}, \quad (22)$$

where the quantities in the integrals are defined by (15). As the simplest case we have

$$J_2(s,t;x,y)=\theta[t-t_+(x,y)]K^{-1}(x,y,t).$$

The double spectral functions can then be written as infinite sums:

$$\rho_{st}(s,t)=\sum_{\substack{n=2\\(\text{even})}}^\infty\rho_{st}^{(n)}(s,t), \quad (23)$$

with

$$\rho_{st}^{(n)}(s,t)=\frac{2^{2n-3}\Gamma(\frac{1}{2}n-\frac{1}{2})}{\pi^{n-1}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}n+1)[s(s-1)]^{n/2-1/2}}$$
$$\times\int_1^\infty\cdots\int_1^\infty dx_1,\cdots,dx_n$$
$$\times F_t^{(n)}(x_1,\cdots,x_n,s)J_n(s,t;x_1,\cdots,x_n), \quad (24)$$

and $\rho_{su}(s,u)$ is obtained from (23) and (24) by replacing

[6] See, for example, Ref. 1.

each t by u. In particular, the first term in the expansion of the double spectral function takes the form

$$\rho_{st}^{(2)}(s,t) = \frac{2}{\pi}\int_1^\infty\int \frac{dxdy[f_t(x,s)f_t(y,s)+f_u(x,s)f_u(y,s)]}{[s(s-1)]^{1/2}K(x,y,t)} \times\theta[t-t_+(x,y)]. \quad (25)$$

The reader will recall from experience with the Mandelstam representation that the integral in (25) is a finite integral for finite values of t; the step function cuts off the integral for sufficiently large values of x or y. Similarly, all of the integrals in (22) and (24) are finite, for finite t, because of the step functions.

It is of interest to compare (25) with the expression for the double spectral function obtained by Mandelstam.[2] Mandelstam's result for the complete double spectral function in the elastic region is obtained by making the replacement

$$[f_t(x,s)f_t(y,s)+f_u(x,s)f_u(y,s)] \to [A_t^*(x,s)A_t(y,s)+A_u^*(x,s)A_u(y,s)]$$

in (25). Since $f_i(x,s)=\mathrm{Re}A_i(x,s)$, $i=(t,u)$ in the elastic region, this comparison indicates two features of the expansion (23). First, the higher terms in the expansion of ρ_{st} correspond to expansions of $\mathrm{Im}A_t$ and $\mathrm{Im}A_u$ in powers of $\mathrm{Re}A_t$ and $\mathrm{Re}A_u$ because the explicit contributions of the real parts are completely contained in $\rho_{st}^{(2)}$. Identical comments apply of course to $\rho_{su}(s,u)$.

Second, because the real parts of the crossed-channel absorptive parts are known to determine completely the double spectral function in a domain bordering the leading Landau curve, the higher-order terms in the expansion (23) must have Landau curves further from the origin in the s-t plane. This is easily verified by direct calculation. We denote the domain where $\rho_{st}^{(n)}$ is nonvanishing by

$$t \geqslant t^{(n)}(s).$$

The leading Landau curve, $t=t^{(2)}(s)$, is determined by the step function in (25) to be

$$t^{(2)}(s)=\min t_+(x,y), \quad 1\leqslant x,\ y<\infty.$$

Since the function $t_+(x,y)$ is monotonic in its two variables [Eq. (15)], we obtain the familiar result

$$t^{(2)}(s)=t_+(1,1)=4s/(s-1).$$

The Landau curves for the higher-order terms are found from (22) and (24) to satisfy the recurrence relation

$$t^{(n+1)}(s)=t_+[t^{(n)}(s),1],$$

which in turn leads to the inequality $t^{(n+1)}(s)>t^{(n)}(s)$. The first few expressions for the higher Landau curves are computed to be

$$t^{(4)}(s)=16s(s+1)^2/(s-1)^3,$$
$$t^{(6)}(s)=4s(3s+1)^2(s+3)^2/(s-1)^5,$$

and the asymptotic limits of the curves are

$$t^{(n)}(s)\underset{s\to1}{\longrightarrow}4^{n-1}/(s-1)^{n-1}, \qquad t^{(n)}(s)\underset{s\to\infty}{\longrightarrow}n^2. \quad (26)$$

It follows that for any point in the s-t plane that the sum (23) is a finite sum, and the number of terms in the sum, by (26), is roughly $\frac{1}{2}\sqrt{t}$.

The unitary scattering amplitude constructed in Sec. III yields double spectral functions in the elastic region that are determined by two real functions in contrast to the two complex functions of the conventional Mandelstam representation. It will be shown in the following sections that these expressions for the double spectral functions are unchanged by the presence of subtraction terms in the Froissart-Gribov formula or by the existance of resonances in the partial-wave amplitudes.

