Dispersive Approximations for $\pi\pi$ Partial Waves with Essential Singularities*

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We show that despite the apparent essential singularities in Veneziano $\pi\pi$ partial waves with $I=0$ and $I=2$, these partial waves can be approximated to any desired accuracy over a substantial region about threshold by functions which do satisfy dispersion relations of the usual form, although the number of subtractions required increases without limit as the desired accuracy tends toward exactness. Two subtractions are shown to be sufficient to approximate the S waves within 10% up to $E_{\text{c.m.}} = 700$ MeV. A dispersive method is proposed for unitarizing the Veneziano amplitudes up to 1 GeV or more.

Many of the theoretical analyses of $\pi\pi$ elastic scattering which have been carried out over the last decade have been based on dispersion relations for the partial-wave amplitudes. However, it has recently been strongly argued that Veneziano $\pi\pi$ partial waves with $I=0$ and 2 contain essential singularities at infinity,[1] and thus do not satisfy dispersion relations with any finite number of subtractions. In this work we shall demonstrate that despite the apparent essential singularities for $I=0$ and 2, Veneziano $\pi\pi$ partial waves can be approximated to any desired accuracy over a substantial region about threshold by functions which *do* satisfy dispersion relations, although the number of subtractions required to achieve a given accuracy increases without limit for $I=0$ and 2 as the desired accuracy tends toward exactness. We will also show that the Veneziano S waves can be approximated within 10% below 700 MeV by functions which satisfy twice-subtracted dispersion relations, so that certain calculations in the literature[2] are viable up to this energy even if physical S waves contain essential singularities like those in the Veneziano S waves. Finally, we will propose a dispersive method for unitarizing Veneziano $\pi\pi$ amplitudes which should be reliable up to 1 GeV or more.

Consider the following Veneziano representation for $\pi\pi$ elastic scattering amplitudes[3]:

$$A^0 = \tfrac{1}{2}F(t,u) - \tfrac{3}{2}[F(s,t)+F(s,u)], \tag{1a}$$

$$A^1 = F(s,u) - F(s,t), \tag{1b}$$

$$A^2 = -F(t,u), \tag{1c}$$

where the superscript on A^I denotes s-channel isospin. We shall restrict our discussion to the leading term of the Veneziano series, so that

$$F(x,y) = \beta\frac{\Gamma(1-\alpha(x))\Gamma(1-\alpha(y))}{\Gamma(1-\alpha(x)-\alpha(y))}, \tag{2}$$

where β is a real constant, and $\alpha(\xi)\equiv a+b\xi$, where a and b are real constants. We shall use units wherein $m_\pi = \hbar = c = 1$ (except where MeV is explicitly stated), and we shall sometimes use the variable $\nu = |\vec{q}_{\text{c.m.}}|^2 = \frac{1}{4}(s-4)$. As a final remark on notation, we shall denote s-channel partial-wave projections of the Veneziano amplitudes (1a)–(1c) by $V^{(l)I}$.

Let us consider the series representation[4]

$$F(x,y) = \beta\sum_{K=1}^{\infty}\frac{(-1)^K T_K(1-\alpha(x)-\alpha(y))}{\Gamma(K)} \times\left[\frac{1}{\alpha(x)-K}+\frac{1}{\alpha(y)-K}\right], \tag{3}$$

where $T_K(\xi)$ is the Kth order Pochhammer polynomial:

$$T_K(\xi) = \xi(\xi+1)\cdots[\xi+(K-1)] = \Gamma(K+\xi)/\Gamma(\xi). \tag{4}$$

The series (3) converges when $\text{Re}[\alpha(x)+\alpha(y)] > 0$.[4] If we replace x in Eq. (3) by s and y by t or u, then the resulting series for $F(s,t)$ or $F(s,u)$ converges for $\text{Re}s > -2a/b \cong -(1.0\text{ GeV})^2$,[5] provided that $|\cos\theta_s| \le 1$. If we replace x in Eq. (3) by t and y by u, then the resulting series for $F(t,u)$ converges for $\text{Re}s < (2a/b+4) \cong (1.1\text{ GeV})^2$.[6]

