DETERMINATION OF EXCHANGE FORCES IN THE $\pi\pi$ P WAVE

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A new representation is derived for the $\pi\pi$ P wave. This representation enables us to compute the second and all higher negative moments of the discontinuity across the left cut, strictly in terms of physical-region absorptive parts. We construct N/D solutions for the P wave, using a model for the left cut which incorporates generous estimates for the second, third, and fourth negative moments. We find that despite generosity of our estimates, the left cut is too weak to generate a resonance when elastic unitarity is assumed, and also when the possibility of substantial inelasticity is taken roughly into account. We conclude that the ρ resonance is due either to presently unknown details of the inelasticity, or to a bound state in some other channel.

It has long been tempting to suppose that the ρ meson is a dynamically bound state of two π mesons, in the sense that the forces corresponding to the left cut of the $\pi\pi$ P wave are sufficient to generate the ρ resonance. Indeed, numerous authors have proposed models for the left cut which resulted in the generation of ρ resonances [1]. However, the validity of such models has always been open to question - partly because of the difficulty which has been experienced in obtaining satisfactory values for the mass and width, and especially because no method has heretofore been known for obtaining reliable information about the discontinuity to the left of $|\boldsymbol{q}_{\mathrm{c.m}}|^2 = -9m_\pi^2$.

In this letter, we derive a new representation for the $\pi\pi$ P wave. We compare this new representation with the standard dispersion relation, and thereby obtain a powerful new condition on the left cut. This new condition enables us to compute the second and all higher negative moments of the discontinuity across the left cut, strictly in terms of physical-region absorptive parts.

Using a simple model for the left cut which incorporates generous estimates for the second, third, and fourth negative moments, we construct N/D solutions for the $\pi\pi$ P wave. We find that the left cut is too weak to generate a resonance when elastic unitarity is assumed, and also when the possibility of substantial inelasticity is taken roughly into account. Our work therefore indicates that the ρ resonance is due either to presently unknown details of the inelasticity, or to a bound state in some other channel. The analysis proceeds as follows.

Let us denote the $\pi\pi$ elastic amplitudes by $A^I(\nu, \cos\theta)$, where I denotes s-channel isospin, and $\nu \equiv |\boldsymbol{q}_{\mathrm{c.m}}|^2$. We shall use units wherein $m_\pi = \hbar = c = 1$, and our normalization is such that the partial waves $A^{(l)I}$ satisfy

$$A^{(l)I}(\nu) = R_l^I(\nu)\sqrt{\frac{\nu+1}{\nu}}\exp(\mathrm{i}\delta_l^I)\sin\delta_l^I$$

for $\nu > 0$, where R_l^I denotes the ratio of elastic to total parital-wave cross sections, and the δ_l^I are real. Bose symmetry implies that if I is odd (even), then $A^{(l)I}$ vanishes unless l is also odd (even).

To facilitate the remainder of our discussion, we shall denote the combination of amplitudes with $I = 1$ in the t-channel by

$$T^1(\nu, \cos\theta) \equiv \sum_{I=0}^{2} \beta_{1I} A^I(\nu, \cos\theta) ,$$

where $\beta_{1I} = \frac{1}{3}$, $\frac{1}{2}$ and $-\frac{5}{6}$ for $I = 0$, 1 and 2, respectively. We shall also denote the sum over $A^{(l)1}$ with $l \geqslant 3$ by

$$\tilde{A}^1(\nu) \equiv \sum_{l=3}^{\infty} (2l+1)A^{(l)1}(\nu) .$$

To obtain a representation wherein $A^{(1)1}$ can be computed from physical-region absorptive parts, we write the P wave as

$$A^{(1)1}(\nu) = \tfrac{1}{3}\left[A^1(\nu,1) - \tilde{A}^1(\nu)\right] . \tag{1}$$

The forward amplitude $A^1(\nu,1)$ satisfies the representation [2]

$$A^1(\nu,1) = \frac{\nu}{\pi}\int_0^\infty \frac{d\nu'}{(\nu'+1)(\nu'+\nu+1)} \times$$

$$\times\left[\mathrm{Im}\,T^1(\nu',1) + \frac{(\nu+1)(2\nu'+1)}{\nu'(\nu'-\nu)}\mathrm{Im}A^1(\nu',1)\right]. \quad (2)$$