V. SITUATION WITH SUBTRACTIONS

The results obtained so far have been based upon the Froissart-Gribov formula [Eqs. (10) and (11)] written without subtractions for any of the partial-wave amplitudes. This is equivalent to assuming that the fixed-s dispersion relation holds without subtractions. It is known from the work of Jin and Martin[7] that two subtractions are sufficient for s in the unphysical region $0\leqslant s\leqslant 1$, and the optical theorem suggests that two subtractions are necessary in this region as well. For s in the elastic region, it is very plausible that a finite number (and probably a small number) of subtractions suffice. Even in the case of indefinitely rising Regge trajectories,[8] with absorptive parts $A_t(t,s)\sim t^{\alpha(s)}$, the number of subtractions required is expected to increase at most linearly with s. We will assume that there is a finite range for s, including the elastic region, in which a finite number of subtractions in the fixed-s dispersion relation is sufficient.

This assumption permits us to write the fixed-s dispersion relation (for s in the specified range) in the form

$$A(s,t,u)=G(s,t,u)+\frac{(t-t_0)^m}{\pi}\int_1^\infty\frac{dt'A_t(t',s)}{(t'-t_0)^m(t'-t)} +\frac{(u-u_0)^m}{\pi}\int_1^\infty\frac{du'A_u(u',s)}{(u'-u_0)^m(u'-u)}, \quad (27)$$

where $G(s,t,u)$ consists of polynomials in t and u of order $(m-1)$ with coefficients that are, in general, complex functions of s. The simple observation that we wish to

[7] Y. S. Jin and A. Martin, Phys. Rev. **135**, B1375 (1964).
[8] See, for example, S. Mandelstam, Phys. Rev. **166**, 1539 (1968).

make is that the presence of the subtraction terms modifies the Froissart-Gribov formula for $l<m$ only by an additional function of s,

$$A_l(s)=B_l(s)+2[(s-1)\pi]^{-1}\int_1^\infty dt\, A_t(t,s)\times Q_l[1+2t/(s-1)]+2(-1)^l[(s-1)\pi]^{-1}\times\int_1^\infty du\, A_u(u,s)Q_l[1+2u/(s-1)]. \quad (28)$$

$B_l(s)$ contains not only the partial-wave projection of $G(s,t,u)$ but also terms involving integrals over the absorptive parts (but without Legendre functions). These integrals have the property of cancelling the divergences of the explicit integrals in (28) so that $A_l(s)$ is finite, as it must be; for $l\geqslant m$, $B_l(s)=0$.

Since the partial-wave amplitudes must be unitary, the construction procedure for the unitary scattering amplitude goes through as in Sec. III. One expands $\mathrm{Im}A_l(s)$ using (7) or (8), depending upon the magnitude of the phase shift (this point is discussed in Sec. VI), in powers of $\mathrm{Re}A_l(s)$ as given by (28). The Legendre sum is then evaluated and the result can be expressed formally as

$$A(s,t,u)=A_0(s,t,u)+\sum_{l=0}^{m-1} h_l(s)P_l[1+2t/(s-1)], \quad (29)$$

where $A_0(s,t,u)$ represents the scattering amplitude for the unsubtracted case given by (17). The functions $h_l(s)$, unfortunately, are quite complicated expressions involving $\mathrm{Re}B_l(s)$ and integrals over $f_t(x,s)$ and $f_u(x,s)$ which combine with the terms of $A_0(s,t,u)$ to ensure the convergence of all of the integrals appearing in the final representation. For example, the real parts of $h_l(s)$ are such that $\mathrm{Re}A(s,t,u)$ in (29) is identical to the real part of the original fixed-s dispersion relation [Eq. (27)].

While the unitary scattering amplitude is more complicated in the subtracted case, it is evident from (29) that the expressions of Sec. IV for the double spectral functions are unchanged. It also follows that the validity of the m-times subtracted fixed-s dispersion relation implies the singularity structure in the momentum-transfer variables of the Mandelstam representation, at least for s in the elastic region.

VI. SITUATION WITH RESONANCES

In the development of the scattering amplitude in Sec. III, the restrictive assumption was made that all partial-wave phase shifts were less than $\frac{1}{4}\pi$ in magnitude. The purpose of this section is to relax this restriction and to study the role of resonances in the Mandelstam representation. The restriction arose from the fact that the partial-wave elastic unitarity condition gives two expansions for $\mathrm{Im}A_l(s)$ in powers of $\mathrm{Re}A_l(s)$. Both expansions are absolutely convergent but only one of them converges to the correct value, and the appropriate choice depends upon the magnitude of the phase shift.

