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# Zeros of Hankel Functions and Poles of Scattering Amplitudes*

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The complex zeros $\nu_n(z)$, $n = 1, 2, \cdots$ of $H_\nu^{(1)}(z)$, $dH_\nu^{(1)}(z)/dz$ and $dH_\nu^{(1)}(z)/dz + iZH_\nu^{(1)}(z)$ are investigated. These zeros determine the poles in the scattering amplitudes resulting from scattering of various kinds of waves by spheres and cylinders. Formulas for $\nu_n(z)$ are obtained for both large and small values of $|z|$ and for large values of $n$. In addition, for $H_\nu^{(1)}(z)$ and $dH_\nu^{(1)}(z)/dz$, numerical solutions are found for real $z$ in the interval $0.01 \leq z \leq 7$ and $n = 1, 2, 3, 4, 5$. The resulting loci of $\nu_n(z)$ in the complex $\nu$ plane are presented. These loci are the trajectories of the so-called Regge poles for scattering by spheres and cylinders.

## 1. INTRODUCTION

IN 1918 Watson[1] discovered that a certain scattering amplitude in electromagnetic theory had poles at the values of $\nu$ for which $H_\nu^{(1)}(z) = 0$. Here $H_\nu^{(1)}(z)$ is the Hankel function of the first kind of order $\nu$ and argument $z$. Similar poles have since been found in other scattering amplitudes at the zeros of other transcendental functions. Recently Regge[2] has examined them in quantum-mechanical potential scattering and this has stimulated many other investigations. Because of the importance of these poles and their trajectories, we have considered some special cases in detail and have obtained asymptotic formulas and numerical results for them.

Mathematically our investigation concerns the roots $\nu_n(z)$, $n = 1, 2, \cdots$ of the following three equations:

$$H_\nu^{(1)}(z) = 0, \tag{1}$$

$$(d/dz)H_\nu^{(1)}(z) = 0, \tag{2}$$

$$(d/dz)H_\nu^{(1)}(z) + iZH_\nu^{(1)}(z) = 0. \tag{3}$$

In (3) $Z$ is either a given constant or a given function of $z$ and $\nu$. Each root $\nu_n(z)$ of each equation is a complex function of the complex argument $z$. We present some old and some new expansions of $\nu_n(z)$ for both large and small values of $|z|$ as well as for large values of $n$ for any $z$. In addition, with the aid of an electronic computer, we have computed the first five roots of (1) and (2) for real $z$ in the range $.01 \leq z \leq 7$ and have plotted graphs of them. [See Figs. (1) and (2).] We have also compared these "exact" numerical values with the expansions for large and small values of $|z|$, thus determining the accuracy and range of validity of these expansions.

Equation (1) determines the poles in the quantum-mechanical scattering by a rigid sphere or cylinder, i.e., by a potential which is infinite within a sphere or cylinder and zero outside it. It also determines the poles in the scattering of an acoustic wave by an acoustically soft sphere or cylinder. In addition, it determines some of the poles in the scattering of an electromagnetic wave by a perfectly conducting sphere or cylinder. Equation (2) determines the poles in the scattering of an acoustic wave by a rigid cylinder and some of the poles in electromagnetic scattering by a perfectly conducting cylinder. Equation (3) determines the poles in acoustic or electromagnetic scattering by a cylinder of surface impedance $Z$. In all cases $z = ka$, where $a$ is the radius of the sphere or cylinder and $k = 2\pi/\lambda$, with $\lambda$ being the incident wavelength.

Because of the importance of the scattering problems just mentioned, some studies have been made of the Eqs. (1)–(3). The most complete study of (1) is that of Magnus and Kotin,[3] which led to the present work. In part, our analysis is similar to theirs. However, we succeeded in obtaining expansions of $\nu_n(z)$ for $|z|$ small which they did not find. These expansions show that Theorems 6.1 and 6.2 of reference 3 are false, and it is then not difficult to locate the flaws in the proofs. Fortunately none of their subsequent results depend upon these theorems. In addition, we have found a number of misprints in their formulas on p. 243 for $\nu_n(z)$ for large $n$. The correct formulas are given below.

