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Dual, Crossing-Symmetric Amplitude with Mandelstam Analyticity*

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We present a class of scattering amplitudes which are dual and crossing symmetric, have Regge asymptotic behavior and second-sheet poles for resonances in all channels, and satisfy a Mandelstam representation with correct double-spectral-function boundaries. The Regge trajectories and residues are essentially arbitrary.

In the following we describe a class of scattering amplitudes. They possess the set of properties connected with crossing symmetry and duality that has recently led to interesting algebraic and symmetry-type results,[1] and they simultaneously possess some dynamical properties related to unitarity. In particular, all resonances occur as second-sheet poles, the threshold behavior is correct, and the amplitudes can be written in the form of Mandelstam double dispersion relations with correct double-spectral-function boundaries.

Before we present the detailed results we would like to note several implications of this work. First, it finally gives us a scattering amplitude that is Regge behaved, is crossing symmetric, and has a Mandelstam representation. To find such an amplitude has been a basic problem in particle physics. It may lead to new insight into the meaning and uniqueness of bootstrap ideas to have an amplitude that satisfies all the conditions that are usually postulated except unitarity. This amplitude has still an essentially arbitrary trajectory and residue; they will be related by unitarity.

Second, we show that a dual, crossing-symmetric, Regge-behaved, etc., amplitude can be constructed; it has all of the conditions that are put on the Veneziano amplitude, but is is not unique at all. Thus many speculations on the meaning of the Veneziano model and on duality must be reconsidered. Fermion channels may be included in our model by $\sqrt{u}$ dependence in the residue and trajectory.

Third, the present amplitude is suitable for phenomenology without modifying any of its good properties; e.g., it already has second-sheet poles. Thus any relevance to experimental data can be tested unambiguously.

We proceed by exhibiting the amplitude and its properties, with proofs briefly indicated. Consider, for equal-mass scalar-meson scattering [for simplicity assume a process with exotic u channel; or else add $M(s,u)$ and $M(u,t)$ to $M(s,t)$ just as in the Veneziano model],

$$M(s,t)=\int_0^1 dx\, x^{-\alpha(tx')-1}(1-x)^{-\alpha(sx)-1}f(sx)f(tx'),\quad x'\equiv 1-x, \tag{1}$$

where α represents a (complex) Regge trajectory, $f(y)$ and $\alpha(y)$ are real for $y<4m^2$ and analytic at $y=0$, and have threshold branch points for $y=4m^2$. $f(y)$ satisfies the condition that as y becomes infinite anywhere in the physical sheet f vanishes faster than any inverse power of y. The condition is sufficient to guarantee the convergence of all integrals of interest. For example,[2] one might use $\alpha(y)=\alpha_0+\alpha' y+\gamma(4m^2-y)^{1/2}$, and $f(y)=\exp[-\beta(4m^2-y)^{1/4}]$. For trajectories with a positive intercept f will also contain a factor proportional to $\alpha(y)$, removing the spin zero pole.

The properties of $M(s,t)$ include the following:

(1) Regge-pole asymptotic behavior in all channels. This is easily shown by a change of variables $\mu=-sx$, which gives $M=(-s)^{\alpha(t)}G(t)$ in the limit as $|s|\to\infty$, with G a convergent integral, $G(t)=f(t)\times\int_0^\infty \mu^{-\alpha(t)-1}f(-\mu)d\mu$.

(2) Crossing symmetry. After putting $s\leftrightarrow t$, change variables $x\leftrightarrow 1-x$.

(3) Second-sheet resonance poles in all channels. The poles arise from the end-point singularity near $x=0$, and are of the form $1/(n-\alpha)$. Since $\mathrm{Im}\,\alpha\neq 0$ they are explicitly second-sheet poles.

(4) The pole residues in one variable are polynomials in the other, even with complex trajectories. There are no "ancestors."

(5) The amplitude is exchange degenerate. The appropriate representation for exchange-nondegenerate processes is clear.

(6) The correct threshold behavior is explicitly included in the choice of α and f.