Partial-wave projections of the series (3) for $F(s,t)$ and $F(s,u)$ are nontrivial, because the arguments of the Pochhammer polynomials T_K depend on $\cos\theta_s$. However, in the series for $F(t,u)$, the argument of T_K is independent of $\cos\theta_s$, and the series (3) together with Eq. (1c) leads immediately to

$$V^{(l)2}(\nu) = \frac{\beta}{b\nu}\sum_{K=1}^{\infty}\frac{(-1)^K T_K(1-2a+4b\nu)}{\Gamma(K)} Q_l\left(1+\frac{K-a}{2b\nu}\right), \tag{5}$$

where Q_l is the Legendre function of the second kind. From the asymptotic behavior of Q_l and Γ, it is straightforward to establish that the series (5) converges if $\text{Re}s < [(l+2a)/b+4] \cong 1.2(l+1)$

GeV^2.[7]

Now let us approximate $F(s,t)$, $F(s,u)$, and $F(t,u)$ in Eqs. (1a)–(1c) by the Nth partial sum of the series (3) with appropriate substitutions for x and y, and denote the resulting approximations to A^I by A_N^I. The A_N^I are analytic and crossing-symmetric by construction. If we denote the s-channel partial-wave projections of A_N^I by $A_N^{(l)I}$, the domains of convergence of the series (3) and (5) imply that

$$\lim_{N\to\infty} A_N^{(l)0}(s) = V^{(l)0}(s) \tag{6a}$$

if $-2a/b < \mathrm{Re}\, s < [(l+2a)/b+4]$,

$$\lim_{N\to\infty} A_N^{(l)1}(s) = V^{(l)1}(s) \tag{6b}$$

if $-2a/b < \mathrm{Re}\, s$,

$$\lim_{N\to\infty} A_N^{(l)2}(s) = V^{(l)2}(s) \tag{6c}$$

if $\mathrm{Re}\, s < [(l+2a)/b+4]$.

Within the above-stated domains of convergence of the sequences $\{A_N^{(l)I}(s)\}$, the $V^{(l)I}$ can be approximated to any desired accuracy by the functions $A_N^{(l)I}$ with sufficiently large index N.

From inspection of Eq. (3), it is evident that the singularities in A_N^I are simply the poles corresponding to the first N towers of resonances[8] in the direct and crossed channels. Thus $\mathrm{Im}\, A_N^{(l)I}$ for $\nu > 0$ simply consists of the δ-function absorptive parts corresponding to whatever resonances exist in the $(l)I$ channel within the first N towers of resonances. To obtain the left cut of $A_N^{(l)I}$, we note that analyticity and crossing symmetry imply that[9]

$$\mathrm{Im}A^{(l)I}(\nu) = \frac{1}{\nu}\int_0^{-\nu-1} d\nu' P_l\left(1+2\frac{\nu'+1}{\nu}\right)\sum_{I',l'} \alpha_{II'}(2l'+1) \times \mathrm{Im}A^{(l')I'}(\nu') P_{l'}\left(1+2\frac{\nu+1}{\nu'}\right), \tag{7}$$

where

$$\alpha_{II'} = \begin{pmatrix} \frac{2}{3} & 2 & \frac{10}{3} \\ \frac{2}{3} & 1 & -\frac{5}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{pmatrix}.$$

Thus $\mathrm{Im}A_N^{(l)I}$ for $\nu < -1$ may be obtained[9] by substituting into the right-hand side of Eq. (7) the δ-function absorptive parts corresponding to the resonances in the first N towers.[8]

Since N is the highest spin value which occurs in the first N towers of resonances, it is evident from Eq. (7) that N subtractions are necessary and sufficient to guarantee convergence of the integral over the left cut in a dispersion relation for $A_N^{(l)I}$. From Eq. (3), it is clear that the asymptotic behavior of $A_N^{(l)I}$ in all directions is such that the contour integral at infinity in such a dispersion relation can be dropped, so $A_N^{(l)I}$ satisfies a dispersion relation of the usual form, albeit with N subtractions.

