cross section for a hydrogen target at this kinematical point is 1.57×10^{-31} cm^2/sr.[1] Under the conditions of this experiment, this cross section would provide a yield for hydrogen of 0.126 K^+ mesons per monitor unit.

A rough estimate of the mean free path in nuclear matter can be made by assuming that the production of K^+ mesons depends only on the number of protons in the nucleus, and that the absorption depends on the distance traveled in leaving the nucleus. Accordingly, a plot of the logarithm of the yield per proton versus $A^{1/3}$, where A is the atomic weight, would yield a straight line whose slope is inversely proportional to the mean free path. Figure 2 shows such a plot of the experimental data points with their statistical errors. The line which is drawn is a least-squares fit for the function $\exp[-(r/\lambda)A^{1/3}]$. The resulting slope is 0.095. This yields a value for the mean free path λ equal to about 10 F, assuming a value of 1 F for the nuclear radius r. However, this is only an order of magnitude estimate since a complete analysis would require consideration of effects of nuclear motion, Coulomb scattering, and geometrical factors. The fact that some of the points

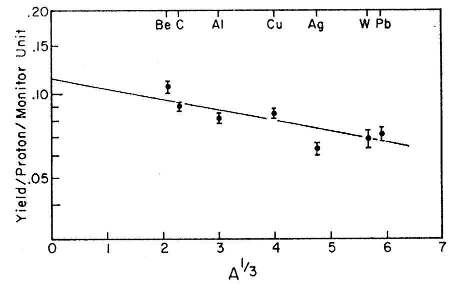

FIG. 2. K^+ meson yield per proton per monitor unit plotted on a logarithmic scale as a function of $A^{1/3}$, where A is the atomic weight. Statistical errors are indicated. The straight line is a least-squares fit for the function $\exp[-(r/\lambda)A^{1/3}]$.

deviate fron the straight line by an amount for outside the statistical error indicates that such systematic effects are present.

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Three-Pion Decays of Unstable Particles*

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The properties of particles of arbitrary spin and parity that decay into three pseudoscalar mesons are surveyed, with primary attention to 3π decays, in order to find efficient means of detecting such particles. Among the topics considered are the general forms of amplitudes subject to invariance and symmetry requirements, the regions of vanishing density in the Dalitz plot, branching ratios, angular correlations among vectors normal to and lying in the production and decay planes, and special decay modes through two-particle resonances. The angular correlations are discussed in detail for processes independent of the intrinsic spin of the production particles, as is appropriate in coherent nuclear processes, and a framework of analysis is provided for more complex problems. A complete characterization of $K\to3\pi$ decays is given in terms of ΔI rules and final-state isospin. The suggestion is made that a second pion, with the same quantum numbers as the ordinary pion, should be found at an energy less than 2 BeV. This prediction is based on the possibility that the pion is primarily a nucleon-antinucleon bound S state and that the force of binding is therefore so strong that it should produce more than one bound S state.

I. INTRODUCTION

THE multiplicity of recently discovered baryon resonances raises the possibility, perhaps even the confident expectation, of a similar richness and regularity in the family of meson resonances. Some of these mesons will decay into two particles; others will prefer a three-particle mode of decay, thus complicating the experimental analysis.

A number of interesting unstable mesons are likely to decay into three pions. We undertake here a phenomenological description of such processes in order to find efficient methods for identifying the quantum numbers of the particles from experimental data. The 3π states have, of course, the symmetry imposed by Bose statistics. This symmetry, properly exploited, supplies information in addition to that deduced from conservation laws and, according to one's viewpoint, makes the phenomenological procedure more simple or more complex than it is for other three-particle decays. Much of the discussion is, however, directly applicable to general

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three-particle decays. We include, in the final section, some remarks on K decay. Because we aim at a systematic and coherent presentation, some elementary results, already well known, are rederived. The principal results are summarized in several tables.

Of the theoretical reasons for anticipating new 3π resonances, we emphasize two:

(a) One expects Regge recurrences of the π, ω, φ mesons. The baryon trajectories are, in the explored energy region, parallel to one another and appear to be straight lines when spin is plotted against mass squared. For example, the nucleon and its recurrences have spins of $\frac{1}{2}$, $\frac{5}{2}$, $\frac{9}{2}$ and squared masses of 0.88, 2.83, 4.79, in $(\mathrm{BeV})^2$. The optimist is tempted to attribute similar behavior to the mesons and seek an $I=1$, spin-2^- particle at, roughly, 1.46 BeV, and two $I=0$, spin-3^- particles near 1.60 and 1.73 BeV, respectively, as well as still higher spin particles at higher energies.

(b) The occurrence of several bound states of the same symmetry type, differing only in "radial quantum number" is commonplace in atomic physics and in potential-theory models when the forces are very strong. There is no general principle in elementary-particle physics which forbids this phenomenon and, indeed, the Pomeranchuk Trajectory and Igi's second Pomeranchuk Trajectory may together be an example of it.

Let us take seriously the view that the pion, and its unitary partners as well, are, like all "elementary" particles, bound states of elementary particles. The force of binding is very large by any standard. Then there is a possibility that a second pion exists (and also a second η and a second K), less strongly bound and carrying the same isospin, spin, and parity as the first particle.

A composite pion would be sought in the channels $\pi\rho$, $\bar{K}K^*$, $\rho\omega$, $\bar{N}N$, and so forth. In the spirit of bootstrap calculations which have already proved qualitatively successful, one might suppose that the channel of lowest mass, $\pi\rho$, is the most important in forming the π, and that π exchange and ω exchange supply the most important forces. If so, then these mechanisms must also be dominant in producing the ω as a bound state of $\pi\rho$ in the appropriate isospin channel. Each particle would be sought in a P wave of the $\pi\rho$ system; the isotopic factors in the forces are the same, so that the forces themselves for the π and ω problems have the same order of magnitude. We then suggest that if a force suffices to produce an ω at 781 MeV, a similar force cannot possibly produce a binding energy large enough to account for a π at 140 MeV. Thus, the $\pi\rho$ channel probably does *not* play a major role in making the π. For similar reasons, of an equally speculative nature, we do not expect that $K\bar{K}^*$ or $\rho\omega$ play a major role. Another interesting possibility is that the pion is largely a composite of three pions in mutual S states. Otherwise, we must retreat to the $\bar{N}N$ channel and other baryon-antibaryon channels.

If it is really true that the pion is primarily a bound state of $\bar{N}N$ and higher energy channels, then the force in these channels has produced a binding energy of almost 2 BeV. It would be the strongest force encountered in elementary-particle physics, much stronger than the forces that in recent bootstrap calculations have accounted for the ρ, K^*, N, Σ, and 33 resonance, and is perhaps capable of producing the second pion with a mass less than, say, two nucleon masses. This new pion would be unstable, decaying into $\pi+\rho$, or more generally, into 3π.

II. CONSTRUCTION OF DECAY AMPLITUDES

1. Formulation

We desire to construct a general amplitude M for the decay

$$X \to 3\pi \tag{2.1}$$

of an X meson of mass m_X. The generality of M is limited by the specification of the isospin, spin, and parity of the 3π final state, and by Bose statistics. We work in the rest frame of X and defer to Sec. IV, Part 3, the consideration of transformation properties under pure Lorentz transformations.

The amplitude may be thought of as a product, or sum of products of the form

$$M=\sum M_I M_F M_{JP}, \tag{2.2}$$

where M_I carries the isotopic spin dependence, M_{JP} carries the spin J and parity P and the remaining energy-momentum dependence is in the "form factor" M_F.

The vectors from which M_{JP} is to be constructed (for $J>0$) are the momenta $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ of the three pions and the pseudovector $\mathbf{q}$,

$$\mathbf{q}=\mathbf{p}_1\times\mathbf{p}_2=\mathbf{p}_2\times\mathbf{p}_3=\mathbf{p}_3\times\mathbf{p}_1. \tag{2.3}$$

The pion momenta and their energies w_i satisfy conservation laws

$$\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0; \tag{2.4}$$

$$w_1+w_2+w_3=m_X. \tag{2.5}$$

A useful consequence is that any function of the energy and momentum variables of the three pions can be expressed again as a function of the variables of any two pions. Then symmetry problems involving three identical particles are reduced to symmetry problems involving two particles, and are trivial.

We note also that any scalar product, e.g., $\mathbf{p}_1\cdot\mathbf{p}_2$, can be expressed again in terms of the pion energies so that the general form factor M_F may be taken as a function only of the energies.

[1] S. Weinberg, Phys. Rev. Letters **4**, 87 (1960).

[2] R. Sawyer and K. Wali, Nuovo Cimento **17**, 938 (1960); Phys. Rev. **119**, 1429 (1960).

The density of the Dalitz plot is proportional to

$$D=\sum|M|^2, \tag{2.6}$$

where the indicated sum is over spin indices (the tensor components) if any.

2. Isotopic Spin and Branching Ratios

To construct forms of M_I, classified by total isospin, we use vector operators **a**, **b**, **c** to represent the isospins of the first, second, and third pions, respectively. The components (a_1,a_2,a_3), etc., transform like rectangular coordinates under isospin rotations. Charge components a_+, a_-, a_0 are defined by

$$a_1=\frac{a_+ + a_-}{\sqrt{2}},\quad a_2=\frac{i(a_- - a_+)}{\sqrt{2}},\quad a_3=a_0, \tag{2.7}$$

with similar definitions for **b** and **c**. A symbol a_+, a_-, or a_0 in an expression for M refers to a process in which pion 1 with momentum $\mathbf{p}_1$ has plus, minus, or zero charge, and so on. In this notation, a_+ and a_- are charge conjugates of each other and the operator $(-a_+, a_0, a_-)$ transforms, under isospin rotations, like components of the $I=1$ representation with $I_3=+1$, 0, -1, respectively. (Note the minus sign.) Scalar and vector products, in the charge notation, are

$$\mathbf{a}\cdot\mathbf{b}=a_0b_0+a_+b_-+a_-b_+, \tag{2.8}$$

and

$$\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}=i\begin{vmatrix} a_+ & b_+ & c_+ \\ a_- & b_- & c_- \\ a_0 & b_0 & c_0 \end{vmatrix}. \tag{2.9}$$

Let E denote a general function of energy-momentum variables which is completely symmetric (even) in the three pions. Let O denote a general completely antisymmetric function. Also let $A\equiv A(23)$ and $\tilde{A}\equiv\tilde{A}(23)$ be general functions of the variables for pions 2 and 3 which are symmetric and antisymmetric, respectively, in these pions. By permutation of the pions, we define associated functions $B=A(31)$, $C=A(12)$, and $\tilde{B}=\tilde{A}(31)$, $\tilde{C}=\tilde{A}(12)$. These functions will be constructed for various spin-parity choices later. We are now ready to consider the isospin possibilities available to 3π states.

Case I=0. The isospin factor is $M_I=\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}$. The general decay amplitude is

$$M=\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}\,O. \tag{2.10}$$

Decays occur only in the neutral mode (π^+,π^-,π^0). There are no $3\pi^0$ decays.

A typical Dalitz triangle is shown in Fig. 1. The physical region, enclosed by the curve within the triangle, is divided by the medians into six "sextant" regions labeled I through VI. The plot density $D=|O|^2$ always has sixfold symmetry for $I=0$, because a permutation of the pions, under which $|O|^2$ is symmetric, permutes the sextant regions.

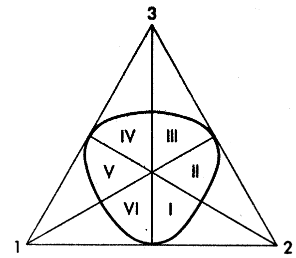

FIG. 1. The sextant regions of the Dalitz plot.

Case I=1. The three possible isospin factors are $\mathbf{a}(\mathbf{b}\cdot\mathbf{c})$, $\mathbf{b}(\mathbf{c}\cdot\mathbf{a})$, and $\mathbf{c}(\mathbf{a}\cdot\mathbf{b})$. The coefficient of $\mathbf{a}(\mathbf{b}\cdot\mathbf{c})$, which is symmetric in pions 2 and 3, must be of type A. The general amplitude is

$$M=\mathbf{a}(\mathbf{b}\cdot\mathbf{c})A+\mathbf{b}(\mathbf{c}\cdot\mathbf{a})B+\mathbf{c}(\mathbf{a}\cdot\mathbf{b})C. \tag{2.11}$$

The amplitudes for the positively charged modes are

$$M(\pi^+\pi^+\pi^-)=A+B; \tag{2.12}$$

$$M(\pi^0\pi^0\pi^+)=C, \tag{2.13}$$

where the indices 1, 2 go with the like particles in each case. The experimental points may all be placed in sextants I, II, III of the Dalitz plot, because pions 1 and 2 are indistinguishable. The branching ratio for the positively charged modes is (apart from the over-all factor due to the slightly different total amounts of phase space for the two processes)

$$\frac{\gamma(\pi^+\pi^+\pi^-)}{\gamma(\pi^0\pi^0\pi^+)}=\frac{\sum_{\mathrm{I+II+III}}|A+B|^2}{\sum_{\mathrm{I+II+III}}|C|^2}. \tag{2.14}$$

Further information is required before quantitative conclusions can be drawn from (2.14). For example, if the decay is dominated by the process $X\to\pi+\rho\to3\pi$, then, as we shall see later, $A+B=-C$, and the branching ratio is 1:1. On the other hand, in the decay of a 0^- particle, A, B, and C will be substantially constant and equal to one another over the Dalitz plot if the Q value of the decay is low. Then we have a branching ratio of about 4:1, as is observed in K^+ decay.

For neutral modes, the amplitudes are

$$M(\pi^+\pi^-\pi^0)=C; \tag{2.15}$$

$$M(3\pi^0)=A+B+C, \tag{2.16}$$

and the branching ratio is

$$\frac{\gamma(\pi^+\pi^-\pi^0)}{\gamma(3\pi^0)}=\frac{\sum_{\mathrm{I+II+III+IV+V+VI}}|C|^2}{\sum_{\mathrm{I}}|A+B+C|^2}=\frac{\sum_{\mathrm{I}}2(|A|^2+|B|^2+|C|^2)}{\sum_{\mathrm{I}}|A+B+C|^2}. \tag{2.17}$$

If $A\approx B\approx C$, as may occur in the 0^- case with low Q value (e.g., η or $K_2{}^0$ decay), then the branching ratio $\approx2:3$.

For a fixed value of the denominator on the right side of (2.17), the numerator clearly achieves a minimum for $A=B=C$. We find then, in general, that the neutral ratio (2.17) is always equal to or greater than 2:3. To place general limits on the charged ratio, we separate A, B, C into their symmetric and nonsymmetric parts by writing

$$A=\tfrac{1}{3}(A+B+C)+\tfrac{1}{3}(2A-B-C),$$

etc., and substituting into (2.14). Because of the symmetry of A, B, C, interference terms such as $(A+B+C)^*(2C-A-B)$ vanish when summed over regions I, II, III of the Dalitz plot. Thus,

$$\frac{\gamma(\pi^+\pi^+\pi^-)}{\gamma(\pi^0\pi^0\pi^+)}=\frac{\sum_{\mathrm{I+II+III}}(4|A+B+C|^2+|2C-A-B|^2)}{\sum_{\mathrm{I+II+III}}(|A+B+C|^2+|2C-A-B|^2)}. \tag{2.14$'$}$$

This version of (2.14) shows that the charged branching ratio always lies between 4:1 and 1:1.