Furthermore, the $A^{(l)1}$ with $l \geqslant 3$ can be obtained from the Froissart-Gribov representation*:

$$A^{(l)1}(\nu) = \frac{2[1-(-1)^l]}{\pi\nu}\int_0^\infty d\nu'\, Q_l\left(1+2\frac{\nu'+1}{\nu}\right)$$

$$\times\,\mathrm{Im}\,T^1\left(\nu',\ 1+2\frac{\nu+1}{\nu'}\right), \quad (3)$$

which is valid for Re $\nu < 0$. Using the relation

$$Q_l(Z) = -\tfrac{1}{2}\int_{-1}^{1} dz'\,\frac{P_l(z')}{z'-z}$$

together with

$$\sum_{\text{odd } l=3}^{\infty}(2l+1)\,P_l(z) = \delta(z-1) - \delta(z+1) - 3P_1(z)\ ,$$

we readily sum the $A^{(l)1}$ given by eq. (3) to obtain

$$\tilde{A}^1(\nu) = \frac{\nu}{\pi}\int_0^\infty d\nu'\left[\frac{1}{(\nu'+1)(\nu'+\nu+1)} - \right.$$

$$\left. -\frac{12}{\nu^2}\,Q_1\left(1+2\frac{\nu'+1}{\nu}\right)\right]\mathrm{Im}\,T\left(\nu',\ 1+2\frac{\nu+1}{\nu'}\right)\ . \quad (4)$$

The absorptive parts on the right side of eq. (2) are to be evaluated in the physical region, and the absorptive part on the right side of eq. (4) can be obtained from a convergent Legendre series, provided that $-9 < \nu < 0$**. Therefore, eqs. (1), (2) and (4) enable us to compute $A^{(1)1}$ for $-9 < \nu < 0$, strictly in terms of physical-region absorptive parts.

* The Froissart-Gribov representation is also valid for the P wave itself. However, there is a practical advantage to writing $A^{(1)1}$ in the form of eq.(1). Specifically, $A^{(1)1}$ can be obtained near threshold from total cross sections, simply by using eqs. (1) and (2) together with the optical theorem, while noting that $\tilde{A}^1$ vanishes like ν^3 near threshold.

** We assume that the domain of convergence is determined by the boundaries of the double spectral functions. It follows that the Legendre series for $\mathrm{Im}A^I(\nu',\ 1+2(\nu+1)/\nu')$ converges for all $\nu' > 0$ if $-9 < \nu < 0$, but diverges for $\nu > 2$ if $\nu < -9$ [3].

The standard dispersion relation for $A^{(1)1}$ is

$$A^{(1)1}(\nu) = \frac{\nu}{\pi}\left[\int_{-8}^{-1}d\nu' + \int_0^\infty d\nu'\right]\frac{\mathrm{Im}A^{(1)1}(\nu')}{\nu'(\nu'-\nu)}\ . \quad (5)$$

For $-9 < \nu < -1$, analyticity and crossing symmetry imply [5]

$$\mathrm{Im}A^{(1)1}(\nu) = \frac{2}{\nu}\int_0^{-\nu-1} d\nu'\, P_1\left(1+2\frac{\nu'+1}{\nu}\right)\times$$

$$\times\sum_{I=0}^{2}\beta_{1I}\sum_{l=0}^{\infty}(2l+1)\mathrm{Im}A^{(l)I}(\nu')P_l\left(1+2\frac{\nu+1}{\nu'}\right)\ . \quad (6)$$

Unfortunately, the Legendre series on the right side of eq. (6) diverges over part of the range of integration when $\nu < -9$**. It is this divergence which has heretofore prevented us from obtaining any reliable information about $\mathrm{Im}A^{(1)1}$ for $\nu < -9$.

To obtain further information about $\mathrm{Im}A^{(1)1}$ for negative ν, we note that eq. (5) implies

$$\frac{\nu}{\pi}P\int_{-\infty}^{-1}d\nu'\,\frac{\mathrm{Im}A^{(1)1}(\nu')}{\nu'(\nu'-\nu)} =$$

$$= \mathrm{Re}A^{(1)1}(\nu) - \frac{\nu}{\pi}P\int_0^\infty d\nu'\,\frac{\mathrm{Im}A^{(1)1}(\nu')}{\nu'(\nu'-\nu)}\ . \quad (7)$$