For $l>1$, all $V^{(l)I}$ and $A_N^{(l)I}$ vanish at threshold like ν^l,[10] so the first $(l-1)$ derivatives coincide at threshold. However, if $l=0$ and/or $N>l$, better approximations can be obtained near threshold by constructing the functions

$$V_N^{(l)I} \equiv \sum_{n=0}^{N-1} \frac{\nu^n}{n!} \left.\frac{d^n V^{(l)I}}{d\nu^n}\right|_{\nu=0} + \frac{\nu^N}{\pi}\int_{-\infty}^{\infty} d\nu' \frac{\mathrm{Im}A_N^{(l)I}(\nu')}{\nu'^N(\nu'-\nu)}. \tag{8}$$

Intuitively, one would expect the domains of convergence of the sequences $\{V_N^{(l)I}\}$ to be at least as large as for the $\{A_N^{(l)I}\}$, but perhaps no larger.

To facilitate the remainder of our discussion of the accuracy of the approximations

$$V^{(l)I} \cong V_N^{(l)I}, \tag{9}$$

let us define functions which represent the percentage discrepancies:

$$\Delta V_N^{(l)I} \equiv \frac{V^{(l)I} - V_N^{(l)I}}{\frac{1}{2}(V^{(l)I} + V_N^{(l)I})}. \tag{10}$$

The accuracy of the approximation (9) depends on the values of the Regge parameters a and b. For the sake of definiteness, let us consider the Lovelace values $a=0.483$, $b=0.017$.[3] In Figs. 1(a) and 1(b), we display $\Delta V_1^{(l)I}$ and $\Delta V_2^{(l)I}$, respectively, for $l=0$, 1, and 2. Since $V^{(l)I}$ behaves at threshold like ν^l, $\Delta V_N^{(l)I}$ vanishes at threshold like ν^{N-l} if $N>l$, and behaves like ν^0 if $N \le l$.[11] The zeros in $\Delta V_N^{(0)0}$ and $\Delta V_N^{(1)1}$ at 762 MeV occur because $V_N^{(0)0}$ has the same ϵ resonance pole as $V^{(0)0}$, and $V_N^{(1)1}$ has the same ρ pole as $V^{(1)1}$. The sharp rise in $\Delta V_2^{(1)1}$ above 900 MeV is due to a shrinking denominator in Eq. (10); the difference $V^{(1)1} - V_2^{(1)1}$ is only -0.06 at 1 GeV (assuming $\beta = 0.50$, which corresponds to $\Gamma(\rho) \cong 120$ MeV). The behavior of the $\Delta V_1^{(0)I}$ and $\Delta V_2^{(0)I}$ near 1 GeV suggests that the sequences $\{V_N^{(0)I}\}$ for $I=0$ and 2 diverge at the same energy as the sequences $\{A_N^{(0)I}\}$, namely at 1.08 GeV.[7]

The discrepancies displayed in Fig. 1 for $I=2$ are especially interesting, because the $V_N^{(l)2}$ are determined by their *left* cuts, together with the threshold constraints implied by the subtractions in Eq. (8). Thus the information which determines $V_1^{(0)2}$ is closely analogous to the information which is fed into N/D equations when a single subtraction is performed at threshold, the left cut of N is constructed by feeding ρ and ϵ exchange into Eq. (7), and D contains no poles. The fact that $\Delta V_1^{(0)2} = -0.4$ at 750 MeV suggests that such a prescription is unreliable in the ρ region. Although $\Delta V_1^{(0)0}$ and $\Delta V_1^{(1)1}$ vanish at 762 MeV, this is because $V_1^{(0)0}$ and $V_1^{(1)1}$ are explicitly given the same poles as $V^{(0)0}$ and $V^{(1)1}$, respectively, at this energy. (Analogous constraints could be built into N/D equations by performing two additional subtractions in N, or by inserting a pole with appropriate residue into D at the point where the phase shift reaches 180°.)

