ELECTROMAGNETIC FORM FACTOR OF THE PION*

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We consider a model for the pion electromagnetic form factor based on duality and Regge asymptotic behavior for strong interactions. We find that $F_\pi(t) \xrightarrow[t\to\infty]{} t^{-1/2}$ and that vector dominance is expressed through second-order poles as t approaches the masses of vector mesons.

We consider a model for the pion electromagnetic form factor that incorporates both the constraints imposed by current algebra and the duality property of strong interactions. We have not yet achieved a completely well-defined model, but the results seem to be sufficiently interesting, both from a theoretical and an experimental point of view, to justify a preliminary report.

Our point of view, adopted as a working hypothesis and not necessarily a fundamental principle, is that the photon interacts with hadrons through pair creation as in field theory. The form factor arises from the strong final-state interactions. Thus the isovector pion form factor is obtained through the Feynman graphs of Fig. 1(a), which give a T-matrix element proportional to

$$p_\mu - \frac{i}{(2\pi)^4}\int d^4q \frac{(2q+p)_\mu A(t,q^2)}{(q^2+2q\cdot p_1)(q^2-2q\cdot p_2)}, \tag{1}$$

where $p_\mu = (p_1 - p_2)_\mu$, and $A(t,s)$ is the invariant amplitude for $\pi\pi$ scattering in the $I=1$ state with squared c.m. energy t and squared four-momentum transfer s.[1] We ignore particles other than pions in the intermediate state but shall comment on their effects later. The integral in (1) is proportional to p_μ, and so is gauge invariant. The $I=1$ $\pi\pi$ amplitude can be written in the form

$$A(t,s) = -\beta[V(t,s) - V(t,u)], \tag{2}$$

where $-\beta V(t,s)$ is a symmetric function of t, s, and represents the amplitude for $\pi^+\pi^-$ scattering in the t channel. The form factor is then found to be

$$F_\pi(t) = 1 + G(t), \tag{3}$$

$$G(t) = \frac{2i\beta}{(2\pi)^4}\int d^4q \left(1 + \frac{2p\cdot q}{p^2}\right) \times \frac{V(t,q^2)}{(q^2+2q\cdot p_1)(q^2-2q\cdot p_2)}. \tag{4}$$

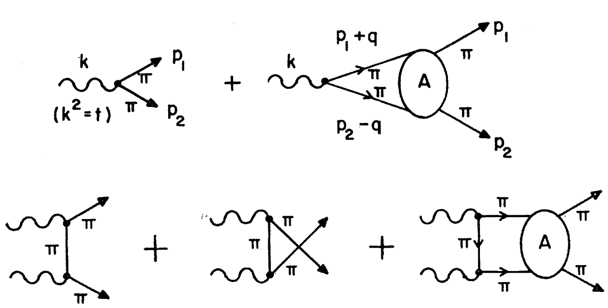

FIG. 1. (a) Feynman graphs for pion electromagnetic form factor. (b) Feynman graphs for pion Compton scattering.

If, in the same spirit, we take pion Compton scattering to be given by the Feynman graphs of Fig. 1(b), we will insure that our form factor is the residue of a fixed J-plane pole in the Compton amplitude, in accordance with current algebra.[2]

We now assume that $V(t,s)$ can be separated into a part $V_\rho(t,s)$ containing only narrow resonances with Regge asymptotic behavior dominated by the ρ trajectory, plus a part $V_P(t,s)$ containing the Pomeranchukon but no resonances.[3] Correspondingly, the form factor is decomposed as

$$F_\pi(t) = 1 + G_P(t) + G_\rho(t). \tag{5}$$

We shall only calculate the final term in the following.

Writing $V_\rho(t,s)$ as a sum of s poles, we immediately obtain $G_\rho(t)$ as a sum of triangular graphs, which can be easily calculated. They can be resummed to yield an expression involving $V_\rho(t,s)$ itself:

$$G_\rho(t) = (\beta/4\pi^2 t)\int_0^\infty dx\, g(t,x) V_\rho(t,-x), \tag{6}$$

$$g(t,x) = -\frac{2y\tau}{\tau+1}\left[\left(1+\frac{1}{y}\right)^{1/2} - 1\right] + \left(\frac{\tau}{\tau+1}\right)^{1/2}\left(\frac{y}{\tau+1} + \frac{1}{2}\right)\ln R, \tag{7}$$

$$R = \left[\frac{(\tau+1)^{1/2} + \tau^{1/2}}{(\tau+1)^{1/2} - \tau^{1/2}}\right] \times \frac{[(\tau+1)(y+1)]^{1/2} - (\tau y)^{1/2}}{[(\tau+1)(y+1)]^{1/2} + (\tau y)^{1/2}}, \tag{8}$$

where $y = x/4\mu^2$ and $\tau = -t/4\mu^2$, μ being the pion mass. As $\mu \to 0$, $g(t,x)$ approaches a simple limit:

$$g(t,x) \xrightarrow[\mu \to 0]{} [\tfrac{1}{2} - x/t] \ln[1 - t/x] - 1. \tag{9}$$

Since $V(t,s) = V(s,t)$, we could also expand it in a series of t poles. Indeed, it may appear that one would thus be led directly to vector dominance. However, this cannot be done, for the Feynman graph corresponding to a t pole is a divergent self-energy graph. Only the sum of such graphs gives a finite number. As we see later, this circumstance leads to vector dominance by second-order instead of simple poles.