If eqs. (1), (2) and (4) are used to evaluate $\mathrm{Re}A^{(1)1}$ on the right side of eq. (7), then a great deal can be inferred about $\mathrm{Im}A^{(1)1}$ for $\nu < -1$. For example, the second and all higher nagtive moments of $\mathrm{Im}A^{(1)1}$ can be obtained by equating the derivatives of the left and right sides of eq. (7) at threshold. The first, second, and third derivatives at threshold are given by

$$\frac{1}{\pi}\int_{-\infty}^{-1}d\nu\,\frac{\mathrm{Im}A^{(1)1}(\nu)}{\nu^2} = \frac{1}{3\pi}\int_0^\infty\frac{d\nu}{(\nu+1)^2}\times$$

$$\times\,\mathrm{Im}\left[T^1(\nu,1) - 3A^{(1)1}(\nu) + \frac{2\nu+1}{\nu^2}\tilde{A}^1(\nu)\right], \quad (8a)$$

$$\frac{2}{\pi}\int_{-\infty}^{-1}d\nu\,\frac{\mathrm{Im}A^{(1)1}(\nu)}{\nu^2} = -\frac{2}{3\pi}\int_0^\infty\frac{d\nu}{(\nu+1)^3}\times$$

$$\mathrm{Im}\left[T^1(\nu,1) - 3A^{(1)1}(\nu) - \frac{\nu^3+(\nu+1)^3}{\nu^3}\tilde{A}^1(\nu)\right], \quad (8b)$$

$$\frac{6}{\pi}\int_{-\infty}^{-1} d\nu \frac{\mathrm{Im}A^{(1)1}(\nu)}{\nu^4}$$

$$= \frac{2}{\pi}\int_0^\infty \frac{d\nu}{(\nu+1)^4} \mathrm{Im}\Big[T^1(\nu,1) - \frac{1}{10}T^1(\nu,1+\frac{2}{\nu}) - 3A^{(1)1}(\nu) + \frac{(2\nu+1)(2\nu^2+2\nu+1)}{\nu^4}\tilde{A}^1(\nu)\Big], \qquad (8c)$$

respectively. Since $\mathrm{Im}A^{(1)1}$ is given by eq. (6) for $-9 < \nu < -1$, eqs. (8a-c) can be used to deduce the corresponding moments of $\mathrm{Im}A^{(1)1}$ over the interval $-\infty < \nu < -9$, again in terms of physical-region absorptive parts.

Having made the preceding general remarks, let us now evaluate the integrals on the right sides of eqs. (8a-c).

We shall assume that below 1.5 GeV, only the S, P and D waves contribute. For the S waves, we shall use the solutions published recently by the present author with $a_0 = 0.25$, $a_2 = -0.046$, $m_\epsilon = 960$ MeV, $\Gamma_\epsilon = 550$ MeV [5]. These solutions are in good agreement with analyses of $\pi N \to \pi\pi N$ data, and the resulting values for the ϵ contributions to the right sides of eqs. (8a-c) are unlikely to be appreciably *smaller* than the correct physical values. As mentioned before, one of the conclusions of our work will be that the left cut of $A^{(1)1}$ appears to be too weak to generate the ρ, and it would only strengthen this conclusion if our values for the ϵ contributions should turn out to be somewhat larger than the physical values ***

We shall assume that the P wave and $I = 0$ D wave are dominated below 1.5 GeV by the ρ and f_0 resonances, respectively. We assume that $m_\rho = 765$ MeV, $\Gamma_\rho = 125$ MeV, $m_f = 1260$ MeV, and $\Gamma_f = 125$ MeV, and we represent the absorptive parts by Breit-Wigner formulas with correct threshold behavior [6]. The $I = 2$ D wave is quite small over this region, and we shall neglect it.

Next we consider energies above 1.5 GeV. We begin by noting that the integrals on the right sides of eqs. (2), (4), (8a) and (8c) are dominated in the asymptotic region by $\mathrm{Im}T^1$, and that the integrals on the right sides of eqs. (8b) and (8c) converge so rapidly that contributions from above 1.5 GeV are only of secondary importance.

Olsson has combined πN charge-exchange data with Sakurai's universality to infer that $\mathrm{Im}\,T^1(\nu,1) \approx 0.29\sqrt{\nu}$ for large ν [2] †. If one uses this formula above 1.5 GeV, the resulting contribution to the right side of eq. (8a) is 0.0019.