Although $\Delta V_1^{(0)2}$ reaches 0.26 at 500 MeV, $\Delta V_2^{(0)2}$ is less than 0.10 below 700 MeV, while $\Delta V_2^{(0)0}$ and $\Delta V_2^{(1)1}$ are less than 0.10 below 900 MeV.[12] Thus even if physical $A^{(l)I}$ with $I=0$ and 2 contain essential singularities like those in $V^{(l)I}$, our present work indicates that $A^{(0)2}$ can be well approximated below 700 MeV by a solution to a twice-subtracted dispersion relation, while $A^{(0)0}$ and $A^{(1)1}$ can be well approximated below 900 MeV by solutions to

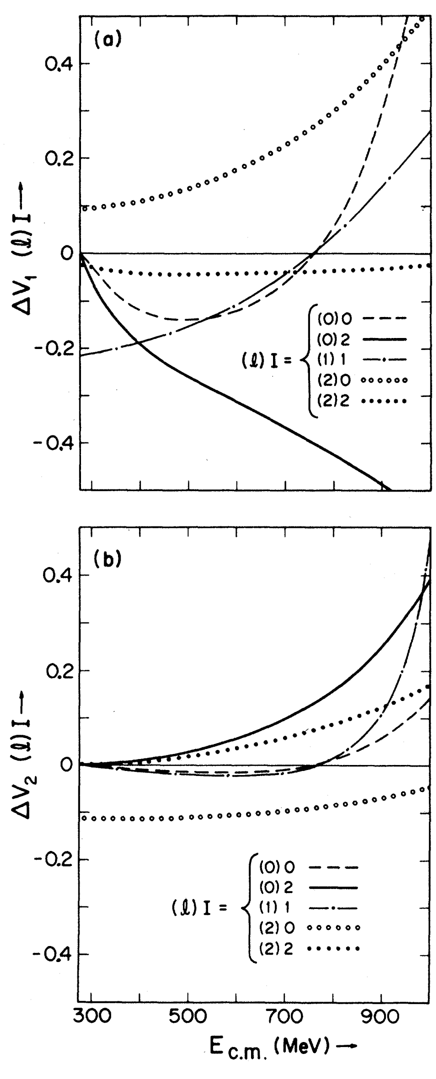

FIG. 1. (a) Values of $\Delta V_1^{(l)I}$ for $l=0$, 1, and 2. (b) Values of $\Delta V_2^{(l)I}$ for $l=0$, 1, and 2.

twice-subtracted dispersion relations (provided that the solutions are constrained to contain ϵ and ρ resonances with correct masses and widths). Since $\Delta V_2^{(2)I}$ is less than 0.12 below 1 GeV for $I=0$ and 2, similar remarks hold for the D waves. Thus the dispersive models with two subtractions proposed earlier by the present author[2] are reliable over the aforementioned ranges of energy even if physical $A^{(l)I}$ with $I=0$ and 2 contain essential singularities like those in $V^{(l)I}$.[1]

To see how one might construct analytic, exactly crossing-symmetric $\pi\pi$ amplitudes which have the same resonance spectrum and essential singularities as the Veneziano amplitudes but which have unitary partial waves for $l \leq L$ (where L is finite but arbitrary), let us consider the functions $\Delta A^{(l)I}(\nu)$ defined by

$$\Delta A^{(l)I} \equiv A^{(l)I} - V^{(l)I}. \tag{11}$$

The $\Delta A^{(l)I}$ are real analytic functions with the same crossing properties as $A^{(l)I}$ and $V^{(l)I}$. Unitarity implies that every $\Delta A^{(l)I}$ has a right cut, and crossing symmetry implies that each $\Delta A^{(l)I}$ has a left cut related to the right cuts of all the $\Delta A^{(l')I'}$ by Eq. (7) (with $\Delta A^{(l)I}$ substituted everywhere for $A^{(l)I}$).