## 2. ZEROS OF $H_\nu^{(1)}(z)$

Let us begin by expressing $H_\nu^{(1)}(z)$ in terms of Bessel functions by the formula

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[1] G. N. Watson, Proc. Roy. Soc. (London) **A95**, 83 (1918).

[2] T. Regge, Nuovo Cimento **14**, 951 (1958).

[3] W. Magnus and L. Kotin, Numerische Math. **2**, 228 (1960).

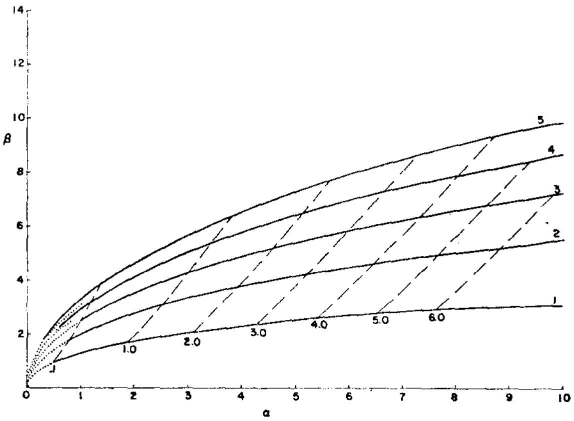


FIG. 1. The zeros $\nu_n(z)$ of $H_\nu^{(1)}(z)$ in the complex $\nu$ plane for $z$ real. Re $\nu_n$ is plotted horizontally and Im $\nu_n$ is plotted vertically for $n = 1, 2, 3, 4, 5$, and $0 \le z \le 7$. The solid lines are the loci of $\nu_n(z)$ for fixed $n$ as functions of $z$. The dashed lines connect values of $\nu_n(z)$ for fixed $z$ and different values of $n$. The zeros are symmetric about $\nu = 0$ so there is a similar set of curves in the third quadrant.

$$i \sin \nu\pi H_\nu^{(1)}(z) = J_{-\nu}(z) - J_\nu(z) e^{-i\nu\pi}. \qquad (4)$$

The power series for $J_\nu(z)$ is

$$J_\nu(z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{z}{2}\right)^{\nu+2m} [m!\, \Gamma(\nu + 1 + m)]^{-1}. \quad (5)$$

Upon using (5) for $J_\nu$ and $J_{-\nu}$ in (4), it becomes for $0 < |z| \ll 1$ or $|\nu| \gg 1 + |z|^2$,

$$i \sin \nu\pi \Gamma(\nu + 1)\left(\frac{z}{2}\right)^\nu H_\nu^{(1)}(z) = \frac{\Gamma(\nu + 1)}{\Gamma(\nu - 1)}\left[1 + O\left(\frac{z^2}{\nu + 1}\right)\right] - \left(\frac{z}{2}\right)^\nu e^{-i\nu\pi}\left[1 + O\left(\frac{z^2}{\nu + 1}\right)\right]. \qquad (6)$$

When $H_\nu^{(1)}(z) = 0$ we transpose the quotient of gamma functions in (6) and take logarithms of the two sides of the resulting equation, obtaining

$$2\nu\left(\log \frac{z}{2} - i\frac{\pi}{2}\right) = -2\pi i n + \log \frac{\Gamma(\nu + 1)}{\Gamma(1 - \nu)} + O\left(\frac{z^2}{\nu + 1}\right). \qquad (7)$$

Here $n$ is an integer.