Case $I=2$. The two possible isospin amplitudes may be labeled $M_2^{(a)}$ and $M_2^{(s)}$. $M_2^{(a)}$ is formed by combining pions 1 and 2 to form an $I=1$ state, then adding in pion 3 to get a total isospin of two. $M_2^{(a)}$ is antisymmetric in 1 and 2. $M_2^{(s)}$ is formed by first combining pions 1 and 2 into an $I=2$ state and is symmetric in these pions. For the singly charged mode $I_z=1$, we have

$$M_2^{(a)}=\sqrt{3}\{(a_0b_+-a_+b_0)c_0+(a_+b_--a_-b_+)c_+\}; \tag{2.18a}$$

$$M_2^{(s)}=(a_0b_++a_+b_0)c_0-2(a_+b_+)c_- +(a_+b_-+a_-b_+-2a_0b_0)c_+, \tag{2.18b}$$

and for the neutral mode $I_z=0$,

$$M_2^{(a)}=(a_0b_+-a_+b_0)c_-+2(a_-b_+-a_+b_-)c_0 +(a_-b_0-a_0b_-)c_+; \tag{2.19a}$$

$$M_2^{(s)}=\sqrt{3}\{(a_+b_0+b_0a_+)c_--(a_-b_0+a_0b_-)c_+\}. \tag{2.19b}$$

$M_2^{(a)}$, $M_2^{(s)}$ serve as a basis for the two-dimensional irreducible representation of the permutation group on three objects. The permutation P_{12} which interchanges 1 and 2 is represented in this basis by

$$P_{12}=\begin{pmatrix}-1 & 0\\ 0 & +1\end{pmatrix}. \tag{2.20a}$$

Other permutations are

$$P_{13}=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix},\quad P_{23}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix}. \tag{2.20b}$$

These matrices have a simple geometric interpretation. Let x, y be Cartesian coordinates of a reference frame for Fig. 1, with origin at the center of the triangle. Permutations of 1, 2, 3 are simply reflections and rotations by 120° in the xy plane and correspond to the matrix representations (2.20).

TABLE I. Decay amplitudes classified by the isotopic spin of the 3π state.

I (isospin)		3π decay amplitude
0		$(\mathbf{a}\times\mathbf{b}\cdot\mathbf{c})O$
1		$\mathbf{a}(\mathbf{b}\cdot\mathbf{c})A+\mathbf{b}(\mathbf{c}\cdot\mathbf{a})B+\mathbf{c}(\mathbf{a}\cdot\mathbf{b})C$
2		$M_2^{(a)}\sqrt{3}(A-B)+M_2^{(s)}(2C-A-B)$
	or	$M_2^{(a)}(2\tilde{C}-\tilde{A}-\tilde{B})+M_2^{(s)}\sqrt{3}(\tilde{A}-\tilde{B})$
3		M_3E

Because the representation is irreducible, the coefficients of $M_2^{(a)}$, $M_2^{(s)}$ in the general amplitude M are not independent. Suppose we begin by writing $M(123)=M_2^{(s)}C$ and then symmetrize, forming $M=M(123)+M(321)+M(132)$. This is done with the aid of (2.20b). The result, apart from an over-all factor, is

$$M=M_2^{(a)}\sqrt{3}(A-B)+M_2^{(s)}(2C-A-B). \tag{2.21a}$$

Alternatively, we may start with $M_2^{(a)}\tilde{C}$ and symmetrize, obtaining the equivalent general form

$$M=M_2^{(a)}(2\tilde{C}-\tilde{A}-\tilde{B})+M_2^{(s)}\sqrt{3}(\tilde{A}-\tilde{B}). \tag{2.21b}$$

The amplitudes for singly charged decays are

$$M(\pi^+\pi^+\pi^-)=M(\pi^0\pi^0\pi^+)=-2(2C-A-B). \tag{2.22}$$

The Dalitz plots for $(\pi^+\pi^+\pi^-)$ and $(\pi^0\pi^0\pi^+)$ are identical for $I=2$ and the branching ratio is 1:1.

In the neutral case,

$$M(\pi^+\pi^-\pi^0)=-2\sqrt{3}(A-B). \tag{2.23}$$

Decay into $3\pi^0$ does not occur for $I=2$.

Case $I=3$. The only isospin factor, call it M_3, is symmetric in all pions. Then $M=M_3E$ and the Dalitz plot density has sixfold symmetry as in the $I=0$ case. The singly charged amplitude is

$$M_3=a_-b_+c_+-2a_+b_0c_0+a\leftrightarrow b+a\leftrightarrow c, \tag{2.24a}$$

and the neutral one is

$$M_3=a_+(b_-c_0+b_0c_-)+a\leftrightarrow b+a\leftrightarrow c-2a_0b_0c_0. \tag{2.24b}$$

These results are summarized in Table I.

3. Spin and Parity

Because of the negative intrinsic parity of 3π, particles X with spin-parity assignments 0^-, 1^+, 2^-, 3^+, $\cdots$ will have M_{JP}'s built out of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ only; they will be called particles of "normal parity." Particles X of type 1^-, 2^+, 3^-, etc., will have M_{JP}'s linear in $\mathbf{q}$; they have "abnormal parity." More than one factor of $\mathbf{q}$ in an M_{JP} is unnecessary for quadratic occurrences of $\mathbf{q}$ can be expressed again in terms of $\mathbf{p}_1$, $\mathbf{p}_2$, to wit:

$$q_iq_j=\tfrac{1}{3}\delta_{ij}q^2-p_2^2[(\mathbf{p}_1)_i(\mathbf{p}_2)_j-\tfrac{1}{3}\delta_{ij}p_1^2] -p_1^2[(\mathbf{p}_2)_i(\mathbf{p}_2)_j-\tfrac{1}{3}\delta_{ij}p_2^2]+(\mathbf{p}_1\cdot\mathbf{p}_2) \times[(\mathbf{p}_1)_i(\mathbf{p}_2)_j+(\mathbf{p}_2)_i(\mathbf{p}_1)_j-\tfrac{2}{3}\delta_{ij}(\mathbf{p}_1\cdot\mathbf{p}_2)]. \tag{2.25}$$

To describe a state of angular momentum J, we use a tensor $T_{i_1 i_2 \cdots i_J}$ of the Jth rank in three-dimensional space which is symmetric and traceless in every pair of indices. The general symmetric Jth-rank tensor in 3-space has $\frac{1}{2}(J+1)(J+2)$ independent components. The requirement of tracelessness imposes $\frac{1}{2}J(J-1)$ constraints leaving $2J+1$ independent components, as it should. To construct a Jth-rank tensor, one chooses J vectors $\mathbf{p}_a$, $\mathbf{p}_b$, $\cdots \mathbf{p}_J$ from among $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{q}$ (with repetitions among the $\mathbf{p}_i$), writes down the basic tensor

$$T_{i_1 i_2 i_3 \cdots i_J} = (\mathbf{p}_a)_{i_1} (\mathbf{p}_b)_{i_2} \cdots (\mathbf{p}_J)_{i_J},$$

subtracts off enough contracted terms to produce tracelessness in any pair of indices, and then symmetrizes it. This is always easier than using Clebsch-Gordan coefficients and the general theory of angular momentum. We label the tensors $T(11223)$, $T(12q)$, and so forth, indicating in an obvious way the ranks of the tensors and how many times $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{q}$ are used in constructing them. Some tensors for spin 2 are

$$T_{ij}(11) = (\mathbf{p}_1)_i (\mathbf{p}_1)_j - \tfrac{1}{3}\delta_{ij} p_1^2,$$
$$T_{ij}(1q) = \tfrac{1}{2}[(\mathbf{p}_1)_i (\mathbf{q})_j + (\mathbf{q})_i (\mathbf{p}_1)_j].$$

For the lower spins, the enumeration of possible choices for tensors is quite easy:

(0^+) no choice.
(0^-) one choice: $M_{JF}=1$.
(1^+) three choices: $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, related by (2.4).
(1^-) one choice: $\mathbf{q}$.
(2^+) three choices: $T(1q)$, $T(2q)$, $T(3q)$, related by $T(1q)+T(2q)+T(3q)=0$.
(2^-) three independent choices which may be taken as $T(11)$, $T(22)$, $T(33)$, or as $T(12)$, $T(23)$, $T(13)$, with $T(12)=\frac{1}{2}[T(33)-T(11)-T(22)]$, etc.
(3^+) four independent choices: e.g., $T(111)$, $T(222)$, $T(112)$, and $T(122)$.
(3^-) three independent choices: $T(11q)$, $T(22q)$, $T(33q)$, and so on.

4. Form Factors

We have observed that form factors are generally functions $M_F(w_1,w_2,w_3)$ of the pion energies. It is convenient to use, instead of the w_i, variables s_1, s_2, s_3 with

$$s_i = w_i - \tfrac{1}{3}(w_1+w_2+w_3) = w_i - \tfrac{1}{3}m_X, \qquad (2.26)$$

so that

$$s_1+s_2+s_3=0. \qquad (2.27)$$

Then $s_1=s_2=s_3=0$ marks the center of the Dalitz plot. The vertical median is $s_1=s_2$. At its base s_3 reaches its minimum value of $-\frac{1}{3}Q$. ($Q=m_X-3m_\pi$ is the Q value.)

The maximum value of each s is given by

$$s_M = \tfrac{1}{6}Q(m_X+3m_\pi)/m_X, \qquad (2.28)$$

which lies between $Q/3$, the nonrelativistic limit and $Q/6$, the relativistic limit. The equation for the boundary of the physical region is

$$8m_X s_1 s_2 s_3 + \tfrac{2}{3}(s_1^2+s_2^2+s_3^2)(m_X^2+3m_\pi^2) = (m_X^2-9m_\pi^2)^2/27. \qquad (2.29)$$

These formulas ignore the small mass difference between charged and neutral pions.

For decays where the Q value, and hence the range of s values, is small compared to the relevant scale of mass, the form factors can be approximated by the first few terms of a power series in s. An example is K decay, where $Q/3<0.2m_\pi$.

For conciseness, we use the following abbreviations:

(a) f_e denotes any completely symmetric function of s_1, s_2, s_3.
(b) f_3, g_3 denote any functions of the two variables s_1, s_2, which are symmetric in these variables. If f_1, f_2 occur in the same expression with f_3, it is understood that they are formed from f_3 by permutation of the variables.
(c) $s_{123}=(s_1-s_2)(s_2-s_3)(s_3-s_1)$.
(d) h_{12} denotes an arbitrary function, $h(s_1,s_2)$, unrestricted by symmetry requirements. Companion functions $h_{21}=h(s_2,s_1)$, h_{13}, etc., are constructed by permutations.

Any function of type f_e can depend only on the symmetric combinations $s_1s_2s_3$, $s_1^2+s_2^2+s_3^2$, and $s_1+s_2+s_3$. Because the last combination vanishes, f_3 may be regarded as an arbitrary function of the first two. A suitable representation of f_e, for low energies, is

$$f_e = f_e(s_1^2+s_2^2+s_3^2,\, s_1s_2s_3) = c_1 + c_2(s_1^2+s_2^2+s_3^2) + c_3 s_1s_2s_3 + O(s^4). \qquad (2.30)$$

Similarly, we may regard f_3 as an arbitrary function of s_3 and s_1s_2 and expand it as

$$f_3 = a + bs_3 + cs_3^2 + ds_1s_2 + O(s_i^3). \qquad (2.31)$$

An antisymmetric function of the s_i vanishes when any two of them are equal and so has s_{123} as a factor. This factor is itself antisymmetric. Hence the general antisymmetric function has the form $s_{123}f_e$.

The general antisymmetric function in two variables s_1, s_2 can be written $(s_1-s_2)f_3$.

Weinberg[1] has used an expansion like (2.31) in a discussion of K^+ decay. Other authors[2] have asserted that Weinberg's procedure is invalid, because of a singularity of the decay amplitude arising from its dependence on 2π scattering amplitudes. The latter depend on, for example, the variable $(\nu_1)^{1/2}$, where ν_1 is the square of the relative three-momentum of pions 2 and 3 in the rest frame of these two pions. The claim is that $(\nu_1)^{1/2}$ is a singular function of the Dalitz plot variables, so that an expansion of $(\nu_1)^{1/2}$, and hence of the decay amplitude, is unpermissible. However, one may calculate that

$$(\nu_1)^{1/2} = [2m_X(s_M-s_1)]^{1/2}. \qquad (2.32)$$

TABLE II. The general forms which appear in 3π amplitudes for different spins and parities of the decay particle.

Spin	O	A	E
0^-	$s_{123}f_e$	f_1	f_e
1^+	$\mathbf{p}_1(f_2-f_3)+\mathbf{p}_2(f_3-f_1)+\mathbf{p}_3(f_1-f_2)$	$h_{32}\mathbf{p}_2+h_{23}\mathbf{p}_3$	$f_1\mathbf{p}_1+f_2\mathbf{p}_2+f_3\mathbf{p}_3$
2^-	$(s_2-s_3)f_1T(11)+(s_3-s_1)f_2T(22)$ $+(s_1-s_2)f_3T(33)$	$f_1T(11)+h_{32}T(22)+h_{23}T(33)$	$f_1T(11)+f_2T(22)+f_3T(33)$
3^+	$(s_2-s_3)f_1T(111)+(s_3-s_1)f_2T(222)$ $+(s_1-s_2)f_3T(333)+f_eO(3^+)$	$f_2T(222)+f_3T(333)+h_{23}T(233)$ $+h_{32}T(322)$	$f_1T(111)+f_2T(222)+f_3T(333)$ $+s_{123}f_eO(3^+)$
	where $O(3^+)=T(112)-T(122)+T(223)-T(233)+T(331)-T(311)$		
1^-	$f_e\mathbf{q}$	$(s_2-s_3)f_1\mathbf{q}$ (and $\tilde{A}=f_1\mathbf{q}$)	$s_{123}f_e\mathbf{q}$
2^+	$f_1T(1q)+f_2T(2q)+f_3T(3q)$	$h_{32}T(2q)-h_{23}T(3q)$	$(s_2-s_3)f_1T(1q)+(s_3-s_1)f_2T(2q)$ $+(s_1-s_2)f_3T(3q)$
3^-	$f_1T(11q)+f_2T(22q)+f_3T(33q)$	$(s_2-s_3)f_1T(11q)+h_{32}T(22q)$ $-h_{23}T(33q)$	$(s_2-s_3)f_1T(11q)+(s_3-s_1)f_2T(22q)$ $+(s_1-s_2)f_3T(33q)$

The feared singularity occurs at a point on the edge of the Dalitz plot. The coefficients in (2.31) are still well defined by an expansion about the center of the plot which converges in the physical region, but their magnitudes may deviate from what dimensional considerations relating to the decay process would suggest. As a practical matter, quadratic terms are not yet detected in K-decay experiments. As data improve, one may wish to fit a plot density like $|f_3|^2$ by representing f_3 as

$$f_3=a'+b's_3+b''(1-s_3/s_M)^{1/2}+b'''(1-s_1/s_M)^{1/2}+b'''(1-s_2/s_M)^{1/2}+\cdots. \quad (2.33)$$

Then b/a, which is still the interesting quantity (see Ref. 1 and Sec. VI), is given by

$$b/a=[b'-\tfrac{1}{2}(b''-b''')/s_M]/(a'+b''+2b'''). \quad (2.34)$$

The point is that if data which represent $(1-s_3/s_M)^{1/2}$ are fitted to a linear function in the region $-s_M\leqq s_3\leqq s_M$, the coefficient of s_3 for best fit may turn out closer to $-0.7s_M^{-1}$ than to its correct value of $-\frac{1}{2}s_M^{-1}$. The difficulty can also be avoided by restricting data to a region about the center of the Dalitz plot, so that higher powers of s carry less weight. Our conclusion is that the energy expansions are valid, but the data analyst must treat them with care.