If one imposes unitarity on the $A^{(l)I}$ with $l \leq L$ for some finite L but sets $\mathrm{Im}(\Delta A^{(l)I})=0$ for $\nu>0$ and all $l>L$, then Eq. (7) is valid for all $\nu<-1$.[9] The resulting $\mathrm{Im}(\Delta A^{(l)I})$ grow like ν^{L-1} as $\nu \to -\infty$, so every $\Delta A^{(l)I}$ satisfies a dispersion relation with L subtractions. If the subtraction constants are chosen in a way consistent with crossing symmetry, then the full amplitudes

$$A^I(\nu, \cos\theta) = \sum_{l=0}^{\infty}(2l+1)[V^{(l)I} + \Delta A^{(l)I}]P_l(\cos\theta) \tag{12}$$

are analytic, exactly crossing-symmetric, and approximately unitary over the range of energies where $(\mathrm{Re}A^{(l)I})^2 \ll 1$ for all $l>L$.

If the resonance spectrum of physical $\pi\pi$ amplitudes agrees with that of the Veneziano amplitudes, then resonance contributions to $\mathrm{Im}(\Delta A^{(l)I})$ *vanish* in the sense of local averages. Then it is reasonable to conjecture that amplitudes of the form (12) with $\Delta A^{(l)I}$ constructed in accordance with the dispersive method just outlined might be good approximations to Nature up to 1 GeV or more for small values of L.

For the case $L=1$, only one of the subtraction parameters remains independent when crossing symmetry is imposed.[9] Thus the model implies a relation between the S-wave scattering lengths a_0 and a_2. Upon solving the model[13] with $L=1$ for different values of the independent subtraction parameter and different masses and widths of the ϵ resonance consistent with Chew-Low extrapolations of $\pi N \rightarrow \pi\pi N$ data, one obtains the "universal curve" relation between a_0 and a_2 of Morgan and Shaw.[14] The model possesses solutions for the S waves and P wave which are in excellent agreement with experiment at least to 1 GeV. The $I=2$ S wave is quite insensitive between 500 MeV and 1 GeV to reasonable variations of the subtraction parameter and variations of the ϵ mass and width, so a definite prediction is made for the $I=2$ S wave.

If in addition to crossing symmetry one imposes a well-known sum rule on $2a_0 - 5a_2$,[15] then only one subtraction parameter remains independent[2] for the case $L=2$. Thus twice-subtracted dispersion relations could be used to obtain the $\Delta A^{(l)I}$ with only one independent subtraction parameter. The resulting approximations for the $\Delta A^{(l)I}$ might be expected to be valid up to 1 GeV or more, especially since the $V^{(l)I}$ with $l \leq 2$ can be approximated despite their essential singularities by solutions to twice-subtracted dispersion relations over the regions indicated by Fig. 1(b).

The success of the model with $L=1$ in generating S waves and P wave consistent with experiment up to 1 GeV can be understood from the fact that a_0 and a_2 *automatically*[13] satisfy the sum rule for $2a_0 - 5a_2$.[16] Thus the S waves and P wave with $L=1$ are essentially equivalent to those which would be generated by the model with $L=2$, when the latter is constrained by the sum rule for $2a_0 - 5a_2$.[17]

We remark that if one had never heard of Veneziano amplitudes but simply wrote twice-subtracted dispersion relations for the $A^{(l)I}$ and determined one of the two independent subtraction parameters by imposing the sum rule for $2a_0 - 5a_2$, then the resulting amplitudes would agree with the results of the aforementioned model for unitarized Veneziano amplitudes within about 10% below 700 MeV for $A^{(0)2}$, and below about 900 MeV for $A^{(0)0}$ and $A^{(1)1}$. Thus below these energies, the unitarized Veneziano amplitudes have a much greater generality than does the Veneziano model itself.

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[1]It was suggested without proof by F. Drago and S. Matsuda, Phys. Rev. 181, 2095 (1969), that all Veneziano partial waves contain essential singularities at infinity. However, R. T. Park and B. R. Desai, Phys. Rev. D 2, 786 (1970), have shown that $I = 1$ Veneziano $\pi\pi$ partial waves satisfy unsubtracted dispersion relations. Hence the plausibility argument of Drago and Matsuda is not valid. Recently the present author has shown [E. P. Tryon, preceding paper, Phys. Rev. D 4, 1216 (1971)] that the $I = 0$ and 2 S and D waves can be well approximated over the left half s plane, at least out to $|s| = 10^3 m_\pi^2$, by exponentials which reach values of the order of $10^5 m_\pi^{-1}$ for $\mathrm{Re}\, s = -10^3 m_\pi^2$. On the basis of this and more detailed considerations, it has been argued that all Veneziano $\pi\pi$ partial waves with $I = 0$ and 2 are likely to contain essential singularities at infinity.