For $|z| \ll 1$ it is convenient to use the following series for the logarithm of the quotient of gamma functions:

$$\log \frac{\Gamma(1 + \nu)}{\Gamma(1 - \nu)} = -2\gamma\nu - 2 \sum_{m=1}^{\infty} \frac{\zeta(2m + 1)}{2m + 1} \nu^{2m+1}. \quad (8)$$

Here $\gamma$ is Euler's constant and $\zeta$ is the Riemann zeta function. With the aid of (8), (7) can be rewritten as

$$\log \frac{z}{2} = -\frac{i\pi n}{\nu} + \frac{i\pi}{2} - \gamma - \sum_{m=1}^{\infty} \frac{\zeta(2m + 1)}{2m + 1} \nu^{2m} + O\left(\frac{z^2}{\nu}\right). \qquad (9)$$

Upon reverting the series (9) for $n \neq 0$ we obtain $\nu$ as a power series in $[\log (z/2)]^{-1}$. For $n = 0$ there is no root of (9) for which $|\nu| \ll 1$. Denoting the value of $\nu$ by $\nu_n$ and setting $z = re^{i\varphi}$, we may write the result as the following series in $[\log (r/2)]^{-1}$

$$\nu_n = -i\pi n[\log (r/2)]^{-1}\left\{1 + \left[i\left(\frac{\pi}{2} - \varphi\right) - \gamma\right] \times [\log (r/2)]^{-1} + \left[i\left(\frac{\pi}{2} - \varphi\right) - \gamma\right]^2 [\log (r/2)]^{-2} + \left(\left[i\left(\frac{\pi}{2} - \varphi\right) - \gamma\right]^3 - \zeta(3)\pi^2 n^2/3\right)[\log (r/2)]^{-3} + O(z^2/\log (z/2))\right\}, \qquad |z| \ll 1. \qquad (10)$$

This result for $\nu_n$, which appears to be new, shows that all the roots $\nu_n$ tend to zero as $z$ tends to zero. For $z$ real this was shown to be true by Magnus and Kotin.[3] However, their theorems 6.1 and 6.2, which describe the manner in which $\nu_n$ tends to zero, are in disagreement with (10) and are incorrect.

To determine $\nu_n$ for $n$ large and $z$ fixed, we again proceed from (7). We assume that $|\nu| \gg 1$ and use Stirling's formula for the gamma functions in (7),

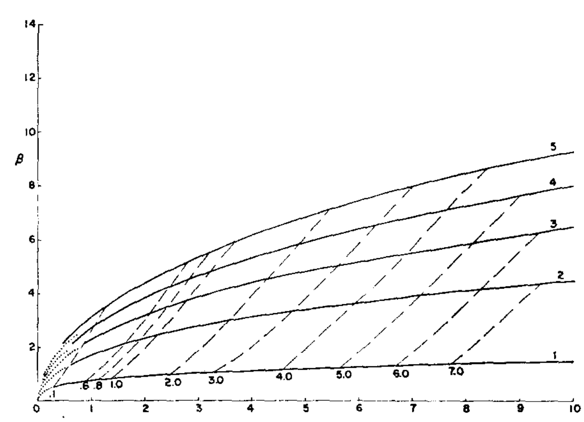


FIG. 2. The zeros $\nu_n(z)$ of $dH_\nu^{(1)}(z)/dz$ in the complex $\nu$ plane for $z$ real. Re $\nu_n$ is plotted horizontally and Im $\nu_n$ is plotted vertically for $n = 1, 2, 3, 4, 5$, and $0 \le z \le 7$. The solid lines are the loci of $\nu_n(z)$ for fixed $n$ as functions of $z$. The dashed lines connect values of $\nu_n(z)$ for fixed $z$ and different values of $n$. The zeros are symmetric about $\nu = 0$ so there is a similar set of curves in the third quadrant.

which yields

$$\log \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} = 2\nu(\log \nu - 1 + i\pi/2) - i\pi/2 + O(\nu^{-1}). \quad (11)$$