III. DISTRIBUTIONS ON THE DALITZ PLOT

1. General Form of the Decay Amplitude (Tables I and II)

An amplitude characterized by definite isospin must be of one of the forms given in Table I. The functions O, E, A, $\tilde{A}$, etc., whose symmetry types were defined in the previous section, depend on the spin and parity state and are constructed with the aid of the possible M_{JP}'s already enumerated. For example, in the 1^- case, the general functions symmetric and antisymmetric in pions 2 and 3 are, respectively, $A=(s_2-s_3)f_1\mathbf{q}$ and $\tilde{A}=f_1q$, so the amplitudes for $I=1$ and $I=2$ decay can always be written

$$M(1^-,I=1)=[\mathbf{a}(\mathbf{b}\cdot\mathbf{c})(s_2-s_3)f_1+\mathbf{b}(\mathbf{c}\cdot\mathbf{a})(s_3-s_1)f_2+\mathbf{c}(\mathbf{a}\cdot\mathbf{b})(s_1-s_2)f_3]\mathbf{q}, \quad (3.1)$$

and

$$M(1^-,I=2)=[M_2^{(a)}(2f_3-f_2-f_1)+M_2^{(s)}\sqrt{3}(f_1-f_2)]\mathbf{q}. \quad (3.2)$$

As another example, for 1^+, we may take $\mathbf{p}_3$ and $(\mathbf{p}_1-\mathbf{p}_2)$ as basic tensors. Then, in constructing O, the coefficient of $(\mathbf{p}_1-\mathbf{p}_2)$ must be f_3 and in constructing E, the coefficient of $\mathbf{p}_3$ must be f_3. Hence the complete forms can be written

$$O=f_1(\mathbf{p}_2-\mathbf{p}_3)+f_2(\mathbf{p}_3-\mathbf{p}_1)+f_3(\mathbf{p}_1-\mathbf{p}_2), \quad (3.3a)$$

$$E=f_1\mathbf{p}_1+f_2\mathbf{p}_2+f_3\mathbf{p}_3, \quad (3.3b)$$

and in terms of these, we may write $M(1^+,I=0)$ and $M(1^+,I=3)$.

Table II contains a complete list of the functions for $J\leqq3$. Perhaps an additional word of explanation is desirable for the 3^+ entries. At this stage of complexity, it is helpful (though not essential) to treat with the representations of the permutation group on three objects. Let $\bar{E}$, $\bar{O}$, and $(\bar{\Delta}^{(a)},\bar{\Delta}^{(s)})$ denote basis elements for the even, the odd, and the two-dimensional irreducible representations, respectively. The general form factors for these representations are

$$\bar{E}=f_e, \quad (3.4a)$$

$$\bar{O}=s_{123}f_e, \quad (3.4b)$$

$$\bar{\Delta}^{(a)}=\sqrt{3}(f_1-f_2), \quad (3.4c)$$

$$\bar{\Delta}^{(s)}=2f_3-f_2-f_1. \quad (3.4d)$$

The four independent tensors of spin 3^+ can be classified as follows:

$$E(3^+)=T(111)+T(222)+T(333)\equiv T(123), \quad (3.5a)$$

$$O(3^+)=T(112)-T(122)+T(223)-T(233)+T(331)-T(311), \quad (3.5b)$$

$$\Delta^{(a)}(3^+)=\sqrt{3}[T(111)-T(222)], \tag{3.5c}$$

$$\Delta^{(s)}(3^+)=2T(333)-T(111)-T(222). \tag{3.5d}$$

Now, $\Delta\times\Delta=\Delta+E+O$. Thus E can be obtained from (3.4) and (3.5) by multiplying $E(3^+)$ by $\bar{E}$, $\bar{O}(3^+)$ by $\bar{O}$, or $\Delta(3^+)$ by $\bar{\Delta}$. The first and third possibilities are included in the expression

$$f_1T(111)+f_2T(222)+f_3T(333), \tag{3.6a}$$

and the second is represented by

$$s_{123}f_aO(3^+). \tag{3.6b}$$

By these techniques, the catalog of energy-momentum functions for 3^+ and higher spins may be derived straightforwardly.

2. Regions of the Dalitz Plot Where the Density Must Vanish (Fig. 2)

Vanishing at the periphery. The most obvious feature of the abnormal parity states $1^+, 2^-, 3^+, \cdots$ is that each spin tensor is linear in $\mathbf{q}$. The plot density has a factor of q^2 and vanishes all along the periphery. A decay having points on or near the periphery is easily identified as a normal parity type, e.g., $0^-, 1^+, 2^-, \cdots$. The possibility of a normal parity decay also vanishing along the entire periphery cannot be logically excluded, but is rather unlikely.

Vanishing at the center. Because $s_1=s_2=s_3$ at the center, we must try to build functions of the required symmetry types using the M_{JP} alone, without the help of form factors. If this cannot be done, the plot density must vanish at the center. This situation occurs for some low-spin decay modes, as may be seen by inspecting Table II. But vanishing at the center is *not* obligatory for any decay mode with $J\geqq 4$.

To see this, we first exhibit sample tensors of types E and O for cases 4^- and 4^+ which do not vanish at the center:

$$E(4^-)=T(1111)+T(2222)+T(3333), \tag{3.7a}$$

$$E(4^+)=T(111q)+T(222q)+T(333q), \tag{3.7b}$$

$$O(4^-)=T(1112)-T(2221)+T(2223) \\ -T(3332)+T(3331)-T(1113), \tag{3.7c}$$

$$O(4^+)=T(112q)-T(221q)+\text{similar terms}. \tag{3.7d}$$

Therefore, when $J=4$, the density need not vanish for $I=0$ or $I=3$, and *a fortiori*, need not vanish for $I=1$, 2, where the symmetry requirements are less stringent. This result holds for all higher J, because appropriate E and O tensors can be built from (3.7) by putting in as many extra $\mathbf{q}$'s in the arguments of the T's as necessary.

Vanishing at the head of the vertical median. At the head, $s_1=s_2$ and $\mathbf{p}_1=\mathbf{p}_2=-\frac{1}{2}\mathbf{p}_3$, so that any amplitude is proportional to a single tensor, of the type $T(111\cdots)$ or $T(111\cdots q)$. In the normal (abnormal) parity case, the tensor is even (odd) under $1\leftrightarrow 2$. Then the plot

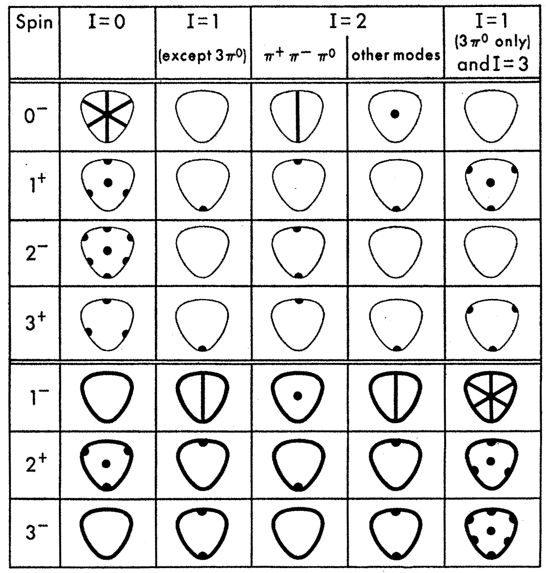

FIG. 2. Regions of the 3π Dalitz plot where the density must vanish because of symmetry requirements are shown in black. The vanishing is of higher order (stronger) where black lines and dots overlap. In each isospin and parity state, the pattern for a spin of $J+$ even integer is identical to the pattern for spin J, provided $J\geqq 2$. (Exception: vanishing at the center is not required for $J\geqq 4$.)

density vanishes in each normal parity case for $I=0$ and the neutral $I=2$ mode. For abnormal parity, there is an especially strong vanishing (i.e., in addition to that imposed by the $\mathbf{q}$ factor) for $I=1$, the charged $I=2$ modes, and $I=3$. The sixfold symmetry of $I=0$, $I=3$ implies additional vanishings at the other median heads.

Vanishing at the base of the vertical median. Here $\mathbf{p}_3=0$, $\mathbf{p}_1=-\mathbf{p}_2$. The results are the same as at the head for (J^-) but reversed for (J^+), because $\mathbf{p}_1$ now changes sign under $1\leftrightarrow 2$, as does $\mathbf{q}$.

These results are summarized in Fig. 2.

3. Further Energy Dependence of the Dalitz Plot Density

If there is no evidence of distortion of the 3π phase space by strong 2π interactions, the form factors in M may, perhaps, be assumed to vary slowly and approximated by one or two terms of a power series expansion. The variety of forms an amplitude may have is then greatly reduced. This approach is generally valid for decays with small Q value. (Stevenson *et al.*[3] have analyzed $I=0$ decays with spins of 0, 1, and 2 in this limit. We note from Fig. 2 that their predictions of the regions of vanishing density are still valid for amplitudes with arbitrary form factors.)

At the other extreme is the possibility of very strong 2π interactions leading to an intermediate two-particle decay, for example, $\pi+\rho$. This can be analyzed by special methods on which we comment in Sec. V. We

[3] M. Lynn Stevenson, L. W. Alvarez, B. C. Maglíc, and A. H. Rosenfeld, Phys. Rev. 125, 687 (1962).

expect that the general decay of a high-spin, high-mass particle can be represented well as a superposition of one or more two-particle decay modes and a "background" with form factors only weakly dependent on the pion energies.

4. Partial-Branching Ratios

The expansion of form factors in powers of energy is always valid in a sufficiently small region, say, a neighborhood of the center of the plot. Some branching ratios are well defined, if calculated only for events lying at the center, although they are not specified uniquely by our phenomenology, if all the events are counted. For example, for 0^-, the partial ratios for (2.14) and (2.17) are rigorously 4:1 and 3:2, respectively. For 1^+, 2^+, 3^-, we have $A+B+C=0$ at the center, as is evidenced by the vanishing of the density at the center in the $I=3$ column of Fig. 2. In these cases, the partial ratios for (2.14) and (2.17) are 1:1 and 1:0, respectively. The branching ratio at the center is not uniquely determined by phenomenology for other spin-parity cases.

IV. ANGULAR CORRELATIONS

1. Formulation

Consider production of an unstable X meson in a collision of the type

$$P+N\rightarrow X+N'. \tag{4.1}$$

The X subsequently decays according to

$$X\rightarrow \pi_1+\pi_2+\pi_3. \tag{4.2}$$

Experimentally, one measures the counting rate for the combined process

$$P+N\rightarrow \pi_1+\pi_2+\pi_3+N', \tag{4.3}$$

which is supposed to proceed through the nearly stable intermediate state $X+N'$. In speaking of angular correlations, we refer to the dependence of the counting rate on the relative orientations in space of the production particles P, N, N' and the decay particles π_1, π_2, π_3. To reduce the ambiguity of the discussion, we shall first suppose that (a) P represents a pseudoscalar meson, (b) the initial and final target states N and N' have the same parity, and (c) the cross sections do not depend on the spins of N, N', if any. The resulting formulas are applicable to a collision in which P is a π or K meson and N, N' are states of a nucleus which scatters the meson coherently. Such processes, taking place in a heavy-liquid bubble chamber, may be very useful for the exploration of higher meson resonances.

Some remarks on other production processes will be made at the end of this section in Part 4 and general "maximum complexity theorems" are derived.

Because the triplet P, N, N' is assumed here to have the same intrinsic parity as 3π, the classification of tensors built out of momentum vectors into normal and abnormal parity cases is the same as in the previous sections.

The particle momenta are $\mathbf{P}$, $\mathbf{N}$, $\mathbf{N}'$, $\mathbf{X}$, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$. P and w_P will denote the momentum magnitude and energy of the P particle, and so on for other particles. Let M_P and M_D be the invariant amplitudes for the production process (4.1) and the decay process (4.2), respectively. (M_D was called M in previous sections.) We suppose that these amplitudes are constructed with particle densities normalized to $(2\pi)^{-3}$ particles per unit volume. This convention fixes the powers of 2π in the formulas below in accordance with the usual rules. Define M_{PD} by

$$M_{PD}=\sum M_PM_D\,, \tag{4.4}$$

where the summation is over the polarization of X and is only present if X has spin. With M_D, M_P specified by tensors, as defined in the previous sections, the polarization sum is the scalar product between M_P, M_D. The invariant amplitude for the total process (4.3) is now given by the expression

$$M(PN\rightarrow 123N')=(2\pi)^3M_{PD}\frac{i(2\pi)^{-4}}{X_\mu{}^2-(m_X-\frac{1}{2}i\Gamma)^2}, \tag{4.5}$$

where

$$X_\mu\equiv(w_X,\mathbf{X})=(w_1+w_2+w_3,\,\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\,, \tag{4.6}$$

and Γ is the full width of X. The incident flux factor $\mathfrak{F}$ for (4.3) in a general frame is

$$\mathfrak{F}=w_Pw_N[(P_\mu N_\mu)^2-m_P{}^2m_N{}^2]^{-1/2}.$$

In terms of these quantities, the differential cross section for (4.3) has the form

$$\begin{aligned}d\sigma(PN\rightarrow 123N')&\\ =(2\pi)^2&\frac{\mathfrak{F}|M(PN\rightarrow 123N')|^2}{(2w_P)(2w_N)}\frac{d\mathbf{N}'}{2w_{N'}}\frac{d\mathbf{p}_1}{2w_1}\frac{d\mathbf{p}_2}{2w_2}\frac{d\mathbf{p}_3}{2w_3}\\ &\times\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{N}'-\mathbf{P}-\mathbf{N})\\ &\times\delta(w_1+w_2+w_3+w_{N'}-w_P-w_N)\,.\end{aligned} \tag{4.7}$$

For the intermediate state to have meaning at all, we must have $\Gamma\ll m_X$. Then

$$\begin{aligned}\left|\frac{1}{X_\mu{}^2-(m_X-\frac{1}{2}i\Gamma)^2}\right|^2&\approx\frac{1}{(w_X{}^2-X^2-m_X{}^2)^2+m_X{}^2\Gamma^2}\\ &\approx\frac{2\pi}{2\Gamma m_X}\delta(w_X{}^2-X^2-m_X{}^2)\\ &=\frac{2\pi}{2\Gamma m_X}\frac{\delta(w_X-(X^2+m_X{}^2)^{1/2})}{2(X^2+m_X{}^2)^{1/2}}.\end{aligned} \tag{4.8}$$

We now specialize to the reference frame where the X

meson is at rest. The cross section becomes

$$d\sigma(PN\to 123N')=[(P_\mu N_\mu)^2-m_P^2m_N^2]^{-1/2}2^{-8}\times\left(\frac{2\pi}{\Gamma m_X^2}\right)|M_{PD}|^2d\rho'', \quad (4.9)$$

where

$$d\rho''=\frac{d\mathbf{p}_1d\mathbf{p}_2d\mathbf{p}_3}{w_1w_2w_3}\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\delta(w_1+w_2+w_3-m_X)\times\frac{d\mathbf{N}'}{w_N}\delta(m_X+w_{N'}-w_P-w_N). \quad (4.10)$$

Let us count the variables that enter into a description of the collision. The momenta of the three decay particles are specified by nine coordinates, but the four conservation laws

$$\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0, \quad w_1+w_2+w_3=m_X \quad (4.11)$$

reduce the number of independent degrees of freedom to five. Three of these are angles which orient the decay configuration in space. The other two are internal degrees of freedom and may be chosen to be the energy variables s_1, s_2 already used in the previous sections. Similarly, the production particles P, N, N' obey conservation laws

$$\mathbf{P}+\mathbf{N}-\mathbf{N}'=0, \quad w_P+w_N-w_{N'}=m_X. \quad (4.12)$$

Thus, the production configuration is also characterized by three angles of orientation and two internal variables.