(7) Remarkably, the amplitude also has Mandelstam analyticity. To see this, note that for $s, t < 4m^2$ we have $M(s,t)$ real. For $s \geq 4m^2$ and $t < 4m^2$ we find a discontinuity arising from Imα or from Imf, from the region of integration $x \geq 4m^2 s$. This discontinuity itself will have a nonvanishing discontinuity in t when $t(1-x) \geq 4m^2$ with $x \geq 4m^2/s$, or with the boundary curve

$$(1-4m^2/s)t \geq 4m^2$$

which is the conventional boundary[3] of the double-spectral-function region for scalar-meson scattering.

(8) The generalization to the N-particle problem[4] is similar to the corresponding situation in the conventional Veneziano model. The N-particle amplitude is given by

$$M \sim \int d\mu \, \Pi_u \, [u^{-\alpha(s_u u')-1} f(s_u u')], \quad u' \equiv 1-u, \tag{2}$$

where the product is taken over all planar channels and $d\mu$ is the invariant Veneziano measure.[4] The explicit form of the five-point function is then

$$\int_0^1 dx_1 dx_2 dx_3 dx_4 dx_5 \, \delta(x_2+x_1x_3-1)\delta(x_3+x_2x_4-1)\delta(x_5+x_1x_4-1) \prod_{i=1}^{5} [x_i^{-\alpha(s_{i,i+1}x_i')-1} f(s_{i,i+1}x_i')],$$

$$x_i' \equiv 1-x_i,$$

where $s_{i,i+1} = (p_i + p_{i+1})^2$. The factorized form of the integrand in (2) is sufficient to guarantee factorization at spin-0 and spin-1 poles on the leading trajectory. Beginning at spin 2, degeneracies appear[5] (still no ancestors). We do not want to speculate on the physical meaning of such degeneracies.

(9) The amplitude satisfies duality in the conventional sense. The same term contains Regge poles in the crossed channel at high energy and resonances at low energy.

(10) Amplitudes with Regge-pole asymptotic behavior in one channel and Regge-cut asymptotic behavior in another, or cut behavior in both, can be simply constructed, without affecting the Mandelstam analyticity. The obvious generalization of (1) to include Regge cuts is

$$M(s,t) = \int_0^1 dx \int_{-\infty}^{J_{\max}} dj \int_{-\infty}^{J_{\max}} dj' \, x^{-j-1}(1-x)^{-j'-1} h(j, sx) h(j', t(1-x)),$$

with appropriate conditions on h. $h(j, sx) = f(sx)\delta(j-\alpha(sx))$ gives formula (1), while $h(j, sx) = f_1(sx) \times \delta(j-\alpha(sx)) + f_2(j, sx)\theta(\alpha_c(sx)-j)$ would give a pole plus cut model. In a paper in preparation we will discuss the combination of terms that can describe experimental data simultaneously at high energies (where it is known that strong absorptive cuts are needed) and in the resonance region.

(11) At fixed angle θ, as $s \to \infty$, the amplitude behaves like the input function $f(s)$, up to a polynomial in s. So this function not only falls faster than any inverse power of s, but it can be chosen, as in our example, to obey also the Martin lower bound $M(s,\theta) > c_1 \exp(-c_2\sqrt{s})$. This latter property is not shared by the Veneziano model.

(12) The Adler zero[6] can be included. If $m_\pi = 0$, then as the four-momentum of one pion becomes zero, $M(s,t) \to M(0,0)$, and

$$M(0,0) = f(0)^2 \int_0^1 dx \, x^{-\alpha(0)-1}(1-x)^{-\alpha(0)-1} = f(0)^2 \Gamma^2(-\alpha(0))/\Gamma(-2\alpha(0)).$$

For $\alpha(0) = \frac{1}{2}$, $M(0,0) = 0$.