[2]Approximate solutions to twice-subtracted dispersion relations for $\pi\pi$ S waves with current-algebra threshold behavior were obtained by E. P. Tryon, Phys. Rev. Letters 20, 769 (1968). This calculation was refined and extended to include the P wave and D waves by E. P. Tryon, in *Proceedings of a Conference on $\pi\pi$ and $K\pi$ Interactions at Argonne National Laboratory, 1969*, edited by F. Loeffler and E. Malamud (Argonne National Laboratory, Argonne, Ill., 1969). In the latter work, it was shown that solutions exist for a range of values of the ϵ and ρ masses and widths. Solutions were constructed with different types of ϵ resonances like those suggested by Chew-Low extrapolations of $\pi N \rightarrow \pi\pi N$ data.

[3]C. Lovelace, Phys. Letters 28B, 264 (1968).

[4]D. Sivers and Joel Yellin, Ann. Phys. (N. Y.) 55, 107 (1969).

[5]If one uses the Lovelace values $a = 0.483$, $b = 0.017$ (Ref. 2), then $-2a/b = -(1.04\ \mathrm{GeV})^2$.

[6]With $a = 0.483$, $b = 0.017$ (Ref. 3), $2a/b + 4 = (1.08\ \mathrm{GeV})^2$.

[7]With $a = 0.483$, $b = 0.017$ (Ref. 3), $(l + 2a)/b + 4 = (1.12l + 1.16)\ \mathrm{GeV}^2$.

[8]By the first tower, we mean all resonances with the mass of the ρ; by the second tower, all resonances with the mass of the f_0; etc.

[9]G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960). If the A^I contain double-spectral functions bounded by the curves derived by Mandelstam, then the Legendre series on the right-hand side of Eq. (7) diverges over part of the range of integration if $\nu < -9$. The Veneziano amplitudes and the A_N^I contain no double-spectral functions, and Eq. (7) is valid for all $\nu < -1$ for these functions. (The convergence of the series for all $\nu < -1$ is guaranteed by the fact that only finitely many $\mathrm{Im}\, V^{(l')I'}$ and $\mathrm{Im}\, A_N^{(l')I'}$ are different from zero over any finite range of integration with $\nu' > 0$.)

[10]Since $t = -2\nu(1 - \cos\theta_s)$ and $u = -2\nu(1 + \cos\theta_s)$, the factors $[\alpha(t) - K]^{-1}$ and $[\alpha(u) - K]^{-1}$ in Eq. (3) can be expanded as power series in t and u, respectively, if ν is near zero and $|\cos\theta_s| \leq 1$. Thus for small ν and physical $\cos\theta_s$, each term in the series on the right-hand side of Eq. (3) can be written as a series in $\cos\theta_s$ wherein every factor of $\cos\theta_s$ has a cofactor of ν. Thus $V^{(l)I}$ and $A_N^{(l)I}$ behave like ν^l near $\nu = 0$.

[11]It is an anomaly that $\Delta V_2^{(2)2}$ is so small at $\nu = 0$.

[12]Since Park and Desai (Ref. 1) have shown that for $I = 1$ the exact $V^{(l)I}$ satisfy unsubtracted dispersion relations, the functions $V_N^{(l)I}$ are clearly not the *best* dispersive approximations to $V^{(l)1}$. However, most dispersive models for $A^{(l)1}$ in the literature bear a closer resemblance to functions of the form of $V_N^{(l)1}$ than to the exact $V^{(l)1}$, so our discussion is of interest even for $I = 1$.

[13]E. P. Tryon, Columbia University Report No. NYO-1932(2)-153, 1969 (unpublished); Columbia University Report No. NYO-1932(2)-196, 1971 (to be published in Phys. Letters B).