Let θ', φ' be spherical-coordinate angles of $\mathbf{N}'$ relative to $\mathbf{P}$ as a polar axis. Then the three angles of orientation may be taken as the two angles which fix the direction of $\mathbf{P}$ in space and the azimuthal angle φ'. For the internal degrees of freedom, we choose θ' and the total incident energy w_0 of the scattering system.

Let θ_{ij} be the angle between $\mathbf{p}_i$, $\mathbf{p}_j$. Let $C_{12}=C_{12}(s_1,s_2)$ be the value which $\cos\theta_{12}$ is constrained to have, in terms of the internal variables, in virtue of the conservation laws (4.11.) Then

$$\delta(w_1+w_2+w_3-m_X)=(w_3/p_1p_2)\delta(C_{12}-\cos\theta_{12}). \quad (4.13)$$

Now, integrate the cross section $d\sigma(PN\to 123N')$ over $N^2dN/w_N=Ndw_N$ and $d\mathbf{p}_3$ and set

$$dp_1=p_1w_1dw_1d\Omega_1=p_1w_1ds_1d\Omega_1,\ \text{etc.} \quad (4.14)$$

The effect is to replace $d\rho''$ in (4.9) by $d\rho'$, where

$$d\rho'=N'ds_1ds_2d\Omega_1d\Omega_2\delta(C_{12}-\cos\theta_{12})d\cos\theta'd\varphi'. \quad (4.15)$$

Finally, we must introduce a transformation which manifests the relation between the decay configuration and the production configuration. The decay configuration has two independent vectors, say, $\mathbf{p}_1$, $\mathbf{p}_2$, and one pseudovector $\mathbf{q}$. Similarly, the production configuration has two independent vectors $\mathbf{P}$, $\mathbf{N}$, and one pseudovector $\mathbf{Q}$,

$$\mathbf{Q}=\mathbf{P}\times\mathbf{N}=\mathbf{N}'\times\mathbf{N}=\mathbf{P}\times\mathbf{N}'. \quad (4.16)$$

It is easier to work with the set $\mathbf{P}$, $\mathbf{R}$, $\mathbf{Q}$, where

$$\mathbf{R}=\mathbf{Q}\times\mathbf{P}=\mathbf{N}(P^2)-\mathbf{P}(\mathbf{P}\times\mathbf{N}), \quad (4.17)$$

so that $\mathbf{R}$ is a vector and orthogonal to $\mathbf{P}$ and $\mathbf{Q}$. The set of unit vectors $\hat{P}$, $\hat{R}$, $\hat{Q}$ is a right-handed system, with $\hat{P}$, $\hat{R}$ defining the production plane. Let $\hat{n}_1$, $\hat{n}_2$, $\hat{n}_3$ be mutually orthogonal unit vectors defined in terms of the unit vectors $\hat{p}_i$ by the equations

$$\hat{p}_i=c_{ij}\hat{n}_j;\quad i,j=1,2,3. \quad (4.18)$$

The c_{ij} should be functions of the internal decay variables s_1, s_2. Some useful ways of doing this are considered immediately below. Let $\hat{m}_1$, $\hat{m}_2$, $\hat{m}_3$ be an analogous set of orthonormal vectors, set up in terms of $\mathbf{P}$, $\mathbf{R}$, $\mathbf{Q}$. The relation between decay and production configurations is now specified by Euler angles α, β, γ, where β is the angle between $\hat{m}_3$, $\hat{n}_3$, and α, γ are the angles between the line of nodes and $\hat{m}_1$, $\hat{n}_1$, respectively. The connection between the unit vectors and the Euler angles is, as is well known,[4]

$$\hat{n}_i\cdot\hat{m}_j=ij\text{th matrix element of}\begin{pmatrix}\cos\alpha\cos\gamma-\sin\alpha\sin\gamma\cos\beta & \sin\alpha\cos\gamma+\cos\alpha\sin\gamma\cos\beta & \sin\gamma\sin\beta\\ -\cos\alpha\sin\gamma-\sin\alpha\cos\gamma\cos\beta & -\sin\alpha\sin\gamma+\cos\alpha\cos\gamma\cos\beta & \cos\gamma\sin\beta\\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta\end{pmatrix}. \quad (4.19)$$

It follows that

$$d\Omega_1d\Omega_2=\sin\beta d\alpha d\beta d\gamma d(\cos\theta_{12}). \quad (4.20)$$

The invariant amplitude M_{PD} is now a function of s_1, s_2, w_0, θ', α, β, γ. In particular, it is independent of $d\varphi'$. Using (4.20) and (4.15) and integrating over $d(\cos\theta_{12})d\varphi'$, we reach a form for $d\sigma(PN\to 123N')$ which is identical with (4.9) except that $d\rho''$ is replaced by $d\rho$,

$$d\rho=2\pi N'ds_1ds_2\sin\beta d\alpha d\beta d\gamma d(\cos\theta'). \quad (4.21)$$

Equation (4.9) with the phase space given by (4.21) controls the discussion of angular correlations. If we are interested in the experimental counting rate for the process (4.3) as a function of an angle β between two vectors, one in the decay system and one in the production system, then β should be chosen as the polar Euler angle. The number of observed events in the interval between $\cos\beta$ and $\cos\beta+d(\cos\beta)$ is, apart from a β-independent factor, given by

$$Z(\beta)d(\cos\beta)=d(\cos\beta)\times\int ds_1ds_2\frac{d\alpha}{2\pi}\frac{d\gamma}{2\pi}d(\cos\theta')|M_{PD}|^2. \quad (4.22a)$$

[4] See, for example, H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1950), p. 109.

TABLE III. Normalized angular distributions between production and decay particles in the process $P+N\to N'+X\to N'+3\pi$ for different spin-parity assignments of X. The production is presumed to be independent of the intrinsic spins of P, N, N' as is appropriate in coherent nuclear collisions of a pion beam. The intrinsic parity of (P,N,N') is taken negative. Directions defining the correlation angles are measured in the rest frame of X. $\hat{Q}$ and $\hat{q}$ are unit normals to the production and decay planes, respectively. $\hat{P}$ and $\hat{p}_1$ are the directions of the incident beam and of one of the decay pions, respectively. Results not restricted to a definite isospin state of the decay pions are valid for decays into any three 0^- particles. K is a constant parameter.

Spin	Cosβ	$\hat{Z}(\beta)$ $\left(\frac{1}{2}\int_0^\pi \hat{Z}(\beta)d\cos\beta=1\right)$	Restrictions and remarks (FP≡forward production)
0^-	any	1	
1^+	$\hat{q}\cdot\hat{P}$	$\frac{3}{2}\sin^2\beta+\frac{3}{4}K(3\cos^2\beta-1)$	$0\leqq K\leqq 1$. (In FP, $K=0$)
	$\hat{q}\cdot\hat{Q}$	$\frac{1}{2}\hat{Z}(\beta)+\frac{1}{2}\hat{Z}(\pi-\beta)=\frac{3}{4}(1+\cos^2\beta)$	In FP, $\hat{Z}(\beta)=\hat{Z}(\pi-\beta)$
2^-	$\hat{q}\cdot\hat{P}$	$(15/8)\sin^4\beta+K(5\cos^4\beta-2\cos^2\beta-\frac{1}{3})$	FP only. Same value of K,
	$\hat{q}\cdot\hat{Q}$	$\frac{15}{64}(3\cos^4\beta+2\cos^2\beta+3)$ $+\frac{1}{8}K[15\cos^4\beta-22\cos^2\beta+(13/3)]$	$K\geqq 0$. For isospin zero, $K=15/32$
3^+	$\hat{q}\cdot\hat{P}$	$(35/16)\sin^6\beta$	FP, isospin zero, center region of Dalitz plot
1^-	$\hat{q}\cdot\hat{P}$	$\frac{3}{2}\sin^2\beta$	
	$\hat{p}\cdot\hat{Q}$	$\frac{3}{2}\sin^2\beta$	
	$\hat{q}\cdot\hat{Q}$	$3\cos^2\beta$	
	$\hat{p}_1\cdot\hat{P}$	$\frac{3}{4}(1+\cos^2\beta)$	
2^+	$\hat{q}\cdot\hat{P}$	$(5/4)(4\cos^4\beta-3\cos^2\beta+1)-(5/4)K(5\cos^4\beta-3\cos^2\beta)$	$0\leqq K\leqq 1$. (In FP, $K=0$)
	$\hat{q}\cdot\hat{Q}$	$\frac{1}{2}\hat{Z}(\beta)+\frac{1}{2}\hat{Z}(\pi-\beta)=(5/4)(4\cos^4\beta-3\cos^2\beta+1)$	In FP, $\hat{Z}(\beta)=\hat{Z}(\pi-\beta)$
3^-	$\hat{q}\cdot\hat{P}$	$(35/8)\sin^2\beta(9\cos^4\beta-2\cos^2\beta+1)$ $+K\sin^2\beta(84\cos^4\beta-40\cos^2\beta+\frac{4}{3})$	FP only. Same value of K, $K\geqq 0$
	$\hat{q}\cdot\hat{Q}$	$(35/128)(27\cos^6\beta-26\cos^4\beta+3\cos^2\beta+4)$ $+K[63\cos^6\beta-106\cos^4\beta+(243/5)\cos^2\beta-4]$	For $3\pi^0$ and $I=3$ states, $K=105/1024$
	$\hat{q}\cdot\hat{P}$	$(35/256)\sin^2\beta[135\cos^4\beta-46\cos^2\beta+(43/5)]$	FP. Decay via intermediate $\pi+\rho$ state. $\hat{p}_1=\rho$ momentum
	$\hat{p}_1\cdot\hat{P}$	$\frac{7}{64}(225\cos^6\beta-305\cos^4\beta+111\cos^2\beta+1)$	Ignore data in interference region of Dalitz plot
J^P	any	Polynomial of degree $2J$ in $\cos\beta$	Valid for arbitrary spins, parities of production and decay particles
	any except $\hat{q}\cdot\hat{Q}$	Polynomial of degree J in $\cos^2\beta$	Valid for arbitrary spins, parities of production and decay particles. In FP, valid also for $\cos\beta=\hat{q}\cdot\hat{Q}$
	$\hat{p}_1\cdot\hat{P}$	$\lvert P_J(\cos\beta)\rvert^2$	FP, normal parity decay, periphery of Dalitz plot

It may also be useful to consider correlations for fixed values of s_1, s_2, that is, in a particular region of Dalitz plot, azimuthal distributions, and in recalcitrant cases where many data are available, simultaneous correlations in two or more angles. Thus, we may also consider more general correlation functions $Z(\alpha,\beta)$, $Z(\alpha,\beta,s_1,s_2)$, etc.

A summary of the results to be obtained below for the polar angle distributions is given in Table III. The tabulated functions are normalized correlations, $\hat{Z}(\beta)$, related to the correlations calculated below by

$$\hat{Z}(\beta)=Z(\beta)\Big/\frac{1}{2}\int_0^\pi Z(\beta')d(\cos\beta'). \tag{4.22b}$$

We shall consider angular correlations between four pairs of vectors: (a) between $\mathbf{q}$ and $\mathbf{Q}$, (b) between $\mathbf{p}_1$ and $\mathbf{Q}$, (c) between $\mathbf{q}$ and $\mathbf{P}$, and (d) between $\mathbf{p}_1$ and $\mathbf{P}$. The relevant angles will be called β_{qQ}, β_{1Q}, β_{qP}, β_{1P}, respectively. Correlations among $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{N}$, $\mathbf{N}'$ may also be interesting but do not require a separate discussion.

To study $Z(\beta_{qQ})$, we define the unit vectors by

$$\begin{aligned}\hat{p}_1&=\hat{n}_1,\\ \hat{p}_2&=\cos\theta_{12}\hat{n}_1+\sin\theta_{12}\hat{n}_2,\\ \hat{p}_3&=\cos\theta_{13}\hat{n}_1-\sin\theta_{13}\hat{n}_2,\\ \hat{q}&=\hat{n}_3,\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}\hat{P}&=\hat{m}_1,\\ \hat{R}&=\hat{m}_2,\\ \hat{Q}&=\hat{m}_3.\end{aligned} \tag{4.24}$$

The Euler angles are defined by (4.19), and β_{qQ} is the polar Euler angle, as desired. If we prefer to study $Z(\beta_{1P})$, we use

$$\begin{aligned}\hat{p}_1&=\hat{n}_3,\\ \hat{p}_2&=\cos\theta_{12}\hat{n}_3-\sin\theta_{12}\hat{n}_2,\\ \hat{p}_3&=\cos\theta_{13}\hat{n}_3+\sin\theta_{13}\hat{n}_2,\\ \hat{q}&=\hat{n}_1,\end{aligned} \tag{4.25}$$

and

$$\hat{P}=\hat{m}_3\,,\quad \hat{R}=-\hat{m}_2\,,\quad \hat{Q}=\hat{m}_1. \tag{4.26}$$

Similarly, the definitions (4.23) and (4.26) are convenient for obtaining correlations in β_{qP}, whereas (4.24) and (4.25) would be used for β_{1Q}.

The tensor M_D is now constructed out of the $\mathbf{p}_i$, s_i, as described in Secs. II and III. The tensor M_P is built out of $\mathbf{P}$, $\mathbf{N}$, w_0, θ' according to the same rules, except that its generality is not limited by symmetry requirements.

Finally, the scalar product $M_{PD}=\sum M_P M_D$ is obtained and converted into a function of s_1, s_2, w_0, θ', α, β, γ. Given that M_D is traceless and symmetric, M_P need not have these properties as the scalar product will project out the correct angular-momentum part. Thus, a possible M_{PD} is obtained by multiplying the general M_D for the assumed spin J of X by J factors of $\hat{P}$, $\hat{R}$, $\hat{Q}$ to produce a scalar. The general M_{PD} is the sum of such terms with coefficients depending on w_0, θ'.

2. Properties of Correlation Functions for Various Spins and Parities

Case 0^-. The correlation function is independent of the Euler angles, regardless of which definition is used. If $Z(\beta)$ is graphed against β, rather than against $\cos\beta$, then the density-of-states factor $\sin\beta$ appears in the distribution for all the spin cases.

Case 1^-. This is the simplest nontrivial case. M_{PD} is proportional to $\mathbf{q}\cdot\mathbf{Q}$. Then, in the β_{qQ} scheme,

$$Z(\alpha,\beta_{qQ},\gamma)=Z(\beta_{qP})=\cos^2(\beta_{qQ})\,. \tag{4.27a}$$

Moreover,

$$Z(\alpha,\beta_{qP},\gamma)=\sin^2\alpha\,\sin^2\beta_{qP}\,, \tag{4.27b}$$

$$Z(\alpha,\beta_{1Q},\gamma)=\sin^2\gamma\,\sin^2\beta_{1Q}\,, \tag{4.27c}$$

$$Z(\alpha,\beta_{1P},\gamma)=(\cos\alpha\cos\gamma-\sin\alpha\sin\gamma\cos\beta_{1P})^2\,, \tag{4.27d}$$

and

$$Z(\beta_{qP})=\tfrac{1}{2}\sin^2\beta_{qP}\,, \tag{4.28a}$$

$$Z(\beta_{1Q})=\tfrac{1}{2}\sin^2\beta_{1Q}\,, \tag{4.28b}$$

$$Z(\beta_{1P})=\tfrac{1}{4}(1+\cos^2\beta_{1P})\,. \tag{4.28c}$$

In each case, the subscript on β indicates the scheme used, and α, γ are defined within that scheme, although we do not bother to put subscripts on them. We note that (a) the correlations are valid for each set of values of w_0, θ', s_1, s_2 as well as for the sum over all of them, (b) azimuthal distributions are given also, (c) the Z's differ from the observed counting rate by a constant factor, of course, but the *same* factor applies to each of the Z's above. We have then quite a number of relations with which to check a hypothesis of spin 1^- for an observed X.