Two preliminary points must be made. First, all phase shifts start from zero at threshold so there is certainly a region in s in which the previous representations are valid. Second, the Froissart-Gribov formula indicates that $A_l(s)$ decreases in magnitude as l increases,[9] so there is some value of l above which all partial-wave phase shifts remain less than $\frac{1}{4}\pi$ in magnitude throughout the elastic region. We need to consider, then, the effects of a finite number of partial waves for which the formula (12) is invalid in certain energy ranges.

For convenience we denote the expansion

$$S_l(s)=\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{1}{2})4^k(s-1)^{k+1/2}}{\Gamma(k+2)\Gamma(\frac{1}{2})s^{k+1/2}}[\mathrm{Re}A_l(s)]^{2k+2}.$$

Consider the first partial-wave phase shift that passes through $\pm\frac{1}{4}\pi$ as the energy increases from threshold. Let the energy at which this occurs be s_1. The phase shift might then continue to grow in magnitude and pass through $\pm\frac{3}{4}\pi$, it might decrease back through $\pm\frac{1}{4}\pi$, or it might do neither. Let the energy at which one of the first two possibilities occurs be s_2. It then follows from the elastic unitarity condition that $\mathrm{Im}A_l(s)$ for this partial-wave can be written

$$\mathrm{Im}A_l(s)=S_l(s)+\theta[(s-s_1)(s_2-s)]\times\{[s/(s-1)]^{1/2}-2S_l(s)\}. \quad (30)$$

This expression will be valid up to an energy, s_3, say, with $s_3>s_2$, at which the phase shift again enters a region where (8) is the appropriate expansion. At this point another step-function term would have to be added to the right-hand side of (30), and so forth.

In the elastic region, then, the formula (12) would be modified for a finite number of partial-wave amplitudes by the addition of step-function terms, each involving different energies s_1 and s_2. The effect of this modification upon the unitary scattering amplitude is easily seen. The first term on the right-hand side of (30) is valid for all partial-wave amplitudes and the Legendre sum over this term reproduces the absorptive part of the scattering amplitude of Sec. III. The Legendre sum over the finite number of step-function terms, on the other hand, contributes "subtractionlike" terms such as the sum on the right-hand side of (29). We emphasize that this finite number of additional terms cannot contribute to the t and u discontinuities, so the expressions for the double spectral functions of Sec. IV are unaffected by the presence of resonances.

We are led to the somewhat unfortunate conclusion that any description of resonances or, more strongly, any partial-wave amplitude with $|\delta_l|>\frac{1}{4}\pi$ in terms of a unitary Mandelstam representation must involve sub-

[9] See, for example, A. W. Martin, Phys. Rev. 173, 1439 (1968).

tractions. The double spectral functions in a unitary representation cannot, by themselves, lead to resonances. Furthermore, the single spectral function terms that will appear in the representation in the form

$$\frac{1}{\pi}\int_1^\infty \frac{ds'\sigma_l(s')}{(s'-s)} P_l[1+2t/(s-1)],$$

where possible subtractions have been suppressed, will have weight functions $\sigma_l(s)$ that include the step functions in (30).

These conclusions indicate that the bootstrap problem of determining resonances through the combination of unitarity and crossing symmetry in the framework of a Mandelstamlike representation becomes more ambiguous. The possibility of anticipated resonances must be incorporated at the start by including single spectral function terms for the appropriate partial wave. The single spectral functions must also anticipate the energies at which the phase shifts pass through certain values. Finally, the single spectral function terms for the crossed channels will contribute to $A_t(t,s)$ and $A_u(u,s)$ and thereby to the double spectral functions, thus affecting the energy dependence of the phase shifts. So it appears that the bootstrap problem becomes very complicated in such a framework.

In concluding this section we note that there seems to be a correlation between these results and the recent conjectures of Freund[10] and Harari[11] concerning the Pomeranchuk trajectory and finite-energy sum rules. Briefly, these authors suggest that the absorptive parts of the amplitudes appearing in the finite-energy sum rules be separated into two pieces, a piece composed primarily of resonance contributions and a remainder piece that can be typified as "background." The resonance contributions then sum to given all of the t-channel Regge poles except the Pomeranchukon, which is given by the integral over the background.

In the language of this paper, the background piece of the low-energy absorptive part can be interpreted as the contribution of the double spectral functions, which must be present in all amplitudes, resonances or no, and which may have something of a universal character. The resonance contributions, on the other hand, can be viewed as arising from the single spectral functions, which necessarily contain the predominant portion of the resonant absorptive parts. While this separation appears natural, and may be correct, it is clearly somewhat premature to stress it in the absence of definitive results on the behavior of the double spectral functions. It is also necessary to determine the amplitude in the inelastic region since this region must contribute substantially to the finite-energy sum rules. This extension to the inelastic region is treated in Sec. VI.