Table 1
Negative moments of $\mathrm{Im}A^{(1)1}$ for $\nu < -1$. K_a, K_b and K_c denote contributions to the right sides of eqs. (8a), (8b) and (8c), respectively.

	$10^2 \times K_a$	$10^3 \times K_b$	$10^3 \times K_c$
S^0	1.19	-6.78	7.34
S^2	-0.29	0.92	-0.72
ρ	-1.02	2.73	-1.42
f_0	0.22	-0.21	0.03
$E > 1.5$ GeV	0.86	0.19	0.01
Total	0.96	-3.15	5.24

Morgan and Shaw have used forward and first-derivative dispersion relations together with phenomenological considerations to construct a set of solitions for $\pi\pi$ amplitudes [7]. For the region above 1.5 GeV, the effective value of $\mathrm{Im}\,T^1(\nu,1)$ is determined by Morgan and Shaw from self-consistency conditions. For the various solutions considered in their work, $\mathrm{Im}T^1$ is such that it would contribute between 0.0066 and 0.0097 to the right side of eq. (8a) from above 1.5 GeV.

As our third and last model for T^1, we consider the single-term Veneziano formula [8]. We find that in this model, $\mathrm{Im}T^1$ contributes 0.0097 to the right side of eq. (8a) from above 1.5 GeV.

In the context of our claim that the left cut of $A^{(1)1}$ is weak, a conservative estimate for $\mathrm{Im}T^1$ corresponds to a generous estimate ***. Therefore, we shall use the Veneziano model for $\mathrm{Im}\,T^1$ above 1.5 GeV.

As we metioned earlier, the high-energy contributions of $\mathrm{Im}A^{(1)1}$ and $\mathrm{Im}\tilde{A}^1$ to the right sides of eqs. (2), (4) and (8a-c) are only of secondary importance. For the sake of definitess, we shall use the Veneziano model for $\mathrm{Im}A^{(1)1}$ and $\mathrm{Im}\tilde{A}^1$ above 1.5 GeV.

In table 1, we present indiviual contributions and net values for the right sides of eqs. (8a-c).

We remark that the P-wave scattering length a_1 is given by

$$a_1 = \frac{1}{\pi}\left[\int_{-\infty}^{-1} d\nu + \int_0^\infty d\nu\right] \frac{\mathrm{Im}A^{(1)1}(\nu)}{\nu^2}. \qquad (9)$$

*** Postive contributions to the right side of eq. (8a) correspond to attractive forces, and we claim that the net attractive force appears to be too weak to generate a resonance.

† This value for $\mathrm{Im}T^1$ is only about 20% as large as the Veneziano value, and the discrepancy seems worth investigating.

The contribution to a_1 from the left cut of $A^{(1)1}$ is precisely the same as the quantity which appears on the left side of eq. (8a), and is given by table 1 as 0.0096. If we use the same values for the ρ parameters as before, then a_1 receives 0.027 from the ρ contribution to the right cut of $A^{(1)1}$. The integral over the right cut is rapidly convergent and should be dominated by the ρ contribution. Assuming ρ dominance of the right cut, we obtain a met value of $a_1 = 0.037$, in good agreement with the current-algebra prediction of Weinberg [9].

To see whether the left cut of $A^{(1)1}$ is capable of generating the ρ resonance, we next contruct P waves driven by left cuts which are consistent with the preceding equations. We assume that $A^{(1)1} = N/D$, where

$$N(\nu) = \frac{\nu}{\pi}\int_{-\infty}^{-1} d\nu' \frac{D(\nu')\,\mathrm{Im}A^{(1)1}(\nu')}{\nu'(\nu'-\nu)}\,, \tag{10a}$$

$$D(\nu) = 1 - \frac{\nu}{\pi}\int_0^{\infty} d\nu' \frac{N(\nu')\,[\nu'/(\nu'+1)]^{\frac{1}{2}}}{\nu'(\nu'-\nu)\,R_1^1(\nu')}\,. \tag{10b}$$

The presence of R_1^1 in the integrand of eq. (10b) enables us to include the effects of inelasticity by the method of Froissart [10].