Case 1^+. The tensors may be written in the form

$$M_D=\hat{p}_1F_1+\hat{p}_2F_2+\hat{p}_3F_3\,, \tag{4.29}$$

$$M_P=\hat{P}G_P+\hat{R}G_R\,, \tag{4.30}$$

where the F_1, F_2, F_3 are related in different ways for different isospin cases, and G_P, G_R are functions of w_0, θ'. The observed correlations will depend on the integrated quantities

$$g_P=\frac{1}{2}\int|G_P|^2d\cos\theta'\,, \tag{4.31}$$

$$g_R=\frac{1}{2}\int|G_R|^2d\cos\theta'\,. \tag{4.32}$$

We have

$$|M_{PD}|^2=|(\hat{P}G_P+\hat{R}G_R)\cdot(\hat{p}_1F_1+\hat{p}_2F_2+\hat{p}_3F_3)|^2. \tag{4.33}$$

In the β_{qP} scheme, we have

$$\hat{p}_1\cdot\hat{P}=\hat{n}_3\cdot\hat{m}_1=\sin\gamma\,\sin\beta_{qP}\,, \tag{4.34a}$$

$$\hat{p}_1\cdot\hat{R}=\hat{n}_3\cdot(-\hat{m}_2)=-\sin\alpha\,\cos\gamma-\cos\alpha\,\sin\gamma\,\cos\beta_{qP}. \tag{4.34b}$$

The scalar products with $\hat{p}_2$, $\hat{p}_3$ are inferred from (4.34) by replacing γ with $\gamma+\theta_{12}$ and $\gamma-\theta_{13}$, respectively. In computing azimuthal averages of (4.33), the following formulas are useful:

$$\langle\cos(\gamma+\theta)\,\cos(\gamma+\theta')\rangle_\gamma \equiv\int_0^{2\pi}\frac{d\gamma}{2\pi}\cos(\gamma+\theta)\,\cos(\gamma+\theta')=\tfrac{1}{2}\cos(\theta-\theta')\,, \tag{4.35a}$$

$$\langle\sin(\gamma+\theta)\,\sin(\gamma+\theta')\rangle_\gamma =\langle\cos(\gamma+\theta)\,\cos(\gamma+\theta')\rangle_\gamma=\tfrac{1}{2}\cos(\theta-\theta')\,, \tag{4.35b}$$

$$\theta_{12}+\theta_{23}+\theta_{31}=2\pi. \tag{4.36}$$

The average over α eliminates interference between $\hat{P}$ and $\hat{R}$ terms. The interference between a $\hat{p}_i$ term and a $\hat{p}_j$ term is always proportional to $\cos\theta_{ij}$, that is, to $\hat{p}_i\cdot\hat{p}_j$. Thus

$$\langle|M_{DP}|^2\rangle_{\alpha\gamma}=\{\tfrac{1}{2}|G_P|^2\sin^2\beta_{qP}+\tfrac{1}{4}|G_R|^2(1+\cos^2\beta_{qP})\}|M_D|^2, \tag{4.37}$$

and, again disregarding a constant factor,

$$Z(\beta_{qP})=g_P\sin^2\beta_{qP}+\tfrac{1}{2}g_R(1+\cos^2\beta_{qP}). \tag{4.38}$$

The corresponding calculation for $Z(\beta_{qQ})$ yields an interference between $\hat{P}$ and $\hat{R}$ terms. It is proportional to $\cos\beta$, and not proportional to $|M_D|^2$. However, if we add the distributions in β and $(\pi-\beta)$, we get formulas like (4.37), (4.38), with

$$Z(\beta_{qQ})+Z(\pi-\beta_{qQ})=(g_P+g_R)(1+\cos^2\beta_{qQ}). \tag{4.39}$$

The g_P, g_R in (4.39) are the same as in (4.38) for the same experiment, and the over-all constant factor is the same. It is especially interesting that, apart from normalization, the function $Z(\beta_{qQ})+Z(\pi-\beta_{qQ})$ is completely determined for spin 1^+, simply from invariance requirements.

$Z(\beta_{1Q})$ is like (4.38) with constants depending on the decay variables. $Z(\beta_{1P})$ is the least interesting of the four correlations, because neither its dependence on decay variables nor on production variables separates out as an over-all multiplicative factor.

Case 2^+. The amplitudes can be written

$$M_D=T(1\hat{q})F_1+T(2\hat{q})F_2+T(3\hat{q})F_3, \tag{4.40}$$

$$M_P=T(\hat{P}\hat{Q})G_P+T(\hat{R}\hat{Q})G_R. \tag{4.41}$$

The situation is quite similar to the spin-1^+ case regarding the appearance of $|M_D|^2$ as a common factor after the azimuthal averaging and the cancellation of interferences between $\hat{P}$ and $\hat{R}$ terms. The corresponding angular distributions are

$$Z(\beta_{qP})=\tfrac{1}{2}g_P(4\cos^4\beta_{qP}-3\cos^2\beta_{qP}+1) +\tfrac{1}{2}g_R(1-\cos^4\beta_{qP}), \tag{4.42}$$

$$Z(\beta_{qQ})+Z(\pi-\beta_{qQ}) =(g_P+g_R)(4\cos^4\beta_{qQ}-3\cos^2\beta_{qQ}+1). \tag{4.43}$$

Again, we find an angular distribution, namely (4.43) which has been completely predicted from invariance requirements. And, if (g_P+g_R) is determined from (4.43) and the experiment, then (4.42) is a fourth-degree polynomial in $\cos\beta_{qP}$ with only one adjustable parameter.

Case 2^-. The simplicity of the correlations begins to dissipate as we approach spin 2^- and higher spins. The spin-2^- amplitudes are

$$M_D=T(11)F_1+T(22)F_2+T(33)F_3, \tag{4.44}$$

$$M_P=T(\hat{P}\hat{P})G_P+T(\hat{R}\hat{R})G_R+T(\hat{R}\hat{P})G_{PR}. \tag{4.45}$$

The correlation functions will now depend on four parameters of the production process, g_P, g_R, g_{PR}, and $\bar{g}$ which are the averages over $d(\cos\theta')$ of $|G_P|^2$, $|G_R|^2$, $|G_{PR}|^2$ and $\frac{1}{2}(G_PG_R^*+G_RG_P^*)$, respectively. The azimuthal averaging cancels interference between $T(\hat{R}\hat{P})$ and the other production tensors.

Moreover, the dependence of $\langle|M_{DP}|^2\rangle_{\alpha,\gamma}$ on the internal-decay variables is less simple. Consider, for example, a typical term of this average:

$$I_{ij}=\langle[T(\hat{p}_i\hat{p}_i):T(\hat{P}\hat{P})] \times[T(\hat{p}_j\hat{p}_j):T(\hat{P}\hat{P})]\rangle_{\alpha\gamma}|G_P|^2F_iF_j^*. \tag{4.46}$$

It turns out that I_{ij} has the form $(X+Y\cos^2\theta_{ij})F_iF_j^*$, where X, Y do not depend on s_1, s_2, s_3. Then I_{ij} can be expressed again as

$$I_{ij}=[X'+Y'(\cos^2\theta_{ij}-\tfrac{1}{3})]F_iF_j^* =[X'+Y'T(\hat{p}_i\hat{p}_i):T(\hat{p}_j\hat{p}_j)]F_iF_j^*, \tag{4.47}$$

so that

$$\textstyle\sum_{ij}I_{ij}=X'|F_1+F_2+F_3|^2+Y'|M_D|^2. \tag{4.48}$$

The reader can now visualize the general form that emerges. We define two integrals over the Dalitz plot:

$$m_D=\int ds_1ds_2|M_D|^2, \tag{4.49a}$$

$$n_D=\int ds_1ds_2|F_1+F_2+F_3|^2. \tag{4.49b}$$

Then, for $Z(\beta_{qP})$, we get the following expression:

$$Z(\beta_{qP})=m_D[g_Pf_1(\beta_{qP})+g_Rf_2(\beta_{qP})+g_{PR}f_3(\beta_{qP}) +\bar{g}f_4(\beta_{qP})]+n_D[g_Pf_5(\beta_{qP})+g_Rf_6(\beta_{qP}) +g_{PR}f_7(\beta_{qP})+\bar{g}f_8(\beta_{qP})], \tag{4.50}$$

where the functions $f_i(\beta_{qP})$ are well defined and derivable by the methods already given. The f_i are quadratic polynomials in $\cos^2\beta_{qP}$. There is a corresponding formula for $Z(\beta_{qQ})+Z(\pi-\beta_{qQ})$.

The details of these formulas are obviously too complicated to be of much use. However, the fact that $Z(\beta_{qP})$ and $Z(\beta_{qQ})+Z(\pi-\beta_{qQ})$ are quadratic in $\cos^2\beta_{qP}$ or $\cos^2\beta_{qQ}$, and are, no doubt, quite different from the spin-2^+ distributions, may well be sufficient to confirm or reject a hypothetical spin-2^- assignment.

If a more detailed test of such an assignment is needed, we may avail ourselves of two other resources: restricting the range of $\cos\theta'$ and restricting the region of the Dalitz plot.

Consider first the restriction to production events in which the scattering is nearly forward; that is, $\cos\theta'\approx 1$ and $\mathbf{P}$, $\mathbf{N}$, $\mathbf{N}'$ are all essentially in the same direction. Because all production tensors are built, basically, out of $\mathbf{P}$, $\mathbf{N}$ with scalar coefficients which are not singular, the tensors dependent on $\hat{R}$ have factors of $|\mathbf{R}|$, $|\mathbf{R}|\approx|\mathbf{N}(P^2)-\mathbf{P}(\mathbf{P}\cdot\mathbf{N})|$, and give a vanishing contribution for forward production relative to the contribution of tensors built out of $\mathbf{P}$.

Berman and Drell[5] have recently emphasized the importance of these forward events for the production of resonances by nuclear targets. They point out that, due to coherent effects, the whole production process will, for sufficiently high energy collisions, be concentrated at angles sufficiently forward that the neglect of what we have called "tensors built with $\hat{R}$'s" is justified.

When X has abnormal parity, the counting rate is proportional to $\mathbf{Q}^2$ and vanishes for precisely forward scattering. One may still look at nearly forward events, particularly if the Berman-Drell argument forbids other events, and in these, the R-independent tensor is expected to dominate.

Berman and Drell also point out that specific production mechanisms may exist which combine $\mathbf{N}$ and $\mathbf{P}$ fortuitiously so as to leave a resultant exactly orthogonal to $\mathbf{P}$, thus defeating the implications of the above argument. Such a mechanism is not expected to domi-

[5] S. Berman and S. Drell, Phys. Rev. Letters **11**, 220 (1963).

nate in strong-interaction production in the multi-BeV region, where many mechanisms may compete. If the dominance of such a mechanism is suspected—it will be evidenced by a vanishing in the forward direction even for normal parity—one may wish to disregard **P** and construct tensors only with **R**.

In what follows, we shall ignore R-dependent production tensors, assuming either that the production process is concentrated in the forward direction, or that we look only at forward events anyway and that the pathology mentioned in the previous paragraph does not occur. We shall refer to this case as forward production or as the forward approximation.

Then, $M_P=T(\hat{P}\hat{P})$. The angular correlations are grossly simplified. We find

$$Z(\beta_{qP})=m_D f_1(\beta_{qP})+n_D f_2(\beta_{qP})\,, \qquad (4.51a)$$

$$Z(\beta_{qQ})=m_D f_3(\beta_{qQ})+n_D f_4(\beta_{qQ})\,, \qquad (4.51b)$$

where

$$f_1(\beta)=\sin^4\beta\,, \qquad (4.52a)$$

$$f_2(\beta)=\tfrac{1}{6}(5\cos^4\beta-2\cos^2\beta-\tfrac{1}{3})\,, \qquad (4.52b)$$

$$f_3(\beta)=\tfrac{1}{8}(3\cos^4\beta+2\cos^2\beta+3)\,, \qquad (4.52c)$$

$$f_4(\beta)=\tfrac{3}{16}(5\cos^4\beta-22\cos^2\beta+\tfrac{13}{3})\,. \qquad (4.52d)$$

A special simplification occurs for zero isotopic spin. The integrations (4.49) are carried out over the whole Dalitz plot and the cross terms in the squared amplitudes integrate to zero because each one is antisymmetric in one pair of s variables. As for the noncrossed terms, we have

$$T(\hat{p}_i\hat{p}_i):T(\hat{p}_i\hat{p}_i)=\tfrac{2}{3}\,,\quad i=1,2,3\,. \qquad (4.53)$$

It follows that

$$m_D=\tfrac{2}{3}n_D\,. \qquad (4.54)$$

Therefore, both $Z(\beta_{qP})$ and $Z(\beta_{qQ})$, in the zero isotopic-spin case, are completely defined distributions, apart from a (common) normalization factor.

Finally, let us imagine that only events near the periphery of the Dalitz plot are counted—close enough to the periphery so that $\cos\theta_{ij}\approx 1$. Then there is essentially only one decay tensor, $T(11)$, and

$$M_{PD}=T(11):T(\hat{P}\hat{P})\sim P_2(\hat{p}_1\cdot\hat{P})\,, \qquad (4.55)$$

where $P_2(x)=\frac{1}{2}(3x^2-1)$ is the Legendre polynomial of order 2. Hence, for this case, we have another simple distribution:

$$Z(\beta_{1P})=|P_2(\cos\beta_{1P})|^2\,. \qquad (4.56)$$

Case 3^-. As this case is very close, computationally, to the one just described, it has the same qualitative features as the spin-2^- problem. We use only the production tensor $(T(\hat{P}\hat{P}\hat{Q})$. Then (4.51a) and (4.51b) are again valid if f_i are now defined as follows:

$$f_1(\beta)=\tfrac{1}{3}\sin^2\beta(9\cos^4\beta-2\cos^2\beta+1)\,, \qquad (4.57a)$$

$$f_2(\beta)=\tfrac{1}{10}\sin^2\beta(63\cos^4\beta-30\cos^2\beta+\tfrac{3}{5})\,, \qquad (4.57b)$$

$$f_3(\beta)=\tfrac{1}{12}(27\cos^6\beta-26\cos^4\beta+3\cos^2\beta+4)\,, \qquad (4.57c)$$

$$f_4(\beta)=(3/40)(63\cos^6\beta-106\cos^4\beta+(243/5)\cos^2\beta-4)\,. \qquad (4.57d)$$

The simplification which occurred in spin 2^- for $I=0$ now occurs for amplitudes fully symmetric in the pion charges, that is, $3\pi^0$ decay in $I=1$ and all $I=3$ decays. It is obtained by setting $m_D=12n_D/5$, because our calculation was carried out in terms of tensors normalized to $T(11\hat{q}):T(11\hat{q})=12/5$.

We note two self-consistency requirements which serve as a check on the correctness of the correlation functions presented. The integral $\int Z(\beta)d\cos\beta$, which represents the total counting rate, must have the same value for a given spin case, regardless of which β scheme is used. Moreover, it must be proportional to the plot integral over $|M_D|^2$ only, because this is the only scalar that can be formed from M_D when the angular integrations are done. Indeed, for the 3^- case, we find

$$\frac{1}{2}\int_0^\pi Z(\beta_{qP})d\cos\beta_{qP}=\frac{1}{2}\int_0^\pi Z(\beta_{qQ})d\cos\beta_{qQ}=\frac{4}{105}m_D\,.$$

One may also verify these self-consistency requirements for the other cases.