[10] P. G. O. Freund, Phys. Rev. Letters **20**, 235 (1968).
[11] H. Harari, Phys. Rev. Letters **20**, 1395 (1968).

VII. CONTINUATION TO THE INELASTIC REGION

In the inelastic region we do not have a simple and concise statement of unitarity, like (1), and we do not intend here to grapple with the difficult problem of three- and four-particle intermediate states. Instead we wish to explore the continuation of the scattering amplitude using the inverse amplitude parametrization for the partial waves and the suggestive structure of the Mandelstam representation. It was noted in Sec. II (and is shown in detail in the Appendix) that the inverse amplitude representation permits a direct continuation of the expansions (7) and (8), which are consequences of elastic unitarity, to values of s outside the elastic region.

We use this fact to suggest that $f_t(x,s)$ and $f_u(x,s)$ develop imaginary parts as s enters the inelastic region. First we note that $\psi_l(s)$ becomes complex at the first inelastic threshold by writing the partial-wave amplitude as

$$A_l(s)=[s/(s-1)]^{1/2}R_l^{-1}(s)\sin\theta_l e^{i\theta_l}, \qquad (31)$$

where θ_l is real and $R_l(s)$ is the ratio of the total to the elastic cross section in the lth partial wave. Equations (6) and (31) can be solved for $\psi_l(s)$ with the result (suppressing subscripts)

$$(s-1)^{1/2}\psi(s)=\{R(s)\cot\theta-i[R(s)-1]\}^{-1},$$

and since $R(s)>1$ if the inelastic partial-wave cross section is nonvanishing, we conclude that $\psi_l(s)$ is complex.

In continuing (12) out of the elastic region we replace $\mathrm{Re}A_l(s)$ by $A_R(s)$ as defined by (9). Now it is possible that $A_R(s)$ does not have a simple Froissart-Gribov representation in the inelastic region, in which case we cannot evaluate the Legendre sum. In the absence of a precise statement of inelastic unitarity this question must remain unanswered. We will assume that (11) remains valid in the inelastic region for $A_R(s)$ which leads to the conclusion that f_t and f_u develop imaginary parts. The structure of the Mandelstam representation suggests a simple way to incorporate these effects.

Crossing symmetry implies that each double spectral function consists of two "wings" that in the present equal-mass case can be written in the form

$$\rho_{st}(s,t)=\theta(st-4s-t)\rho_1(s,t)+\theta(st-4t-s)\rho_2(s,t). \qquad (32)$$

For simplicity we are treating here a problem like pion-pion scattering, where the first inelastic states are the four-pion states. Since the arguments of this section are based upon the Mandelstam representation, it is a simple matter to extend them to the more general situation. The Mandelstam representation also gives the formula for the crossed-channel absorptive parts,

$$A_t(t,s)=\frac{1}{\pi}\int_1^\infty \frac{ds'\rho_{st}(s',t)}{s'-s}+\frac{1}{\pi}\int_1^\infty \frac{du'\rho_{tu}(u',t)}{u'+s+t-1}, \qquad (33)$$

where subtractions have been omitted because they play no role in the argument.

For s and t greater than unity the imaginary part of $A_l(t,s)$ comes from the two wings of ρ_{st}, and the second wing gives an imaginary part only for s in the inelastic region. This leads us to consider the continuation

$$f_t(x,s)=\mathrm{Re}f_t(x,s)+i\theta(sx-4x-s)\rho_2(s,x)\,, \quad (34)$$

with an analogous expression for $f_u(x,s)$. If the subtracted Mandelstam representation is valid, then $\mathrm{Re}f_t(x,s)$ would be given by the real part of the subtracted version of (33). But from the point of view of the unitary scattering amplitude [Eqs. (17), (18), and (19)], it is unnecessary to make any further assumptions about $\mathrm{Re}f_t(x,s)$.

We now wish to examine the consequences of (34) for the singularity structure of the continued scattering amplitude. It is clear by the method of construction that $A^{(1)}(s,t,u)$ [Eq. (18)] contributes to the s-channel absorptive part in the inelastic region and gives a double spectral function with the Landau curve of the second wing in (32). The higher-order terms $A^{(n)}(s,t,u)$, with $n=2, 4, 6, \cdots$, develop real parts and additional imaginary parts because of the presence of the second-wing double spectral function in (34). The formulas get rather complicated and will not be recorded here. There is however an important observation to be made.