Upon using (11) in (7), and assuming that $|\nu| \gg 1 + |z|^2$, we readily find for $n \gg 1$,

$$\operatorname{Re} \nu_n = \pi\left(\frac{\pi}{2} - \varphi\right)(n - \tfrac{1}{4})\left[\log \frac{2\pi(n-\frac{1}{4})}{er}\right]^{-2} \times [1 + O(\log \log n/\log n)],$$

$$\operatorname{Im} \nu_n = \pi(n - \tfrac{1}{4})\left[\log \frac{2\pi(n-\frac{1}{4})}{er}\right]^{-1} \times [1 + O(\log \log n/\log n)], \qquad n \gg 1. \quad (12)$$

This result was obtained by Magnus and Kotin, Theorem 8.1, but their formulas contain a number of misprints. From (12) we see that both Re $\nu_n$ and Im $\nu_n$ become infinite as $n$ becomes infinite, but that Im $\nu_n$ increases more rapidly than does Re $\nu_n$. Consequently, arg $\nu_n$ tends to $\pi/2$ as $n$ increases. This fact has led numerous authors to the false conclusion that $\nu_n$ approaches the imaginary axis of the $\nu$ plane as $n$ increases.

When $|z|$ is large, $|\nu_n|$ is also large. Then for fixed $n$, $\nu_n$ is given by the well known formula obtained by van der Pol and Bremmer[4] with the aid of the Debye expansion for the Bessel function, and refined by Franz[5]:

$$\nu_n = z + 6^{-\frac{1}{3}}e^{i\pi/3}q_n z^{\frac{1}{3}} + \frac{1}{180}6^{\frac{1}{3}}e^{i2\pi/3}q_n^2 z^{-\frac{1}{3}} + O(z^{-1}), \quad |z| \gg n > 0. \quad (13)$$

Here $q_n$ is the $n$th zero of the Airy function $A(q)$,

$$A(q_n) = \int_0^\infty \cos(t^3 - q_n t)\, dt = 0. \quad (14)$$

The first five zeros, as given by Franz,[5] are listed in Table I. For large values of $n$, $q_n$ is given by the asymptotic formula

$$q_n \sim [3\pi(n + \tfrac{3}{4})]^{\frac{2}{3}}6^{\frac{1}{3}}/2, \qquad n \gg 1. \quad (15)$$

Equations (10) and (13) give $\nu_n(z)$ for both large and small values of $|z|$. To obtain $\nu_n(z)$ for intermediate values of $z$ we have solved (1) numerically for $z$ real in the range $0.01 \leq z \leq 7$ and $n = 1, 2, 3, 4, 5$. The resulting values of $\nu_n$ are shown in Fig. 1, which shows the locus of each of the first five roots in the complex $\nu$ plane. As $z$ increases from zero each root moves from the origin upward and to the right. Such loci have recently been called "Regge trajectories" in quantum mechanics.

[4] H. Bremmer, *Terrestrial Radio Waves* (Elsevier Publishing Company, New York, 1949).
[5] W. Franz, Z. Naturforsch. **9a**, 705 (1954).

TABLE I. The first five zeros $q_n$ and $q_n'$ of the Airy function and its derivative, respectively. In terms of them the zeros of $H_\nu^{(1)}(z)$ and $dH_\nu^{(1)}(z)/dz$ can be expressed by (13) and (18).

| $n$ | $q_n$ | $q_n'$ |
|---|---|---|
| 1 | 3.372134 | 1.469354 |
| 2 | 5.895843 | 4.684712 |
| 3 | 7.962025 | 6.951786 |
| 4 | 9.788127 | 8.889027 |
| 5 | 11.457423 | 10.632519 |

We have also compared the values of $\nu_n$ given by (10) and (13) with the numerical results. For $z = 0.01$, (10) yields Re $\nu_1 = 0.205$, Im $\nu_1 = 0.613$, while the numerical solution is Re $\nu_1 = 0.184$, Im $\nu_1 = 0.592$. For larger values of $z$ and $n$ the disagreement is greater. Thus, we conclude that (10) is accurate only for $|z| < 0.01$. On the other hand, for $z = 1$, (13) yields Re $\nu_1 = 1.871$, Im $\nu_1 = 1.706$, while the numerical solution is Re $\nu_1 = 1.880$, Im $\nu_1 = 1.708$. This agreement is very good, and becomes better as $|z|$ increases, but worse as $n$ increases. However, even for $n = 5$ the error in Re $\nu_5$ is only 4% and that in Im $\nu_5$ is only 1% at $z = 2$. At $z = 7$, (13) yields Re $\nu_1 = 8.745$, Im $\nu_1 = 3.126$, while the numerical solution is Re $\nu_1 = 8.746$, Im $\nu_1 = 3.127$.