(13) Below the leading trajectory one finds parallel daughter and higher multipole trajectories (twin daughters, etc.). The latter arise because the expansion of the integrand near $x = 0$ contains terms proportional to

$$(sx)^j x^{-\alpha+n-1} (\ln x)^k, \quad \text{with } n \geq k.$$

These terms lead to contributions $\sim s^j/[j-\alpha+n+k]^{k+1}$. Since $n \geq 0$, and since $\ln x$ is always multiplied by x, these multipoles do not occur on the leading trajectory. It is not clear whether these multipoles are a good or a bad feature. However, we would like to speculate that they are closely connected with the presence of the correct double-spectral-function boundaries. Indeed, both these features arise from the fact that in Eq. (1) s is always multiplied by x and t is always multiplied by $1-x$. Consequently they can be interpreted essentially as a manifestation of unitarity properties. In this connection we note that at $\alpha=1$ at the first daughter level there is a sum of a simple pole and a dipole. If one treats these as the first two terms in an expansion of a shifted simple pole,[7] then the shift is indeed generally in the direction to depress the daughter trajectory. That is, if we put $a/(1-\alpha)+b/(1-\alpha)^2 \approx c/(1-\alpha+\epsilon)$ we find that $1-\alpha+\epsilon$ vanishes beyond the point at which $1-\alpha$ vanishes.

We want to emphasize that the essential improvements here arise by always associating s and t with x and $1-x$, respectively (we call this <u>s-x duality</u>). This enables us to work with dual but nonmeromorphic amplitudes (with no ancestors), to find many features associated with <u>unitarity</u>, and still provides us with a straightforward generalization to <u>N-particle processes</u>.

There have been several previous attempts to "smooth" the Veneziano model. All of these attempts have had difficulties not shared by our model. These difficulties include lack of Regge-pole asymptotic behavior,[8] undesirable Regge cuts,[9,10] essential singularities,[11] and trajectories which are complex straight lines, even below threshold.[12]

We do not consider the lack of uniqueness which characterizes our model [arbitrariness in $\alpha(z)$] as a bad point. Indeed we can argue that since unitarity is not yet fully satisfied the amplitude has no reason to be rigidly fixed, even in a bootstrap context. On the other hand, for phenomenology the lack of uniqueness has to be understood as generality and flexibility. At least we can say that our proposal is a counterexample to several common beliefs related to duality (e.g., "duality implies linear trajectories," "duality and crossing symmetry are sufficient to completely constrain the amplitude,"$\cdots$). These statements are usually made in the context of the meromorphic approximation. We believe that the departure from this approximation is necessary for the full understanding of duality, and of course changes some of its features.

A decade ago it was fashionable to construct scattering amplitudes that approximately satisfied elastic unitarity (at the expense of crossing), perhaps via an N/D calculation. More recently, amplitudes have been studied that have crossing symmetry and even duality, but which badly violate unitarity. Amplitudes of the form of Eq. (1) may allow a synthesis of these viewpoints. They are as useful as the conventional Veneziano amplitudes for studying algebraic and multiparticle techniques. On the other hand, they can be used for a phenomenological description of experimental data in any region since they do not grossly violate any known property of scattering amplitudes. Because of the Regge asymptotic behavior and the existence of a double dispersion relation, they are suitable for N/D calculations where elastic unitarity is exactly satisfied in one channel.

Indeed, one could hope[13] that by including crossing symmetry explicitly, and by including unitarity approximately through the presence of second-sheet resonance poles, correct double-spectral-function boundaries, and Regge-pole plus absorptive Regge-cut asymptotic behavior, the remaining corrections to the amplitude will be sufficiently small that one has a good phenomenological description of experimental data in all regions.

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[1]See, for example, G. Veneziano, in Proceedings of the Fifteenth International Conference on High Energy Physics, Kiev, U. S. S. R., 1970 (to be published).

[2]To guarantee good threshold behavior one must be more careful, using, for example, $\alpha(y)=\alpha' y+\gamma(4m^2-y)^{\frac{1}{2}}+\alpha_0+4m^2+\alpha'$, and $f(y)=\exp\{-\beta[(4m^2-y)^{\frac{1}{2}}+i(5m^2)^{\frac{1}{2}}]^{\frac{1}{2}}\}+\exp\{-\beta[(4m^2-y)^{\frac{1}{2}}-i(5m^2)^{\frac{1}{2}}]^{\frac{1}{2}}\}$.