[14]D. Morgan and G. Shaw, Phys. Rev. D 2, 520 (1970).

[15]M. G. Olsson, Phys. Rev. 162, 1338 (1967); T. Akiba and K. Kang, Phys. Letters 25B, 35 (1967).

[16]This explains why the model with $L = 1$ generates the "universal curve" relating a_0 to a_2, since the derivation by Morgan and Shaw (Ref. 14) is primarily based on the sum rule for $2a_0 - 5a_2$.

[17]The only difference is that with $L = 2$ the D waves are unitarized, whereas with $L = 1$ they are not. However, the D-wave absorptive parts are negligible below 1.5 GeV except for the f_0 (1250), so the $\mathrm{Im}[\Delta A^{(2)I}]$ are negligible up to 1.5 GeV or more. Thus unitarizing the D waves for $L = 2$ would not have an appreciable effect on the S waves or P wave below 1 GeV.

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Eikonal Approximation to the Vertex Function on the Mass Shell*

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The eikonal approximation for graphs representing the exchange of an arbitrary connected amplitude l times in the vertex function is developed for a ϕ^3 theory and quantum electrodynamics when the two external particles are on the mass shell and the momentum transfer is large and spacelike. The simplest case of elementary-particle exchange is calculated. Summing over l, we obtain

$$\Gamma_{\text{scalar}}^{\text{eikonal}}(q^2) = g\exp\left[\frac{g^2}{32\pi^2}\frac{1}{-q^2}\ln^2\left(\frac{-q^2}{m^2}\right)\right] \text{ and } \Gamma^{\mu\,\text{eikonal}}_{\text{QED}}(q^2) = e\bar{u}_{\lambda_f}\gamma^\mu u_{\lambda_i}\exp\left[-\frac{e^2}{16\pi^2}\ln^2\left(\frac{-q^2}{\mu^2}\right)\right],$$

where μ is a photon mass introduced to eliminate infrared divergence problems.

I. INTRODUCTION

The off-mass-shell vertex function at high energy transfer was studied originally by Sudakov,[1] who considered radiative corrections in quantum electrodynamics (QED) by calculating the asymptotic behavior of Feynman integrals. More recent calculations for both on and off the mass shell were done by Jackiw[2] using an infinite-momentum technique similar to Weinberg's.[3] The vertex function, obtained in QED for crossed-ladder radiative corrections by Jackiw, is given by[4]

$$\Gamma^\mu = e\bar{u}\gamma^\mu u\exp\left[\frac{-e^2}{16\pi^2}\ln^2\left(\frac{-q^2}{\mu^2}\right)\right].$$

This result was conjectured from low-order calculations.

Recently the eikonal approximation has aroused much interest as a useful tool in calculating high-energy elastic scattering amplitudes. Abarbanel and Itzykson[5] have clearly demonstrated for QED that in the high-energy limit the eikonal method gives the correct behavior of those graphs considered. (The exact high-energy behavior was calculated for QED by Cheng and Wu[6] in their extensive work.) Later much work was done in applying the eikonal approximation in ϕ^3 theory[7] as well as QED[8] to obtain high-energy amplitudes.

It is natural, therefore, to apply the eikonal method to the graphs of the vertex function.[9,10] In the asymptotic region there are two distinct types of contributions to the Feynman integral for a vertex graph: eikonal contributions and noneikonal contributions. An eikonal contribution corresponds to a region of integration where the large momentum of the incoming particle is carried essentially unchanged by a line of propagators to the vertex and another line of propagators carries the large momentum from the vertex to the outgoing particle. In such a region we can picture the incoming particle as moving through the interaction region, emitting only soft virtual particles, until it reaches the vertex which is a hard interaction. At this point, large momentum is carried away, and then the particle continues through the interaction region, absorbing soft virtual particles, and finally emerging as the outgoing particle.

The distinguishing mark of a noneikonal contribution is that there are hard interactions at places other than the vertex. That is, either a propagator carries both the large incoming momentum and the large outgoing momentum or large momentum is split between two propagators.

Figure 1 illustrates the two types of contributions