We have restricted $n$ to positive values in (12) and (13) and have given only the roots $\nu_n(z)$ with Re $\nu_n \geq 0$ in Fig. 1 because the roots are symmetric about the origin. This follows from the relation $H_{-\nu}^{(1)}(z) = e^{i\nu\pi}H_\nu^{(1)}(z)$.

### 3. ZEROS OF $dH_\nu^{(1)}(z)/dz$

The zeros of $dH_\nu^{(1)}(z)/dz$ can be found by exactly the same methods as were used in the preceding section. Therefore, we shall give only the results. Since the zeros are symmetric about the origin, we shall again give some formulas only for positive $n$, which corresponds to zeros in the half plane Re $\nu \geq 0$.

When $|z|$ is small we find

$$\nu_n = -i\pi(n - \tfrac{1}{2})[\log(r/2)]^{-1}\left\{1 + \left[i\left(\frac{\pi}{2} - \varphi\right) - \gamma\right] \times [\log(r/2)]^{-1} + \left[i\left(\frac{\pi}{2} - \varphi\right) - \gamma\right]^2 [\log(r/2)]^{-2} + \left(\left[i\left(\frac{\pi}{2} - \varphi\right) - \gamma\right]^3 - \zeta(3)\pi^2 n^2/3\right)[\log(r/2)]^{-3} + O[|z|^2/\log(z/2)]\right\}, \qquad |z| \ll 1. \quad (16)$$

For $|\nu| \gg 1 + |z|^2$ we obtain

$$\operatorname{Re}\nu_n = \pi\left(\frac{\pi}{2} - \varphi\right)(n - \tfrac{3}{4})\left[\log\frac{2\pi(n-\frac{3}{4})}{er}\right]^{-2} \times \left[1 + O\left(\frac{\log\log n}{\log n}\right)\right], \quad (n \gg 1) \tag{17}$$

$$\operatorname{Im}\nu_n = \pi(n - \tfrac{3}{4})\left[\log\frac{2\pi(n-\frac{3}{4})}{er}\right]^{-1} \times \left[1 + O\left(\frac{\log\log n}{\log n}\right)\right].$$

Both (16) and (17) are apparently new. The former shows that all $\nu_n$ tend to zero as $z$ tends to zero. The latter shows that both Re $\nu_n$ and Im $\nu_n$ become infinite as $n$ does and that arg $\nu_n$ tends to $\pi/2$.

When $|z|$ is large and $n$ is fixed, $\nu_n$ is given by the formula[5]

$$\nu_n = z + 6^{-\frac{1}{3}}e^{i\pi/3}q_n'z^{\frac{1}{3}} + 6^{\frac{1}{3}}e^{i2\pi/3} \times \left[\frac{(q_n')^2}{180} + \frac{1}{10q_n'}\right]z^{-\frac{1}{3}} + O(z^{-1}), \quad |z| \gg n > 0. \tag{18}$$

Here $q_n'$ is the $n$th zero of $A'(q) = 0$ where $A(q)$ is the Airy function defined in (14). The first five zeros, as given by Franz, are listed in Table I. For $n$ large, $q_n'$ is given by

$$q_n' \sim [3\pi(n + \tfrac{1}{4})]^{\frac{2}{3}}6^{\frac{1}{3}}/2, \qquad n \gg 1. \tag{19}$$

In Fig. 2 are shown the loci of values of $\nu_n(z)$ obtained by solving (2) numerically for $n = 1, 2, 3, 4, 5$, and $.01 \le z \le 7$. Each root moves from the origin, upward and to the right, as $z$ increases from zero. Comparison of the numerical solutions with the results given by (16) and (19) shows about the same agreement as in the preceding case.