Case 3^+. Consider a restriction to data in the neighborhood of the center of the Dalitz plot and use $M_P=T(\hat{P}\hat{P}\hat{P})$. We examine only the case of zero isotopic spin. Then M_D reduces to $O(3^+)$, as seen from Table II. It is not necessary to explicitly symmetrize the tensors $T(112)$, $T(122)$, etc., out of which $O(3^+)$ is formed, because $T(\hat{P}\hat{P}\hat{P})$ is already symmetric. Nor is it necessary to make them traceless for this means subtracting out some $J=1$ terms, and there is no nonvanishing $I=0$, $J=1$ tensor at the center of the Dalitz plot. It is then easy to see that

$$Z(\beta_{qP})=Z(\alpha,\beta_{qP})=(\sin\beta_{qP})^6\,. \qquad (4.58)$$

General Spin. First of all, it is clear that each $Z(\beta)$ is a polynomial in $\cos\beta$, $\sin\beta$ of total degree $2J$, if J is the spin of X. But consider the transformation

$$\alpha\to\alpha+\pi\,,\quad \gamma\to\gamma+\pi\,,\quad \beta\to-\beta\,. \qquad (4.59)$$

This changes the sign of $\sin\alpha$, $\cos\alpha$, $\sin\gamma$, $\cos\gamma$, and $\sin\beta$. It leaves the matrix (4.16) and hence $Z(\beta)$ invariant. Because $Z(\beta)$ was obtained through azimuthal averaging, it cannot be affected by the transformation $\gamma\to\gamma+\pi$, $\alpha\to\alpha+\pi$. Then $Z(\beta)$ is invariant under $\beta\to-\beta$. We conclude that $Z(\beta)$ has no odd powers of $\sin\beta$, or equivalently, each $Z(\beta)$ is a polynomial of degree $2J$ in $\cos\beta$.

In some cases, a stronger result can be obtained by considering transformations of the type

$$\alpha\to\pi-\alpha\,,\quad \gamma\to\pi+\gamma\,,\quad \beta\to\pi-\beta \qquad (4.60a)$$

or

$$\alpha\to\pi+\alpha\,,\quad \gamma\to\pi-\gamma\,,\quad \beta\to\pi-\beta\,. \qquad (4.60b)$$

The transformation (4.60a) changes the signs in the second and third columns of (4.19). This leaves M_{PD} unchanged if these columns are utilized an even number of times in constructing each term of M_{PD}, and changes the sign of M_{PD} if they are utilized an odd number of times in constructing each term. The corresponding $Z(\beta)$ are unchanged in sign in either case. Such $Z(\beta)$ are invariant under $\cos\beta \to -\cos\beta$ and hence are polynomials of degree J in $\cos^2\beta$. Thus, the correlations $Z(\beta_{1P})$ and $Z(\beta_{qP})$ are polynomials in $\cos^2\beta$ since $\hat{R}$, $\hat{P}$ are associated with the second and third columns, respectively. Similarly, (4.60b) changes the signs in the second and third rows, and shows that $Z(\beta_{1Q})$ has this property.

In the forward approximation, since $\hat{R}$ is not used, all the correlations are functions of $\cos^2\beta$ for either parity of X, by similar reasoning.

We have already noted a specific case where the distribution is *not* an even function of the cosine, namely, $Z(\beta_{qQ})$ for spin 1^+.

Finally, we observe that, restricting ourselves to the periphery of the Dalitz plot, again to forward production, and to normal parity resonances, $M_P = T(\hat{P}\hat{P}\cdots\hat{P})$, $M_D = T(11\cdots1)$. Then, $Z(\beta_{1P})$ can be expressed in terms of a Legendre polynomial as has already been done for spin 2^- above:

$$Z(\beta_{1P}) = |T(11\cdots1):T(\hat{P}\hat{P}\cdots\hat{P})|^2 \sim |P_J(\cos\beta_{1P})|^2. \quad (4.61)$$

3. Relativistically Covariant Description

By recasting our formulas, heretofore referred to the rest frame of X, into a manifestly covariant form, the transformation of the formulas from one coordinate frame to another may be simplified. It may be worth emphasizing that, because pure Lorentz transformations are not associated with any conservation laws—other than the mass-shell relations which have already been taken into account—a relativistically covariant formula will not contain any new physical information.

Let P_μ, X_μ, $p_{1\mu}$, etc., be the energy-momentum four-vectors associated with our system. We define new four-vectors

$$\bar{P}_\mu = P_\mu - X_\mu(P_\nu X_\nu)/(m_X)^2, \quad (4.62a)$$

$$\bar{p}_{1\mu} = p_{1\mu} - X_\mu(p_{1\nu}X_\nu)/(m_X)^2, \text{ etc.,} \quad (4.62b)$$

and the pseudo-four-vectors

$$q_\mu = \epsilon^{\mu\nu\rho\sigma} p_{1\nu} p_{2\rho} X_\sigma/(m_X), \quad (4.63a)$$

$$Q_\mu = \epsilon^{\mu\nu\rho\sigma} P_\nu N_\rho X_\sigma/(m_X). \quad (4.63b)$$

In the rest frame of X, these vectors reduce to $(O,\mathbf{P})$, $(O,\mathbf{p}_1)$, $(O,\mathbf{q})$, $(O,\mathbf{Q})$, that is, to the actual vectors we have been using. The relativistic version of any term is obtained by substituting $\bar{P}_\mu$ for $\mathbf{P}$, $p_{1\mu}$ for $\mathbf{p}_1$, and so on. For the Kronecker delta, we substitute

$$\delta_{ij} \to \delta_{\mu\nu} - X_\mu X_\nu/(m_X)^2. \quad (4.64)$$

The energies w_i in the rest frame have the covariant definition

$$w_i = p_{i\mu}X_\mu/m_X. \quad (4.65)$$

In this way, for example, the three-space tensor of rank 2,

$$T_{ij} = P_i P_j - \tfrac{1}{3}\delta_{ij}\mathbf{P}^2, \quad (4.66)$$

is promoted into a four-space tensor,

$$\begin{aligned} T_{\mu\nu} &= \bar{P}_\mu \bar{P}_\nu - \tfrac{1}{3}(\delta_{\mu\nu} - X_\mu X_\nu/m_X^2)(\bar{P}_\sigma \bar{P}_\sigma) \\ &= P_\mu P_\nu - (P_\mu X_\nu + X_\mu P_\nu)(P_\sigma X_\sigma)/m_X^2 \\ &\qquad + X_\mu X_\nu (P_\sigma X_\sigma)^2/m_X^4 \\ &\quad - \tfrac{1}{3}(\delta_{\mu\nu} - X_\mu X_\nu/m_X^2)(P_\sigma P_\sigma - (P_\sigma X_\sigma)^2/m_X^2). \quad (4.67) \end{aligned}$$

$T_{\mu\nu}$ will satisfy the requirements of symmetry, tracelessness, and transversality ($X_\mu T_{\mu\nu} = T_{\mu\nu}X_\nu = 0$) in any frame, because it originally did so in the rest frame.

4. Other Production Modes

The discussion above is directly relevant to coherent nuclear production of resonances. If, however, the target is a nucleon, the possibilities of spin flip and the transformation of the target into a baryon resonance introduces new elements into the analysis. If we consider only final target states of spin $\frac{1}{2}$, there are six independent vectors in the production system, namely

$$\mathbf{P},\ \mathbf{R},\ \boldsymbol{\sigma}\times\mathbf{P},\ \boldsymbol{\sigma}\times\mathbf{R},\ (\boldsymbol{\sigma}\cdot\mathbf{P})\mathbf{Q},\ (\boldsymbol{\sigma}\cdot\mathbf{R})\mathbf{Q} \quad (4.68)$$

and six pseudovectors

$$\mathbf{Q},\ \boldsymbol{\sigma},\ (\boldsymbol{\sigma}\cdot\mathbf{P})\mathbf{P},\ (\boldsymbol{\sigma}\cdot\mathbf{P})\mathbf{R},\ (\boldsymbol{\sigma}\cdot\mathbf{R})\mathbf{P},\ (\boldsymbol{\sigma}\cdot\mathbf{R})\mathbf{R}. \quad (4.69)$$

For forward production, only two vectors and two pseudovectors remain: $\mathbf{P}$, $\boldsymbol{\sigma}\times\mathbf{P}$, and $\boldsymbol{\sigma}$, $(\boldsymbol{\sigma}\cdot\mathbf{P})\mathbf{P}$. Because of the last two, the counting rate for forward production need not vanish in the abnormal parity case.

The angular distributions are found by the same types of calculations as before, only now, there will be a few more arbitrary constants. Consider, for example, a case of medium difficulty: spin 2^-, in the forward approximation. Note that

$$\boldsymbol{\sigma}\times\hat{P} = \boldsymbol{\sigma}\times(\hat{R}\times\hat{Q}) = (\boldsymbol{\sigma}\cdot\hat{Q})\hat{R} - (\boldsymbol{\sigma}\cdot\hat{R})\hat{Q}, \quad (4.70)$$

so that

$$M_P = G_P T(\hat{P}\hat{P}) + H_p[(\boldsymbol{\sigma}\cdot\hat{Q})T(\hat{P}\hat{R}) - (\boldsymbol{\sigma}\cdot\hat{R})T(\hat{P}\hat{Q})]. \quad (4.71)$$

There will be no interferences among the three terms above after $|M_{PD}|^2$ has been averaged over spin. The correlation for $\beta=\beta_{qQ}$ or $\beta=\beta_{qP}$ will be of the form

$$Z(\beta) = g_P[m_D f_1(\beta) + n_D f_2(\beta)] + h_P[m_D f_3(\beta) + n_D f_4(\beta)]. \quad (4.72)$$

The details of these and other correlations are left to the interested reader.

In part 2 of this section, we noted that for any spin J, each $Z(\beta)$ is a polynomial of degree $2J$ in $\cos\beta$. Moreover, the polynomial has only even powers of $\cos\beta$ in a number of cases enumerated. Both the results and lines of reasoning employed are seen to be valid regardless of the spins and parities of the production and decay particles.

5. Other Three-Particle Final States

All the results on angular correlations are valid when the decay system consists of any three pseudoscalar mesons—excepting those results which depended explicitly on isotopic-spin considerations.

V. DECAYS THROUGH A TWO-PARTICLE INTERMEDIATE STATE

1. General Properties

Strong interactions among the final-state pions resulting from a high-energy decay may lead to a process of the type

$$X \to Y+\pi_3, \tag{5.1a}$$

$$Y \to \pi_1+\pi_2, \tag{5.1b}$$

as a dominant or important mode of the X decay. We may ask what the data tells us about the quantum numbers of both the Y and X particles in this circumstance.

If the finite widths Γ_X, Γ_Y of the particles are neglected, the energy w_3 of the pion in (5.1a) is fixed at a value $\bar{w}$,

$$\bar{w}=(m_X^2+m_\pi^2-m_Y^2)/(2m_X), \tag{5.2}$$

and s_3 is fixed at

$$s_3=\bar{s}=\bar{w}-\tfrac{1}{3}m_X. \tag{5.3}$$

But if the widths are not neglected, the plot density will have the structure

$$|M_D|^2=\sigma(s_3)\rho(s_1-s_2), \tag{5.4}$$

where $\sigma(s_3)$ has, approximately, a Breit-Wigner shape, and is peaked for $s_3=\bar{s}$, and $\rho(s_1-s_2)$ describes the density along this line.

When pion symmetry is taken into account, there may be further concentration of density along $s_1=\bar{s}$ and $s_2=\bar{s}$, and interferences among them. The lines $s_1=\bar{s}$, $s_2=\bar{s}$ do cross, yielding interference, if [we utilize (5.2) and (2.28)]

$$(2m_Y^2+m_\pi^2)^{1/2}\leqq m_X\leqq (m_Y^2-m_\pi^2)/m_\pi. \tag{5.5}$$

If Y is a ρ meson of mass 750 MeV, interference occurs for

$$1070\text{ MeV}\leqq m_X\leqq 3900\text{ MeV}; \tag{5.6}$$

that is, over nearly all energies where resonances are currently being sought.

Let us determine $\sigma(s_3)$. In the derivation of the cross section, we replace $(p_1p_2/w_3)d\cos\theta_{12}$ by $d\mathfrak{M}$, where $\mathfrak{M}=w_1+w_2+w_3$ is the mass of X. The distribution in $\mathfrak{M}$ is given approximately by

$$\sigma'(\mathfrak{M})d\mathfrak{M}=\frac{1}{\pi}\frac{\frac{1}{2}\Gamma_X}{(\mathfrak{M}-m_X)^2+(\frac{1}{2}\Gamma_X)^2}d\mathfrak{M}. \tag{5.7}$$

The expression for $\sigma'(\mathfrak{M})$ stems from the left side of (4.8), to which it is proportional, and was treated there as a delta function. Here, we must consider the finite width explicitly.

The decay mode (5.1) contributes to $|M_D|^2$ a factor

$$|Y_u^2-(m_Y-\tfrac{1}{2}i\Gamma_Y)^2|^{-1}, \tag{5.8}$$

which, apart from factors which vary slowly, is equal to

$$\sigma''(\mathfrak{M})=\frac{1}{\pi}\frac{\frac{1}{2}\Gamma_Ym_Y/w_Y}{[\mathfrak{M}-w_3-(w_3^2+m_Y^2-m_\pi^2)^{1/2}]^2+(\frac{1}{2}\Gamma_Ym_Y/w_Y)^2}, \tag{5.9}$$

where

$$w_Y=\mathfrak{M}-w_3\approx m_X-\bar{w}.$$

The factor m_Y/w_Y multiplying Γ_Y expresses the dilatation in the lifetime of an unstable particle in a frame other than its rest frame. It follows that

$$\sigma(s_3)=\int\sigma'(\mathfrak{M})\sigma''(\mathfrak{M})d\mathfrak{M}=\frac{1}{\pi}\frac{\frac{1}{2}(\Gamma_X+m_Y\Gamma_Y/w_Y)}{[m_X-w_3-(w_3^2+m_Y^2-m_\pi^2)^{1/2}]^2+\frac{1}{4}(\Gamma_X+m_Y\Gamma_Y/w_Y]^2}. \tag{5.10}$$

When the widths are not too large, the approximation

$$(w_3^2+m_Y^2-m_\pi^2)^{1/2}+w_3-m_X\approx(m_X/w_Y)(w_3-\bar{w}) \tag{5.11}$$

is good in the region where $\sigma(s_3)$ is large. Then, apart from slowly varying factors, we have the final expression

$$\sigma(s_3)=\frac{1}{\pi}\frac{\frac{1}{2}\gamma}{(s_3-\bar{s})^2+(\frac{1}{2}\gamma)^2}, \tag{5.12}$$

with the width γ on the Dalitz plot given by

$$\gamma=\left(1-\frac{\bar{w}}{m_X}\right)\Gamma_X+\frac{m_Y}{m_X}\Gamma_Y. \tag{5.13}$$

Let Y have spin K. Then its parity is $(-1)^K$. The decay (5.1a) of an X particle of *normal* parity can proceed through states with any one of the following orbital angular momenta (if parity is conserved):

$$J+K,\quad J+K-2,\quad J+K-4,\quad \cdots |J-K|. \tag{5.14}$$

There are, then, $J+1$ or $K+1$ possible invariant forms for M_D, whichever number is less, and there are a corresponding number of coupling constants. The coupling constants are truly constant because all the energies and scalar products are fixed in terms of the masses of the various particles. To construct the invariant forms for M_D, we first write a three-space tensor of rank K which describes the decay of Y in the rest frame of Y. This tensor is $T(\mathbf{tt}\cdots\mathbf{t})$, where $\mathbf{t}=\mathbf{p}_1-\mathbf{p}_2$. Then, using the recipe of Sec. IV, Part 3, we promote T to a four-space tensor $T_{\mu_1\cdots\mu_K}$ in the rest frame of X. This involves the replacement of $\mathbf{t}$ by $t_\mu=(p_1)_\mu-(p_2)_\mu$ because, in the rest frame of Y, t_μ has the components $(O,\mathbf{t})$. The new $T_{\mu_1\cdots\mu_K}$ must then be combined with the vectors $(p_3)_\mu$, Y_μ of the X-decay system to form a tensor of rank J. But, because Y_μ is transverse to T and the M_D tensor must have zero time-like components in the X rest frame, it turns out that only combinations of $\mathbf{p}_3$ and the three-space part of T need be considered explicitly. The upshot is that we merely multiply the three-space part of $T_{\mu_1\cdots\mu_K}$ by a three-space tensor $T(33\cdots3)$ of rank L, where L is one of the orbital angular momenta listed in (5.14), contract over enough indices to leave a tensor of net rank J, then make it symmetric and traceless. Each of the $J+1$ (or $K+1$) forms is obtained the same way.