Consider the term $A^{(2)}(s,t,u)$ [Eq. (20)] with f_t and f_u continued as in (34). In the inelastic region $A^{(2)}$ can be decomposed into imaginary terms involving only $\mathrm{Re}f_t$ and $\mathrm{Re}f_u$, real terms involving these functions and second-wing functions, and imaginary terms involving only second-wing functions. The first of these contribute double spectral functions to the full amplitude with the Landau curves of the first wings. Second, the real terms have t and u discontinuities with Landau curves (in the positive-t case)

$$t=t_+[s/(s-4),\,1]=4(s-1)/(s-4)\,,$$

that is, asymptotic to $t=4$ and $s=4$. The last set of terms, which are imaginary and therefore contribute to the s-channel absorptive parts in the inelastic region, give rise to double spectral functions with the Landau curves,

$$t=t_+[s/(s-4),\,s/(s-4)]=4s(s-2)^2/[(s-1)(s-4)^2]\,,$$

again asymptotic to $t=4$ and $s=4$.

This persists to all orders; the introduction of the second-wing double spectral function in (34) leads to additional singularities for positive t and u, but the Landau curves for these singularities are confined to the region (in the positive-t case) $t\geqslant 4$, $s\geqslant 4$, and they get progressively further from the origin in the s-t plane. As a consequence only $A^{(1)}(s,t,u)$ contributes a double spectral function in the region $1\leqslant t\leqslant 4$, and this is the region determined, in principle, by elastic unitarity in the t channel. We conclude that the prescription (34) is a natural way to extend the unitary scattering amplitude into the inelastic region, giving the additional singularities of the Mandelstam representation required by crossing symmetry.

Finally there is the question of the convergence of the expansion (17) in the inelastic region. This depends upon the magnitudes of $\mathrm{Re}f_t$, $\mathrm{Re}f_u$, and the second-wing double spectral functions, which remain to be determined by inelastic unitarity and crossing symmetry. So at this point, in contrast to the elastic region, we can say nothing about convergence apart from the following observation. The expansions (7) and (8) are absolutely convergent in the elastic region because of unitarity. Outside the elastic region, they are absolutely convergent in complex domains described in the Appendix. Now it is quite possible that a given partial-wave amplitude will have parameters $R_l(s)$ and θ_l in (31) such that the expansions diverge, for example, $R_l\sim 2$ and $\theta_l\sim\pm\frac{1}{2}\pi$.

However, if the Froissart-Gribov formula remains valid for $A_R(s)$ as we have assumed, then all partial-wave amplitudes with sufficiently large l will be within the domain of convergence. This indicates that the troublesome partial waves can be handled by the introduction of a finite number of subtraction terms. In fact, the discussions of Secs. V and VI carry over directly to the inelastic region. The number of subtraction terms is of course an open question.

VIII. CONTINUATION TO THE CROSSED CHANNELS

We turn to the problem of continuing the unitary scattering amplitude to the physical regions of the crossed channels. The t and u dependence of the representation (17) occurs simply in the denominators and causes no trouble. The main question is the continuation of the s dependence below the elastic threshold. We again use the inverse amplitude representation as a guide and assume that $A_R(s)$ satisfies the Froissart-Gribov formula so that the Legendre sums can be evaluated as before.

The fact that $\psi_l(s)$ is real for $0\leqslant s\leqslant 1$ indicates that $f_t(x,s)$ and $f_u(x,s)$ remain real in this range, and likewise $A^{(1)}(s,t,u)$ within the Mandelstam triangle. The terms $A^{(n)}(s,t,u)$, $n=2, 4, 6, \cdots$, have the common factor $i(s-1)^{1/2}$ which is real for $s<1$. The functions $I_n(s,t;x_1,\cdots,x_n)$ [Eqs. (13) and (14)] have well-defined continuations into the Mandelstam triangle and are real there (they develop logarithmic singularities for negative values of s). It follows that the representation (17) can be directly continued into the Mandelstam triangle where it is real and free of singularities, as we would insist, provided the continuations of f_t and f_u are given these properties.

We note in passing the obvious fact that f_t and f_u cannot be identified as the crossed-channel absorptive parts in the region $0<s<1$, since all of the terms in the infinite sum in (17) contribute to the absorptive parts. At this point there are two rather distinct questions that must be asked. What is the nature of the singularity structure as s is continued to negative values, i.e., does the third double spectral function come in with the proper Landau curves? And, what are the convergence properties of the expansion (17) within the Mandelstam triangle? The first question is complicated by the necessity of examining the singularity structure of an infinite number of terms. It is evident that the first term, $A^{(1)}(s,t,u)$, can be endowed with the proper third dsf simply by construction. The higher-order terms have factors $\sqrt{s}$ which give unwanted branch points at $s=0$, but they also have logarithmic singularities (coming from the functions I_n) with branch points at $s=0$.