## 4. ZEROS OF $dH_\nu^{(1)}(z)/dz + iZH_\nu^{(1)}(z) = 0$

To solve (3) we use (6) to obtain for $|z| \ll 1$ or $|\nu| \gg 1 + |z|^2$,

$$2i\sin\nu\pi\,\Gamma(\nu)(z/2)^{\nu+1}\left[\frac{dH_\nu^{(1)}(z)}{dz} + iZH_\nu^{(1)}(z)\right] = -\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}\left(1 - i\frac{Zz}{\nu}\right)\left[1 + O\left(\frac{z^2}{\nu}\right)\right] - \left(\frac{z}{2}\right)^{2\nu}e^{-i\nu\pi}\left(1 + i\frac{Zz}{\nu}\right)\left[1 + O\left(\frac{z^2}{\nu}\right)\right]. \tag{20}$$

Upon equating to zero the right side of (20) and taking logarithms of the resulting equation, we find

$$2\nu[\log(z/2) - i\pi/2] = -i2\pi(n - \tfrac{1}{2}) + \log\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} + O\left(\frac{Zz}{\nu}\right) + O\left(\frac{z^2}{\nu}\right). \tag{21}$$

Let us first suppose that the impedance $Z$ is a finite constant, independent of $z$ and $\nu$. Then it follows from (21) that the zeros of (3) are asymptotically the same as those of (1) in the two cases $|z| \ll 1$ and $|\nu| \gg 1 + |z|^2$. Thus in these two cases the zeros $\nu_n$ of (3) are given by (16) if $|z| \ll 1$ and by (17) if $|\nu| \gg 1 + |z|^2$, with an additional error term $O(Zz/\nu)$. The same result (16) applies if $Z$ is a function of $z$ and $\nu$ such that $Zz/\nu$ tends to zero as $z$ tends to zero with $\nu$ given by (16). Similarly (17) applies if $Zz/\nu$ tends to zero for fixed $z$ as $\nu$ becomes infinite through the sequence (17).

When $|z|$ is large and $n$ is fixed, $\nu_n$ is given by the following formula, obtained by Levy and Keller[6]:

$$\nu_n = z + 6^{-\frac{1}{3}}e^{i\pi/3}q_n(Zz^{\frac{1}{3}})z^{\frac{1}{3}} + O(z^{-\frac{1}{3}}), \qquad |z| \gg n > 0. \tag{22}$$

Here $q_n(Zz^{\frac{1}{3}})$ is the $n$th root of the equation

$$A'(q) = A(q)e^{5\pi i/6}6^{-\frac{1}{3}}Zz^{\frac{1}{3}}. \tag{23}$$

If $|Zz^{\frac{1}{3}}|$ is large, $q_n$ is given by

$$q_n(Zz^{\frac{1}{3}}) = q_n(\infty) + e^{-5\pi i/6}6^{\frac{1}{3}}(Zz^{\frac{1}{3}})^{-1} + O(|Zz^{\frac{1}{3}}|^{-2}). \tag{24}$$

The number $q_n(\infty)$ is the $n$th root of (14), to which (23) reduces when $Zz^{\frac{1}{3}}$ becomes infinite. If $|Zz^{\frac{1}{3}}|$ is small, $q_n$ is given by

$$q_n(Zz^{\frac{1}{3}}) = q_n(0) - e^{5\pi i/6}[3Z/q_n(0)](z/6)^{\frac{1}{3}} + O(|Zz^{\frac{1}{3}}|^2). \tag{25}$$

Here $q_n(0)$ is the $n$th root of $A'(q) = 0$, to which (23) reduces when $Zz^{\frac{1}{3}} = 0$.

[6] B. R. Levy and J. B. Keller, Commun. Pure Appl. Math. **12**, 159 (1959).