For $\nu < -1$, we approximate $\mathrm{Im}A^{(1)1}$ by

$$\mathrm{Im}A^{(1)1}(\nu) = \xi(\nu)\,\Theta(\nu-\Lambda) + \eta\delta(\nu-\bar{\nu})\,, \tag{11}$$

where $\xi(\nu)$ denotes the net contribution of the S waves and P wave to the right side of eq. (6), and Θ denotes the Heaviside function. The cutoff parameter Λ must be finite in order for the N/D equations to posess solutions [4]. This term involving the δ-function on the right side of eq. (11) is simply intended to approximate the difference between $\mathrm{im}A^{(1)1}$ and $\xi(\nu)\,\Theta(\nu-\Lambda)$.

For the input which determines $\xi(\nu)$, we use the S waves and P wave described earlier, in connection with table 1 ††. The resulting $\xi(\nu)$ is displayed in fig. 1. Since the f_0 resonance occurs at $\nu = 20$, D-wave contributions to the right side of eq. (6) would be negligible for $-17 < \nu < -1$ †††. Thus $\xi(\nu)$ is an excellent approximation to $\mathrm{Im}A^{(1)1}$ for $-9 < \nu < -1$, and may be a good approximation for $-17 < \nu < -9$.

Upon using eqs. (8a) and (8b) with talbe 1 to determine η and $\bar{\nu}$ as functions of Λ, we find that $\bar{\nu} < -9$ if and only if $-73 < \Lambda < -10$. We also find that eq. (8c) is satisfied exactly

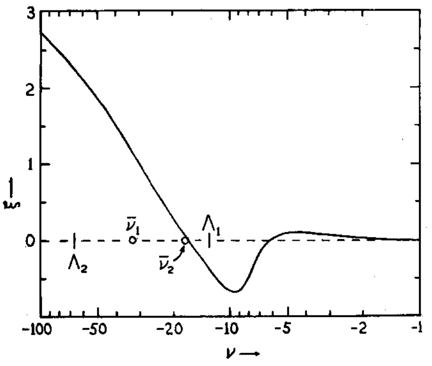

Fig. 1. Net contribution of S waves and P wave to the right side of eq. (6) for $\mathrm{Im}A^{(1)1}$. Also indicated are the two sets of values used for the cutoff parameter Λ and the pole position $\tilde{\nu}$.

if $\Lambda = -12$ or -70, and is satisfied withtin 0.5% if $-73 < \Lambda < -11$.

To obtain a test of the distant left cut which is more sensitive than eq. (8c), we evaluate the right side of eq. (7) at $\nu = -9$, thereby obtaining

$$P\int_{-\infty}^{-1} d\nu \frac{\mathrm{Im}A^{(1)1}(\nu)}{\nu(\nu+q)} = 0.015 \tag{12}$$

We find that eq. (12) is satisfied exactly if $\Lambda = -13$ or -67, but is violated by at least 10% if Λ deviates from either of the aforementioned value by as much as 10%. With $\Lambda = -13$, $\bar{\nu} = -33$ and $\eta = 50$; with $\Lambda = -67$, $\bar{\nu} = -17$ and $\eta = 2.0$.

We have constructed solutions to the N/D equations using both of the aforementioned sets of values for $\Lambda, \bar{\nu}$ and η; as might be expected, the results are insensitive to which set we use. We have assumed that $R_1^1 = 1$ to all energies, and alternatively, that $R_1^1 = 0.5$ above 1 GeV. In neither case does δ_1^1 show any tendency to resonate, at any energy. With $R_1^1 = 1$, δ_1^1 remains less than 7° below 1 GeV, and less than 17° below 2 GeV. With $R_1^1 = 0.5$ above 1 GeV, δ_1^1 remains less than 12° below 1 GeV, and less than

†† It is essential that one use the same S wave and P wave in determining $\xi(\nu)$ as are used on the right sides of eq. (8a-c); otherwise, one may make serious errors when estimating the strength of the distant left cut.

††† If one assumes $m_f = 1260$ MeV, $\Gamma_f = 125$ MeV, and represents the f_0 contribution to $\mathrm{Im}A^{(2)0}$ by a Breit-Wigner formula with correct threshold behavior ***, then the f_0 contribution to the right side of eq. (6) is smaller than 0.001 for $-9 < \nu < -1$, is smaller than 0.005 for $-13 < \nu < -9$, and is smaller than 0.05 for $-17 < \nu < -13$.

36^o below 2 GeV ‡.