It is possible that the infinite set of terms could be arranged so that these unwanted singularities cancel, but this consideration is obviated by an unexpected difficulty arising from the second question, the convergence of the sum. We find that the amplitude as it stands cannot be continued to $s=0$ without the addition of an infinite number of subtraction terms. The basis of this very negative result is as follows. The expansion of the amplitude came from the expansion of the partial-wave amplitudes in powers of $A_R(s)$ and the assumption that the Froissart-Gribov formula was valid for $A_R(s)$ outside the elastic region.

Now, in problems with the appropriate symmetry, Martin has shown[9] that each $\psi_l(s)$ has at least one pole in the interval $0<s<1$, the pole positions approaching $s=0$ as l increases. The quantity $(s-1)\psi_l^2(s)$ for each partial wave then starts from zero at the elastic threshold and decreases to $-\infty$ as s decreases toward zero. The analysis of the Appendix shows that each partial wave passes through a region where the expansions (7) and (8) fail to converge and eventually enters a region where (8) is the appropriate expansion. This indicates that step-function terms must be added to each partial wave, much as in the resonance situation of Sec. VI, to guarantee that the sums converge to the proper values. In contrast to the resonance case, however, an infinite number of subtraction terms is needed here.

This is, to be sure, an indirect approach to the question of continuing (17) to the crossed channels. But in the absence of definite results for $f_t(x,s)$ and $f_u(x,s)$ it provides evidence that the expansion (17) cannot be analytically continued as it stands to $s=0$. Another way to view the situation is this. The poles of the functions $\psi_l(s)$ do not give rise to singularities in the partial-wave amplitudes on the first sheet of the elastic branch cut. But they do lead to poles in $A_l(s)$ on the second sheet,[5] the pole positions approaching $s=0$ as l increases. The full amplitude therefore has an essential singularity at $s=0$ on the second sheet of the elastic cut.

In the construction procedure of Sec. III the square-root-type elastic branch point is given explicitly by the factor $(s-1)^{1/2}$, common to all terms in the sum in (17). It follows that the continuation of the amplitude onto the second sheet differs from that onto the first sheet by a simple sign change of the sum in (17). Since the continuation of $A(s,t,u)$ to the first (second) sheet is given by the sum (difference) of two functions, and since $A(s,t,u)$ has an essential singularity on the second sheet but not on the first, both functions [namely $A^{(1)}(s,t,u)$ and the infinite sum in (17)] must have essential singularities at $s=0$. This means that $f_t(x,s)$ and $f_u(x,s)$ must themselves have essential singularities at $s=0$, and it is not surprising that the preceding discussion indicated the impossibility of the continuation of (17) to the crossed channels.

The fixed-s dispersion relations and the elastic unitarity condition do not contain enough information to deal with crossing symmetry. Further assumptions on the analytic properties in s must be invoked. One way to avoid these difficulties, for example, would be to assume the validity of the Mandelstam representation. The continuation problems then vanish and one has the added feature that the double spectral functions in the elastic region of each of the three channels are given by the expressions of Sec. IV. The determination of the amplitude becomes a problem of coupled integral equations that, unfortunately, is not completely defined without a statement of inelastic unitarity.

IX. CONCLUSIONS

The assumption of the validity of fixed-s dispersion relations for s in the elastic region together with rigorous expansions for the partial-wave amplitudes leads to scattering amplitudes that are manifestly unitary. Moreover, the unitary scattering amplitudes have the singularity structure in the momentum-transfer variables of the Mandelstam representation. By crossing symmetry, this result holds as well in the elastic regions of the crossed channels. The elastic-region double spectral functions are expressed as finite sums involving finite integrals and are determined by the real parts of the crossed-channel absorptive parts, thus providing an alternative description of Mandelstam's results.[2]

It was shown that the presence of subtractions in the fixed-s dispersion relation can be incorporated into the unitary amplitude. One method was described formally in Sec. V. From a practical point of view the simplest way to generate the expansion of the imaginary part of the amplitude in the subtracted case is by the use of the identities (21) and some slight generalizations of those identities. The resulting expressions are more complicated than in the unsubtracted case, but the formulas for the double spectral functions remain unchanged.