When X has *abnormal* parity, the orbital angular momentum of the decay may be

$$J+K-1,\quad J+K-3\cdots|J-K|+1, \tag{5.15}$$

and there are J or K coupling constants and invariant terms for M_D, whichever number is less. The terms are constructed in the manner described above, except that the pseudotensor ϵ_{ijk} must be used once (one index of ϵ_{ijk} contracts with the decay tensor of Y, another contracts with the tensor of the $\mathbf{p}$'s, and third index remains free).

We shall not pursue these general properties further, except to mention the following:

(i) If X has normal parity and either J or $K=0$, or, if X has abnormal parity and either J or $K=1$, then there is only one decay term and, apart from a constant factor, M_D and the density $\rho(s_1-s_2)$ are uniquely specified. An example is noted below.

(ii) In the general case, M_D involves the variables of the first and second pions only through the combinations s_1-s_2 and $\mathbf{p}_1-\mathbf{p}_2$. Hence $|M_D|^2$ and $\rho(s_1-s_2)$ are seen to be polynomials of degree $2K$ in (s_1-s_2).

(iii) Generalizations of (i) and (ii) to cases where the decay products are not all pseudoscalar mesons are easily found by the same methods.

2. Decays Through $\pi+\rho$

We now specialize to the case of most likely importance in which Y is a ρ meson with spin and isospin of unity. Let $M_{3,12}$ be the amplitude, without the isotopic spin factor, for the decay

$$X\to\rho+\pi_3\to\pi_1+\pi_2+\pi_3. \tag{5.16}$$

The isotopic factor for $\rho\to\pi_1+\pi_2$ is $\mathbf{a}\times\mathbf{b}$. This factor is antisymmetric in the first and second pions; hence $M_{3,12}$ is antisymmetric in 1, 2. The amplitudes for (5.16) in the different isospin states are

$$M(I=0)=(\mathbf{a}\times\mathbf{b}\cdot\mathbf{c})M_{3,12}, \tag{5.17a}$$

$$M(I=1)=\mathbf{c}\times(\mathbf{a}\times\mathbf{b})M_{3,12} =[\mathbf{a}(\mathbf{b}\cdot\mathbf{c})-\mathbf{b}(\mathbf{a}\cdot\mathbf{c})]M_{3,12}, \tag{5.17b}$$

$$M(I=2)=M_2{}^{(a)}M_{3,12}. \tag{5.17c}$$

These amplitudes must be symmetrized with respect to the pions. The result is expressed in Table I in terms of functions O, A, B, C. By symmetrizing (5.17), we find that

$$O=M_{1,23}+M_{2,31}+M_{3,12}, \tag{5.18a}$$

$$A=M_{2,31}-M_{3,12}, \tag{5.18b}$$

$$B=M_{3,12}-M_{1,23}, \tag{5.18c}$$

$$C=M_{1,23}-M_{2,31}. \tag{5.18d}$$

The same definitions of A, B, C apply to both $I=1$, $I=2$.

We observe that

$$A+B+C=0, \tag{5.19}$$

so that the branching ratios (2.14), (2.17) for $I=1$ are 1:1 and 1:0 respectively. These branching ratios cannot distinguish between $I=1$ and $I=2$ resonances.

To obtain $M_{3,12}$, we follow the rules of the previous subsection. Apart from irrelevant constants, the abnormal-parity amplitudes are

$$M_{3,12}(1^-)=\alpha_3\mathbf{q}, \tag{5.20a}$$

$$M_{3,12}(2^+)=\alpha_3T(3q), \tag{5.20b}$$

$$M_{3,12}(J^P)=\alpha_3T(33\cdots3q). \tag{5.20c}$$

For normal parity

$$M_{3,12}(0^-)=\alpha_3(s_1-s_2), \tag{5.21a}$$

$$M_{3,12}(1^+)=\alpha_3[x(s_1-s_2)\mathbf{p}_3+y(\mathbf{p}_1-\mathbf{p}_2)], \tag{5.21b}$$

$$M_{3,12}(J^P)=\alpha_3[x(s_1-s_2)T(33\cdots3) +yT(33\cdots3\mathbf{t})], \tag{5.21c}$$

where x, y are constants and, again, $\mathbf{t}=\mathbf{p}_1-\mathbf{p}_2$. The function α_3 concentrates the density near $s_3=\bar{s}$. It is convenient to think of α_3 as a complex function of s_3 whose absolute square is $\sigma(s_3)$ defined in (5.12), although this is not strictly accurate before the integration over $\mathfrak{M}$.

As particular examples of properly symmetrized amplitudes M_D, we have, for either $\pi^+\pi^+\pi^-$ or $\pi^0\pi^0\pi^+$ decay (with either $I=1$ or $I=2$),

$$M_D(0^-)=\alpha_1(s_2-s_3)+\alpha_2(s_1-s_3)\,, \qquad (5.22)$$

$$M_D(1^-)=(\alpha_1-\alpha_2)\mathbf{q}. \qquad (5.23)$$

Note that in the 0^- case, in which we are especially interested because of our suggestion of a "second pion," the density $\rho(s_2-s_3)$ along the line $s_1=\bar{s}$ is $(s_2-s_3)^2$, and thus vanishes in the middle of the line. This distinctive feature is *not* possessed by the plot density for any other spin-parity case.

Another feature of the above amplitudes is that at the interference point, where $s_1\approx\bar{s}$, $s_2\approx\bar{s}$, there is complete destructive interference for 1^-, and complete constructive interference for 0^-. For isospin zero, on the other hand, the roles of 0^- and 1^+ are reversed with respect to interference. For higher spins, only partial interference, of either type, may occur.

3. Angular Correlations for Decays Through $\pi+\rho$

The assumption of an intermediate $\pi+\rho$ states leads to simplified angular correlations, if we restrict the data to events outside the interference region of the Dalitz plot, if there is any. Then M_D may be taken proportional to a single peaking function, say α_1.

The Berman-Drell[5] correlations for decays into 0^-+1^- are valid only if the fraction of data contained in the interference region is small. In their method, which sums over all ρ polarizations (this is equivalent to integrating over the entire "ρ line" in the 3π Dalitz plot), the question of interference cannot be easily formulated. The question can be formulated in our approach, but unless the data from the interference region is excluded, the angular correlations are not simpler than those already presented in the previous section.

For 2^+ and lower spins, the polar angle correlations already derived are so simple that there is no need to study special approximations for them. One can, however, obtain more clean-cut azimuthal distributions.

In the case of spin 2^+, for example, we have

$$M_{PD}=\alpha_1(s)T(1q):[G_PT(\hat{P}\hat{Q})+G_RT(\hat{R}\hat{Q})]. \qquad (5.24)$$

This leads, in the manner already described, to

$$Z(\beta_{qP},\gamma)=g_P[\sin^2\gamma(\sin^2\beta_{qP}-\cos^2\beta_{qP})^2+\cos^2\gamma\cos^2\beta_{qP}] +g_R[\sin^2\gamma(\cos\beta_{qP}\sin\beta_{qP})^2 +\cos^2\gamma\sin^2\beta_{qP}], \qquad (5.25)$$

and

$$Z(\beta_{qQ},\gamma)=(g_P+g_R)[\sin^2\gamma(\cos^2\beta_{qQ} -\sin^2\beta_{qQ})^2+\cos^2\gamma\cos^2\beta_{qP}]. \qquad (5.26)$$

Because M_{PD} is symmetric under the interchange $\mathbf{p}_1\leftrightarrow\mathbf{q}$, it follows that $Z(\beta_{1P},\gamma)$ and $Z(\beta_{1Q},\gamma)$ have the same forms as $Z(\beta_{qP},\gamma)$, $Z(\beta_{qQ},\gamma)$ respectively.

The correlations for the interesting cases 2^- and 3^- are still rather involved, unless we restrict ourselves to forward production, which we do.

The 2^- amplitude is then

$$M_{PD}=\alpha_1[x(s_2-s_3)T(11)+yT(1\mathbf{t})]:T(\hat{P}\hat{P}). \qquad (5.27)$$

Then relations (4.51a), (4.51b) hold at each point along $s_1=\bar{s}$ if we substitute

$$m_D\rightarrow|M_D|^2, \qquad (5.28)$$

$$n_D\rightarrow|\alpha_1(xp_1^2(s_2-s_3)+y(p_3^2-p_2^2)|^2. \qquad (5.29)$$

Because these factors are determined everywhere from their values at one point in the Dalitz plot, additional correlations are provided among the data.

For spin 3^-, we have, simply,

$$M_{PD}=\alpha_1T(11q):T(\hat{P}\hat{P}\hat{Q}). \qquad (5.30)$$

The β_{qP} and β_{1P} distributions are listed in Table III.

VI. THE PHENOMENOLOGY OF $K\rightarrow3\pi$ DECAYS

1. Classification of $K\rightarrow3\pi$ Amplitudes

The phenomenology of $K\rightarrow3\pi$ decays has been worked over by many authors.[1,2,6–9] We shall summarize the situation in our approach which has some advantages of generality. Moreover, it permits one to proceed until the possibilities of the subject have been exhausted, and then stop, knowing that they have been exhausted. The recent data will also be considered.

The general amplitudes for 0^- decay are, in the notation of Sec. II,

$$M(I=0)=(\mathbf{a}\times\mathbf{b}\cdot\mathbf{c})s_{123}f_e\,, \qquad (6.1a)$$

$$M(I=1)=\mathbf{a}(\mathbf{b}\cdot\mathbf{c})f_1+\mathbf{b}(\mathbf{c}\cdot\mathbf{a})f_2+\mathbf{c}(\mathbf{a}\cdot\mathbf{b})f_3\,, \qquad (6.1b)$$

$$M(I=2)=M_2^{(a)}(g_1-g_2)/\sqrt{3}+M_2^{(s)}(2g_3-g_1-g_2)/3\,, \qquad (6.1c)$$

$$M(I=3)=M_3g_e. \qquad (6.1d)$$

We assume that K decays conserve CP. By combining the charge components of a final 3π state of definite I with the charge components of an initial isospinor (K^+,K^0) to form a neutral component of isospin ΔI and symmetrizing with respect to CP, one obtains a decay amplitude corresponding to a "ΔI rule" which conserves charge and CP. A neutral 0^- state of three pions has $CP=(-1)^I$. Hence $K_1^0=[K^0+(CP)K^0]/\sqrt{2}$ decays into 3π states with $I=0$, 2, and $K_2^0=[K^0-(CP)K^0]/\sqrt{2}$ decays into 3π states with $I=1$, 3.

[6] R. H. Dalitz, Proc. Phys. Soc. (London) **A69**, 527 (1956); Rev. Mod. Phys. **31**, 823 (1959).
[7] R. Gatto, Phys. Rev. **106**, 168 (1957).
[8] S. Okubo, R. E. Marshak, and E. C. G. Sudarshan, Phys. Rev. Letters **2**, 12 (1959).
[9] G. Barton, C. Kacser, and S. P. Rosen, Phys. Rev. **130**, 783 (1963).

TABLE IV. The general forms for $K \to 3\pi$ amplitudes, classified by ΔI rules and final-state isospin. The properties of the f's and g's are given in Sec. II.

Rule	Isospin of 3π	K^+ or K^- $\pi^\pm\pi^\pm\pi^\mp$	$\pi^0\pi^0\pi^\pm$	K_2^0 $3\pi^0$	$\pi^+\pi^-\pi^0$	K_1^0 ($\pi^+\pi^-\pi^0$ only)
$\Delta I=\frac{1}{2}$	0	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$s_{123}f_e$
	1	$f_1'+f_2'$	f_3'	$-f_1'-f_2'-f_3'$	$-f_3'$	$\cdots$
$\Delta I=\frac{3}{2}$	1	$f_1''+f_2''$	f_3''	$2(f_1''+f_2''+f_3'')$	$2f_3''$	$\cdots$
	2	$\frac{1}{3}(2g_3''-g_1''-g_2'')$	$\frac{1}{3}(2g_3''-g_1''-g_2'')$	$\cdots$	$\cdots$	$\frac{2}{3}(g_1''-g_2'')$
$\Delta I=\frac{5}{2}$	2	$\frac{1}{3}(2g_3'-g_1'-g_2')$	$\frac{1}{3}(2g_3'-g_1'-g_2')$	$\cdots$	$\cdots$	$-(g_1'-g_2')$
	3	g_e'	$-2g_e'$	$3g_e'$	$-\frac{3}{2}g_e'$	$\cdots$
$\Delta I=\frac{7}{2}$	3	g_e''	$-2g_e''$	$-4g_e''$	$2g_e''$	$\cdots$

A $\Delta I=\frac{1}{2}$ rule can lead to 3π states of $I=0$ or $I=1$ which, following the scheme (6.1), we describe with functions $s_{123}f_e$ and f_1', f_2', f_3'. A $\Delta I=\frac{3}{2}$ rule leads to $I=1$ or 2; we use functions f_1'', f_2'', f_3'' and g_1'', g_2'', g_3'' for them, respectively. $\Delta I=\frac{5}{2}$ gives $I=2$ or 3 for which we use g_1', g_2', g_3', and g_e'. Finally, we use g_e'' for the $I=3$ state allowed by $\Delta I=\frac{7}{2}$.

All the amplitudes for $K\to 3\pi$ modes are given in terms of these functions in Table IV. The index 3 refers to the unlike pion in charged decays and to a π^0 in neutral decays. Table IV embodies all and only those properties of the decays that correspond to the specified transformation properties.