[3]To get the precise boundary for the scalar case a translation is necessary: The boundary is given by $(s-4m^2)(t-4m^2)=16m^2$ in the present case, but actually by $(s-4m^2)(t-4m^2)=4m^4$. We can get the precise boundaries by using, in Eq. (1), $f(sx+2m^2(1-x))$, $f(t(1-x)-2m^2x)$, and the same changes in the argument of α. This changes the residue structure at the daughter level. For real problems such as $\pi\pi$ and πN scattering, where the double-spectral-function boundaries are a sum of two regions, we can have the exact boundaries in the pole-plus-cut models referred to under (10) above. There may not be a reasonable way to put them in terms of the form of Eq. (1).

[4]Chan H.-M., CERN Report No. TH.1057, 1969 (to be published).

[5]Complete factorization along the leading trajectory can be achieved by a suitable choice of functions f,

e.g., $f(sx) = c^{sx}$. This changes the form of the Regge residues and alters the fixed-angle behavior.

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Photon Rest Mass*

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We have performed experiments with very low-frequency parallel resonance circuits which, subject to some questions of theoretical validity, set a new upper limit for the photon rest mass of 10^{-49} g. This value is more than an order of magnitude smaller than the previous limit established by satellite measurements of the Earth's magnetic field.

Experimental and theoretical attacks on the upper limit of the photon rest mass have enjoyed a renaissance that has been very recently and expertly reviewed by Goldhaber and Nieto.[1] In summary, by far the most sensitive tests to date derive from the fact that the existence of a photon rest mass introduces a Yukawa factor, e^{-r/λ_0}, into the $1/r$ terms for the electrostatic and magnetostatic potentials. The length λ_0 is identified as a photon "Compton wavelength," $\lambda_0 = h/mc$, where m is the supposed rest mass. The most recent and sensitive explorations[2] of the validity of Coulomb's law yield an upper limit of 1.05 $\times 10^{10}$ cm for λ_0 (2×10^{-47} g for m), and an analysis[3] of the best satellite measurements of Earth's magnetic field provides a limit of 5.5×10^{10} cm (4×10^{-48} g).

In this paper we wish to describe a "table-top" approach to the experimental problem which, if our theoretical speculation proves sound, has now provided a limit of about 2×10^{12} cm (10^{-49} g) and could readily be extended several orders of magnitude. Our premise springs from early experiments reviewed by Rosa and Dorsey[4] in 1907 in which the phase velocity of light was determined by experiments with resonant circuits, the dimensions and resonant frequencies of which were the measured quantities. (A more traditional way of describing this work is to refer to it as a measurement of the ratio of the electromagnetic to the electrostatic units of charge, which ratio is the velocity of light in massless electromagnetic theory.) We extend the spirit of this method by exploring the behavior of very low-frequency resonant circuits, thereby establishing a limit to deviations of the phase velocity of light and hence an upper limit to the photon rest mass.

It is easy to demonstrate[5] that the free-space phase velocity of light v_φ is related to the angular frequency ω by

$$(v_\varphi/c)^2 = \omega^2/(\omega^2 - \omega_c^2), \qquad (1)$$

where $\omega_c = 2\pi c/\lambda_0$. (This definition of ω_c can be rewritten as $\hbar\omega_c = mc^2$, which provides a glimpse of massiness.) Let us now contemplate an evacuated conducting cavity one of whose resonant modes has frequency ω_0 in the massless electromagnetic case. We seek an expression for the resonance frequency ω' of this mode for the massy case. Since the relation between phase velocity, frequency, and wave vector k is always $v_\varphi = \omega/k$, and since the values of k for the various modes of this cavity are determined by geometry rather than by mass, we see that the change in the resonant frequency is strictly proportional to the change in phase velocity. Specifically, $\omega'/\omega_0 = v_\varphi/c$. Equation (1) now yields promptly the important relation

$$\omega'^2 = \omega_0^2 + \omega_c^2. \qquad (2)$$

This equation predicts, not surprisingly, that the lowest resonant frequency of a cavity is ω_c, irrespective of the size of the cavity. This simply reflects the fact that the phase velocity becomes