The construction procedure also clarifies the relationship of resonances to the Mandelstam representation. It was shown that the double spectral functions in a unitary representation cannot give rise to resonant partial waves and that subtractionlike terms involving single spectral functions must be present. Finally, the continuation of the unitary amplitude away from the elastic region was considered. In the absence of precise statements of inelastic unitarity, only the singularity structure of the continued amplitude could be examined. A simple prescription was found that generates the additional singularities in the inelastic region dictated by crossing symmetry.

A motivation for the construction of unitary scattering amplitudes is the hope that the twin requirements of crossing and unitarity will be more easily satisfied in such a framework. However, as in practically all earlier attempts, the unitary representation constructed here finds crossing symmetry a stubborn obstacle. The amplitude as it stands cannot be continued to the physical regions of the crossed channels to impose the crossing conditions. One way out would be to assume the validity of the Mandelstam representation, but this is a very strong assumption. A more difficult approach would endeavor to sum the expansion of the amplitude [Eq. (17)] with the hope that the necessary continuation properties would then be evident.

APPENDIX

We consider here the expansions of the partial-wave amplitudes using the inverse amplitude representation [Eq. (6)]. Throughout the cut s-plane, the partial-wave amplitude can be written

$$A(s)=A_R(s)+iA_I(s),$$

where $A_R(s)$ and $A_I(s)$ are expressed in terms of $\psi(s)$ by (9). We define the variable

$$z=(s-1)\psi^2(s), \tag{A1}$$

and consider the expansion of $A_I(s)$ in powers of $A_R(s)$ for arbitrary complex values of z.

It is a matter of some algebra to verify the two identities

$$A_I=S_n[A_R]+[s/(s-1)]^{1/2}R_n^{(1)}(z),$$

$$A_I=[s/(s-1)]^{1/2}-S_n[A_R]-[s/(s-1)]^{1/2}R_n^{(2)}(z),$$

where $S_n[A_R]$ is the finite sum

$$S_n[A_R]=\sum_{k=0}^{n}\frac{\Gamma(k+\frac{1}{2})4^k(s-1)^{k+1/2}}{\Gamma(k+2)\Gamma(\frac{1}{2})s^{k+1/2}}[A_R]^{2k+2},$$

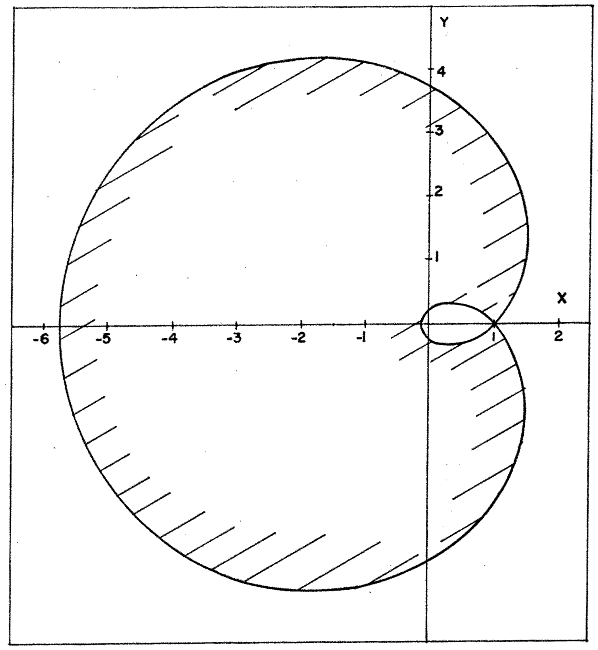

FIG. 1. Locus of points in the complex z plane satisfying $|4z|=|(1+z)^2|$.

and the remainder terms are given by

$$R_n^{(1)}(z)=\frac{z^{n+2}}{(1+z)^{2n+2}}\sum_{m=0}^{n}\frac{2(n-m+1)(2n+1)!}{m!(2n-m+2)!}z^{n-m} \tag{A2}$$

and

$$R_n^{(2)}(z)=\frac{z^{n}}{(1+z)^{2n+2}}\sum_{m=0}^{n}\frac{2(n-m+1)(2n+1)!}{m!(2n-m+2)!}z^{m-n}. \tag{A3}$$

We are interested in the behavior of these quantities in the limit $n\to\infty$.

In this limit, $S_n[A_R]$ becomes a hypergeometric function that converges within and on the unit circle. So the limit exists only for

$$|4(s-1)A_R^2/s|=|4z/(1+z)^2|\leqslant 1. \tag{A4}$$

The locus of points in the complex z plane satisfying the equality $|4z|=|(1+z)^2|$ is plotted in Fig. 1. There are two domains in which $S_n[A_R]$ converges in the limit, the small elliptical region (I) containing the segment $[-3+2\sqrt{2},1]$ and the region (II) lying outside the closed curve of the figure. Recalling that in the elastic energy region,

$$z=(s-1)\psi^2(s)=\tan^2\delta,$$

where δ is the phase shift, we see that the two domains correspond to the regions of validity of (7) and (8), respectively.