It is convenient to discuss "reduced" decay rates γ, related to the experimental decay rates $\bar{\gamma}$ by[10]

$$\gamma(\pi^\pm\pi^\pm\pi^\mp)=\bar{\gamma}(\pi^\pm\pi^\pm\pi^\mp)/(6.282\times10^{-3}), \quad (6.2a)$$

$$\gamma(\pi^0\pi^0\pi^\pm)=\bar{\gamma}(\pi^0\pi^0\pi^\pm)/(7.81\times10^{-3}), \quad (6.2b)$$

$$\gamma(3\pi^0)=\bar{\gamma}(3\pi^0)/(9.925\times10^{-3}), \quad (6.2c)$$

$$\gamma(\pi^+\pi^-\pi^0)=\bar{\gamma}(\pi^+\pi^-\pi^0)/(8.067\times10^{-3}). \quad (6.2d)$$

If the $\Delta I=\frac{1}{2}$ rule were strictly valid, the reduced rates would be

$$\begin{aligned}\gamma(\pi^\pm\pi^\pm\pi^\mp)&=\textstyle\sum_{I+II+III}|f_1'+f_2'|^2\\&=\textstyle\sum_I(|f_1'+f_2'|^2+|f_2'+f_3'|^2\\&\qquad+|f_3'+f_1'|^2), \quad (6.3a)\end{aligned}$$

$$\begin{aligned}\gamma(\pi^0\pi^0\pi^\pm)&=\textstyle\sum_{I+II+III}|f_3'|^2\\&=\textstyle\sum_I(|f_1'|^2+|f_2'|^2+|f_3'|^2), \quad (6.3b)\end{aligned}$$

$$\gamma(3\pi^0)=\textstyle\sum_I|f_1'+f_2'+f_3'|^2, \quad (6.3c)$$

$$\begin{aligned}\gamma(\pi^+\pi^-\pi^0)&=\textstyle\sum_{\text{all sextants}}|f_3'|^2\\&=\textstyle\sum_I 2(|f_1'|^2+|f_2'|^2+|f_3'|^2). \quad (6.3d)\end{aligned}$$

From these equations, one easily verifies the theorem of Okubo, Marshak, and Sudarshan[8] on the equality of (reduced) charged and neutral decay rates and the relation between $(\pi^0\pi^0\pi^\pm)$ and $(\pi^+\pi^-\pi^0)$ decays noted by Sawyer and Wali.[2]

Because the Q value of the decays is small, one naturally considers a series expansion of the f and g functions. We understand that enough data has probably been accumulated to detect quadratic terms in the energy dependence of the plot densities, but that most of it is not yet analyzed. Then, to make comparisons with published data, we expand the functions only up to their linear terms, as follows:

$$\begin{aligned}&f_1'=a'+b's_1, && f_1''=a''+b''s_1,\text{ etc.,}\\&g_1'=c's_1, && g_1''=c''s_1,\text{ etc.,}\\&g_e'=d', && g_e''=d''.\end{aligned} \quad (6.4)$$

The K_1^0 mode makes a negligible contribution to K^0 decay except when data is restricted to very early decay times. We shall ignore it in the next two sections, but consider it briefly in the final section. The resulting simplified amplitudes are given in Table V. There are 8 (complex) parameters in Table V. They determine 4 decay rates and 3 slopes of Dalitz-plot distributions. The $3\pi^0$ decay must have zero slope because of its symmetry.

2. Decay Rates

Because the plots are circular to the order of magnitude being considered, the s_3 terms drop out in the calculation of decay rates. The charged rates depend on

$$a_{\text{ch}}=a'+a'', \quad (6.5a)$$

$$d_{\text{ch}}=d'+d''. \quad (6.5b)$$

The neutral rates depend on

$$a_{\text{n}}=a'-2a'', \quad (6.6a)$$

$$d_{\text{n}}=\tfrac{3}{2}(d'-\tfrac{4}{3}d''). \quad (6.6b)$$

The best current value of the τ'-to-τ branching ratio[11] is 0.299 ± 0.018. Hence[12]

$$\frac{\gamma(\pi^0\pi^0\pi^\pm)}{\gamma(\pi^\pm\pi^\pm\pi^\mp)}=\left|\frac{a_{\text{ch}}-2d_{\text{ch}}}{2a_{\text{ch}}+d_{\text{ch}}}\right|^2=(0.299\pm0.018)\times\frac{6.282}{7.81}\times\text{Coulomb correction}. \quad (6.7)$$

Let us write

$$d_{\text{ch}}/a_{\text{ch}}=\epsilon_1+i\epsilon_2, \quad (6.8)$$

[10] The decimal coefficients represent the total phase space for the different decays. They are taken from G. Alexander, S. Almeida, and F. Crawford, Jr., Phys. Rev. Letters 9, 69 (1962), footnote 20.

[11] G. Giacomelli, D. Monti, G. Quareni-Vignudelli, W. Puschel, and J. Tretge, Phys. Letters 3, 346 (1963).

[12] The Coulomb correction, due to Dalitz, is quoted in Ref. 11. The over-all factor multiplying the experimental ratio in (6.7) is 0.8251.

TABLE V. The amplitudes for $K \to 3\pi$ decay, expanded about the center of the Dalitz plot, and including linear terms. The validity of the expansion is discussed in Sec. II.

Rule	Isospin of 3π	K^+ or K^- $\pi^\pm\pi^\pm\pi^\mp$	$\pi^0\pi^0\pi^\pm$	K_2^0 $3\pi^0$	$\pi^+\pi^-\pi^0$	K_1^0 ($\pi^+\pi^-\pi^0$ only)
$\Delta I=\frac{1}{2}$	1	$2a'-b's_3$	$a'+b's_3$	$-3a'$	$-a'-b's_3$	$\cdots$
$\Delta I=\frac{3}{2}$	1	$2a''-b''s_3$	$a''+b''s_3$	$6a''$	$2a''+2b''s_3$	$\cdots$
	2	$c''s_3$	$c''s_3$	$\cdots$	$\cdots$	$\frac{2}{3}c''(s_1-s_2)$
$\Delta I=\frac{5}{2}$	2	$c's_3$	$c's_3$	$\cdots$	$\cdots$	$-c'(s_1-s_2)$
	3	d'	$-2d'$	$3d'$	$-\frac{3}{2}d'$	$\cdots$
$\Delta I=\frac{7}{2}$	3	d''	$-2d''$	$-4d''$	$2d''$	$\cdots$

and assume ϵ_1, ϵ_2 small.[13] Then, to lowest nonvanishing order in ϵ_1, ϵ_2, (6.7) yields

$$-\epsilon_1+\tfrac{3}{4}\epsilon_2{}^2=0.00\pm0.01. \tag{6.9}$$

The neutral ratio is

$$\frac{\gamma(3\pi^0)}{\gamma(\pi^+\pi^-\pi^0)}=\frac{1}{6}\left|\frac{3a_n-2d_n}{a_n+d_n}\right|^2. \tag{6.10}$$

Roughly speaking, comparisons of charged and neutral data test the presence of various ΔI rules. Ratios among charged data or among neutral data test the presence of various I for the 3π state. In particular, (6.7) and (6.10) are measures of the amount of $I=1$ and $I=3$, but are uninformative about $I=2$. Thus, (6.9) suggests that $I=3$ is excluded in $K^\pm$ decay. This is not at all the same as excluding $\Delta I=\frac{3}{2}$ or even $\Delta I=\frac{5}{2}$, as is sometimes stated. These isospin transfers can occur in such a way as to give $I=2$ final states only, as is indicated by the tables.

We assume, hereafter, that $I=3$ is absent from K_2^0 decay as well as charged decay. This could be verified by measuring (6.10) experimentally and obtaining $\frac{3}{2}$. The remaining comparison of rates that is of interest is

$$\frac{\gamma(\pi^\pm\pi^\pm\pi^\mp)}{\gamma(\pi^+\pi^-\pi^0)}=\frac{1}{2}\left|\frac{2a_{ch}}{a_n}\right|^2=2\left|\frac{a'+a''}{a'-2a''}\right|^2$$
$$=\frac{(4.65\pm0.15)\times10^6}{(1.44\pm0.43)\times10^6}\times\frac{8.067}{6.282}. \tag{6.11}$$

The data in (6.11) are from G. Alexander *et al.*, Ref. 10.[14] Then

$$\left|\frac{a_{ch}}{a_n}\right|^2=\left|\frac{1+a''/a'}{1-2a''/a'}\right|^2=2.07\pm0.62. \tag{6.12}$$

This is the equation of a circle in the plane of the complex variable a''/a', with center on the positive real axis. Moreover $|a''/a'|$ assumes its extreme values for a''/a' real and positive. We have

$$0.11\pm0.08\leq|a''/a'|\leq1.30\pm0.46. \tag{6.13}$$

Equation (6.13) represents a quantitative estimate of the amount of $\Delta I=\frac{3}{2}$ relative to the amount of $\Delta I=\frac{1}{2}$ in K decay.

3. Slopes

The slopes in charged decays depend on

$$b_{ch}=b'+b'', \tag{6.14a}$$

$$c_{ch}=c'+c''. \tag{6.14b}$$

We have

$$|M(\pi^0\pi^0\pi^\pm)|^2=|a_{ch}|^2[1+2(b_{ch}+c_{ch})s_3/a_{ch}], \tag{6.15a}$$

$$|M(\pi^\pm\pi^\pm\pi^\mp)|^2=4|a_{ch}|^2[1-(b_{ch}-c_{ch})s_3/a_{ch}]. \tag{6.15b}$$

If only $I=1$ is present ($c_{ch}=0$), then the ratio of slopes would be -2, which is Weinberg's rule.[1] In (6.15), b_{ch}/a_{ch} and c_{ch}/a_{ch} are assumed real, because only their real parts enter the formula. Comparing with experiment,[15] we have

$$(b_{ch}-c_{ch})/2(b_{ch}+c_{ch})$$
$$=(0.53\pm0.07)/(1.0\pm0.4)=0.53\pm0.22, \tag{6.16}$$

whence

$$|c_{ch}/b_{ch}|<0.25. \tag{6.17}$$

The magnitude of the number in (6.17) can be manipulated by adjusting the definitions of the phenomenological parameters, but generally speaking, we can say that the presence of $I=2$, and hence of $\Delta I=\frac{3}{2}$ in charged K decay is not excluded firmly by current experiments.

Finally, we examine the slope in ($\pi^+\pi^-\pi^0$) decay. To illustrate the situation without belaboring the ambiguities, we drop the limits of error [this emphasizes the case $c_{ch}/b_{ch}\approx0$; see (6.16)] and assume that a_{ch}/a_n is real and has one of the values, by (6.12):

$$a_{ch}/a_n=+1.44 \text{ (minimum amount of } \Delta I=\tfrac{3}{2}), \tag{6.18a}$$

[13] Equation (6.7) is a circle in the complex d_{ch}/a_{ch} plane, and large values for the ratio are not excluded by the data, e.g., $d_{ch}/a_{ch}\approx\frac{4}{3}$. The zero solution seems preferable on esthetic and some physical grounds, and we do not consider the alternative solutions further here.

[14] *Note added in proof.* Recent data [Don Stern *et al.*, preliminary result reported at Weak Interaction Conference, Brookhaven, September, 1963 (unpublished)] indicates that the rate $(1.44\pm0.43)\times10^6$ sec^{-1} used for the neutral decay in (6.11) should be replaced by $(2.84\pm0.71)\times10^6$ sec^{-1}. Then (6.12) becomes $|a_{ch}/a_n|^2=0.97\pm0.24$. This is fully consistent with the $\Delta I=\frac{1}{2}$ rule which requires $|a_{ch}/a_n|^2=1$.

[15] M. Ferro-Luzzi, D. H. Miller, J. J. Murray, A. H. Rosenfeld, and R. D. Tripp, Nuovo Cimento **22**, 1087 (1961). The τ' slope quoted in this paper is based on data by J. K. Bøggild, K. H. Hansen, J. E. Hooper, M. Schaerf, and P. K. Aditya, Nuovo Cimento **19**, 621 (1961).

or

$$a_{\mathrm{ch}}/a_{\mathrm{n}} = -1.44 \text{ (maximum amount of } \Delta I = \tfrac{3}{2}). \quad (6.18\text{b})$$

Then

$$|M(\pi^+\pi^-\pi^0)|^2 = |a_{\mathrm{n}}|^2(1+2b_{\mathrm{n}}s_3/a_{\mathrm{n}}), \quad (6.19)$$

where

$$b_{\mathrm{n}} = b' - 2b'', \quad (6.20)$$

so that[16]

$$\frac{2b_{\mathrm{n}}}{b_{\mathrm{ch}} - c_{\mathrm{ch}}} \frac{a_{\mathrm{ch}}}{a_{\mathrm{n}}} = \frac{1.3}{0.53}. \quad (6.21)$$

Thus, by (6.16), (6.21) and the definitions, we have

$$b''/b' = 0.04 \text{ (minimum amount of } \Delta I = \tfrac{3}{2}), \quad (6.22\text{a})$$

$$b''/b' = 1.6 \text{ (maximum amount of } \Delta I = \tfrac{3}{2}). \quad (6.22\text{b})$$

The ratio b''/b' is a further measure of the amount of $\Delta I = \frac{3}{2}$ relative to $\Delta I = \frac{1}{2}$. These figures are only illustrative, as the errors on the $(\pi^0\pi^0\pi^+)$ and $(\pi^+\pi^-\pi^0)$ slopes are about 40 and 60%, respectively.[16]

In conclusion,[17] we stress that if the strict implications of the experiments are not mixed with special assumptions, and respectful attention is given to stated limits of error, then a final state of $I=1$ is present, and the rules $\Delta I = \frac{1}{2}$ and $\Delta I = \frac{3}{2}$ are present in imprecisely determined amounts, but no isospin states and no rules whatsoever are excluded, not even, strictly speaking, $\Delta I = \frac{7}{2}$.[13] One may expect, however, that in the near future the data will speak much more clearly.[14]

4. Information from $K_1^0 \to 3\pi$ decay

Treiman and Weinberg[18] have pointed out that interference between K_1^0 and K_2^0 decay modes, which occurs in the early life of the K^0, provides further information on the 3π isospin states. To their observations, we add two comments.

Firstly, since the amplitude for the $I=0$ final state of $K_1^0 \to 3\pi$ is explicitly of third order in the energy variables, it will not show up in the data at all, unless, for some unsuspected reason, the energy expansions we have discussed do not give a proper estimate of relative orders of magnitude.

Secondly, the $I=2$ contribution of $K_1^0 \to 3\pi$ can be quantitatively correlated with the other decay measurements already examined. The $K^0 \to 3\pi$ amplitude, with time dependence measured in the laboratory frame, has the structure

$$M(K^0 \to 3\pi) = \frac{1}{\sqrt{2}}[M(K_2^0 \to 3\pi)\exp(-\tfrac{1}{2}\lambda_2 t - iw_2 t) + M(K_1^0 \to 3\pi)\exp(-\tfrac{1}{2}\lambda_1 t - iw_2 t)]. \quad (6.23)$$

Then the time-dependent decay distribution expanded to first order in energy is (we write $c_{\mathrm{n}} = \frac{2}{3}c'' - c'$, as suggested by Table V)

$$|M(K^0 \to 3\pi)|^2 = \tfrac{1}{2}|a_{\mathrm{n}}|^2\{(1+2b_{\mathrm{n}}s_3/a_{\mathrm{n}})e^{-\lambda_2 t} + 2|c_{\mathrm{n}}/a_{\mathrm{n}}| \times (s_1 - s_2)\cos(\Delta w t + \varphi)\exp[-\tfrac{1}{2}(\lambda_1+\lambda_2)t]\}, \quad (6.24)$$

where λ_i, w_i are the lifetimes and energies of the decaying K particles in the lab, $\Delta w = w_1 - w_2$ and φ is the phase between c_{n} and a_{n}. Only the real part of $b_{\mathrm{n}}/a_{\mathrm{n}}$ contributes to the distribution, and we omitted, in (6.24), explicit mention of a phase difference in b_{n}, a_{n}.

If the third term in (6.24) can actually be detected, a possibility about which we are not sanguine, then the information gained about c_{n}, which is closely related to the c_{ch} defined earlier, gives added information about a final $I=2$ state.

VII. CONCLUSION

We have offered recipes by which the spins, parities, and isospins of a wide class of three-particle resonances may be identified. Ideally, the list of recipes ought to be exhaustive, so that if an attempt at identification fails in a given experiment, the experimenter may discard his data with the satisfying feeling that his duty has been done. Our treatment is not exhaustive; however, a general framework has been provided within which many of the questions not explicitly considered can be answered by well-defined procedures.

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[16] The data on the neutral slope is taken from Ref. 9, which attributes it to D. Luers.

[17] An analysis with similar objectives was undertaken in Ref. 9. We do not concur with the view therein that from current data, one may reasonably conclude that the final state in $K^+ \to 3\pi$ is pure $T=1$, although this is certainly consistent with the data.

[18] S. B. Treiman and S. Weinberg, Phys. Rev. **116**, 239 (1959).