From the preceding analysis, we conclude that the left cut of $A^{(1)1}$ is too weak to generate a resonance when elastic unitarity is assumed, and also when the possibility of substantial inelasticity is taken roughly into account. An elementary ρ seems implausible, so we conclude that the ρ resonance is due either to presently unknown details of the inelasticity, or to a bound state in some channel *other* than the $\pi\pi$P wave.

We remark that Collins, Johnson and Squires [11] have given a rather general argument against the validity of any bootstrap which does not include very heavy particles. Their argument is based on the observation that the Regge recurrences of the ρ and A_2 mesosns are too narrow for linearly rising trajectories to be consistent with once-substracted dispersion relations for the trajectories, unless the known mesons are bound states of very heavy particles ‡‡. It is somewhat difficult to compare the analysis of ref. [17] with that of the present paper, but the results have an abvious similarity.

As for the possibility that the ρ is a bound state in some other channel, Ball, Scotti and Wong [12] have argued that it is a bound state of the $N\bar{N}$ system. This conjecture receives some support form work by Mandelstam [13]. Another interesting possibility is that all mesons may be bound states of physical quarks [14].

It is a pleasure to acknowledge informative discussions with Dr. Graham Shaw.

‡ To understand the fact that so many other authors have obtained ρ resonances from N/D equations, it is helpful to keep two points in mind. First, very few authors have claimed that a ρ resonance is *implied* by what is *known* about the left cut of $A^{(1)1}$. Instead, most authors have made the much weaker claim that it is *possible* to obtain a ρ without violating any of the constrains which *they* imposed. Second, the constrains which other authors have imposed on the left cut of $A^{(1)1}$ are much weaker than the constraints implied by eqs. (8a-c) and (12). It is also noteworthy that despite the considerable freedom which has been exercised by other authors in choosing left cuts for $A^{(1)1}$, very few authors have been able to obtain a ρ with $\Gamma_\rho < 250$ MeV. An interesting example of the preceding points is the $\pi\pi$ model of Kang and Lee [1]. They represent the left cuts of the S waves and P wave by a few poles, and determine the pole parameters by requiring that the resulting solutions to N/D equations satisfy the threshold conditions of Weinberg, the crossing conditions of Roskies, and the inequalities of Martin. They also *require* that ρ and ϵ resonances be generated. To describe further the left cut of $A^{(1)1}$ used by Kang and Lee [1], let us denote the contributions to a_1 from the integrals over left and right cuts in eq. (9) by a^L and a^R, respectively, and the contributions to $A^{(1)1}$ from left and right cuts in eq. (5) by A^L and A^R, respectively. Kang and Lee [1] impose the Weinberg value $a_1 = 0.038$, but their ρ is so broad ($\Gamma_\rho > 300$ MeV) that $a^R = 0.086$. Thus $a^L = -0.048$ in their model, whereas eq. (8a) and table 1 indicate that a^L should equal about 0.01. The value of Kang and Lee [1] for A^L is negative between threshold and 850 MeV, but becomes positive and satisfies $A^L > 0.55$ above 1.4 GeV, $A^L > 1.1$ above 2 GeV, and $A^L > 2.0$ above 5 GeV. Since unitarity implies that $\mathrm{Re}A^{(1)1} < \frac{1}{2}(1+1/\nu)^{\frac{1}{2}}$, the value of Kang and Lee [1] for A^R must be such that $\mathrm{Re}A^R < 0$ above 1.4 GeV, $\mathrm{Re}A^R < -0.6$ above 2 GeV, and $\mathrm{Re}A^R < -1.5$ above 5 GeV. This behavior of $\mathrm{Re}A^R$ requires a resonance below 1.4 GeV, and the details of Kang and Lee [1] are such that a broad ρ occurs at 771 MeV. However, our present model for the left cut is such that A^L rises monotonically from zero at threshold to only 0.2 at 2 GeV and 0.4 at 5 GeV, and that is why our solutions for $A^{(1)1}$ do not resonate. (For a detailed critique of Kang and Lee, see Tryon [1].)

‡‡ The criterion for "very heavy" is that the sum of the rest masses of the constituents of the mesons on the ρ and A_2 trajectories must be comparable to or greater than the masses of the heaviest known mesons on these trajectories. Assuming that the T(2200) and U(2380) measons are recurrences of the ρ and A_2, respectively, we see that the $N\bar{N}$ system marginally satisfies this criterion. Since the mass (indeed, the existence) of a physical quark has not yet been established, it is obvious that a $q\bar{q}$ pair could also satisfy this criterion.

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