To complete the demonstration, we turn to the limits of the remainder terms as $n\to\infty$. In region I we have $|z|\leqslant 1$, so in this region the sum in (A2) is bounded in absolute value by

$$\sum_{m=0}^{n}\frac{2(n-m+1)(2n+1)!}{m!(2n-m+2)!}=\frac{(2n+1)!}{n!(n+1)!}. \qquad \text{(A5)}$$

It follows that in region I,

$$|R_n^{(1)}(z)|\leqslant\frac{\Gamma(n+\frac{3}{2})|z|}{2\Gamma(\frac{1}{2})\Gamma(n+2)}\left|\frac{4z}{(1+z)^2}\right|^{n+1},$$

which vanishes in the limit $n\to\infty$ by virtue of (A4). The argument goes through in similar fashion for $R_n^{(2)}$ in region II. In this region, $|z|\geqslant1$, and the sum in (A3) is again bounded in absolute value by (A5). So in region II,

$$|R_n^{(2)}(z)|\leqslant\frac{\Gamma(n+\frac{3}{2})}{2\Gamma(\frac{1}{2})\Gamma(n+2)|z|}\left|\frac{4z}{(1+z)^2}\right|^{n+1},$$

which vanishes in the limit $n\to\infty$ by virtue of (A4).

We conclude that with the replacements $\mathrm{Re}A_l(s)\to A_R(s)$ and $\mathrm{Im}A_l(s)\to A_I(s)$ the expansion (7) converges for complex values of s such that z [Eq. (A1)] lies in region I and the expansion (8) holds for values of z in region II.

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Asymptotic Behavior of the n-Point Function and Some Applications*

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A generalization of the Bjorken limit (for the two-point function) to the three-point and four-point functions is given. Some general features of the asymptotic behavior of the n-point function are also discussed. These results show that in calculating the various Ward identities for the n-point function all currents are "asymptotically conserved." We derive generalized Weinberg sum rules for the three-point functions (these results can be generalized to the n-point functions). We show that the K_L^0-K_S^0 mass difference (in the universal Fermi theory) is quadratically divergent. Making a saturation assumption, we calculate the coefficient of the quadratic divergency and we get a weak-interaction cutoff $\Lambda=4$ BeV, suggesting that weak interactions are strongly nonlocal. By means of a simple power-counting argument, we find that the nth order probably behaves like $n!G(G\Lambda^2)^{n-1}$, and assuming that this is some kind of asymptotic expansion, we find that the series begins to blow up for $n\sim10^4$. The arguments for this do not constitute a proof. We then study the radiative corrections to the decays $\pi\to e\nu$ and $\pi\to\mu\nu$, which involve a three-point function. We find that these decays cannot be discussed within the framework of current algebra. Finally we show that a somewhat generalized version of the Tamm-Dancoff approximation can be justified if we use our results for the n-point function.

1. INTRODUCTION

Some time ago Bjorken proposed[1] a method for calculating the (virtual) asymptotic behavior of the two-point function. This method has been very useful in estimating the radiative corrections to β decay[1,2] (coming from high virtual masses) as well as the electromagnetic mass differences.[1,3] In this paper we shall generalize Bjorken's expansion to the three-point function as well as the four-point function; it is possible to obtain general results for the n-point function also. Such a generalization is required in order to discuss several interesting physical problems, e.g., the K_L^0-K_S^0 mass difference (in the current-current interaction). The main results of this paper are the following:

In Sec. 2 we generalize the Bjorken expansion to the three-point and the four-point functions. We also give a method for calculating the n-point function.

In Sec. 3 we show that the results obtained in the previous section can be used to prove the following theorem: Assuming the ordinary current algebra, all currents are "asymptotically conserved" in the sense that in calculating Ward identities for the n-point function

$$\int\cdots\int d^4x_1\cdots d^4x_n e^{iq_1x_1+\cdots+iq_nx_n}\times\langle A|T(j_{\mu_1}^{\alpha_1}(x_1)\cdots j_{\mu_n}^{\alpha_n}(x_n))|B\rangle, \qquad (1.1)$$

it is correct to assume that in time-ordered products

$$\partial^{\mu_m}j_{\mu_m}^{\alpha_m}(x_m)=0 \qquad (1.2)$$

for all α's in so far as we are only interested in the leading terms of the n-point function. This theorem is evidently of practical importance since it shows that asymptotically the Ward identities allow us to express the n-point function entirely in terms of the $(n-1)$-

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