Under the assumption that all Regge trajectories are linear, the most general $V_\rho(t,s)$ is a linear combination of Veneziano terms.[4,5] We take only a single term:

$$V_\rho(t,s) = \Gamma(1-\alpha_t)\Gamma(1-\alpha_s)/\Gamma(1-\alpha_t-\alpha_s), \tag{10}$$

where $\alpha_t = \alpha_0 + \alpha' t$ is the ρ-f^0 trajectory. As pointed out by Lovelace,[6] this satisfies the Adler self-consistency condition $V_\rho(\mu^2, \mu^2) = 0$, if $\alpha(\mu^2) = \frac{1}{2}$. Together with $\alpha(m_\rho^2) = 1$, this completely determines the trajectory:

$$\alpha_0 = \tfrac{1}{2} - \tfrac{1}{2}\mu^2/(m_\rho^2 - \mu^2) \approx \tfrac{1}{2},$$

$$\alpha' = [2(m_\rho^2 - \mu^2)]^{-1} \approx 1\ (\mathrm{GeV}/c)^{-2}. \tag{11}$$

Through elastic unitarity, the coupling constant β is related to the $\pi\pi$ width Γ_ρ of the ρ meson by

$$\beta = 16\pi(6\Gamma_\rho/m_\rho) \approx 16\pi, \tag{12}$$

where, in the last step, we have used the experimental value $6\Gamma_\rho/m_\rho \approx 1$, from hadronic experiments. Thus there are no unknown parameters in $G_\rho(t)$.

We now describe some properties of $G_\rho(t)$ without proof. It is a meromorphic function in the cut t plane with a logarithmic cut from $4\mu^2$ to ∞. The branch cut is most evident from (9) in the limit $\mu \to 0$. From (6) we see that a factor $\Gamma(1-\alpha_t)$, which has simple poles at $\alpha_t = 1, 2, \cdots$, can be factored out of the integral The remaining integral also has simple poles at the same positions. Consequently $G_\rho(t)$ has a second-order pole plus a simple pole at $\alpha_t = n$ $(n = 1, 2, \cdots)$. The original simple t poles of $V_\rho(t,s)$ become second-order poles in $G_\rho(t)$, because the simple poles couple to the photon through divergent self-energy loops. The residue at the two lowest second-order poles are real and can be read off from the following formulas:

$$G_\rho(t) \xrightarrow[t \to m_\rho^2]{} (\beta/48\pi^2)(1-\alpha_0)[\alpha'(t - m_\rho^2)]^{-2}, \tag{13}$$

$$G_\rho(t) \xrightarrow[t \to m_\rho'^2]{} (\beta/48\pi^2)(2-\alpha_0) \times (3\alpha_0 - 1)[\alpha'(t - m_\rho'^2)]^{-2}, \tag{14}$$

where $m_\rho'^2 = m_\rho^2 + (\alpha')^{-1}$, m_ρ' being the mass of the vector meson ρ'. There are also additive simple poles at the same positions. Asymptotically, both for spacelike and timelike t, we find

$$G_\rho(t) \xrightarrow[|t| \to \infty]{} -(\beta/8\pi^2)\Gamma(1-\alpha_0) \times (-\alpha' t)^{\alpha_0 - 1}[1 + O(\ln \alpha' t)^{-1}]. \tag{15}$$

Finally, at $t = 0$ for small μ^2 we have

$$G_\rho(0) \approx -\mu^2 \alpha' \beta/8\pi, \tag{16}$$

$$G_\rho'(0) \approx (\alpha'\beta/8\pi)\ln(\alpha'\mu^2)^{-1}. \tag{17}$$

The asymptotic behavior (15) is actually independent of the details of the Veneziano model. It only depends on the fact that ρ is the leading trajectory, and that high-energy fixed-angle scattering $(t \to \infty,\ s \to -\infty)$ is exponentially damped. Since the ρ leads all known trajectories except the Pomeranchukon, the asymptotic behavior $t^{\alpha_0 - 1}$ remains valid even if other particles than pions are admitted in the intermediate state.[7]

The Pomeranchukon contribution $G_P(t)$ can only be a subject of speculation. We assume that it is qualitatively like $G_\rho(t)$ except that it has no poles. By analogy with (15) it should approach a negative constant asymptotically, and we can conjecture that it cancels the "contact term" 1 in (5). However, at $t = 0$ it should be $\sim O(\alpha'\mu^2)$, and the contact term furnishes the correct normalization. To make these conjectures more definite, we can try to incorporate the Pomeranchukon into the Veneziano model using, for example, a model suggested by Huang.[8] In that model the ratio ϵ of P-$\pi\pi$ to ρ-$\pi\pi$ effective coupling is of order $\alpha'\Gamma_\rho \sim 10\%$. The Pomeranchukon then represents a small correction except at asymptotic energies. A crude calculation gives a behavior $G_P(t) \to -(\beta/8\pi^2)C = -(2/\pi)C$, where $C \sim \epsilon\Gamma(\epsilon) \sim 1$. Thus, at least in such a model the cancellation of the contact term by the Pomeranchukon is not impossible.

Adopting the picture suggested above, we can make some definite predictions for $F_\pi(t)$. For this purpose we neglect the pion mass wherever possible and set $\alpha_0 = \frac{1}{2}$, $\alpha' = 1\ (\mathrm{GeV}/c)^{-2}$, and $\beta = 16\pi$.

(1) The asymptotic form factor is given by

$$F_\pi(t) \xrightarrow[|t| \to \infty]{} -2\pi^{-1/2}(-\alpha' t)^{-1/2} \times [1 + O(\ln \alpha' t)^{-1}]. \tag{18}$$

Since $F_\pi(t)$ is negative for large spacelike t ($t \to -\infty$), and $F_\pi(0)=1$, it must have a zero at some spacelike t; but we cannot locate it before the Pomeranchukon contribution is better known.

(2) The rms radius of the pion diverges logarithmically as $\mu \to 0$. Taking only the most divergent term, and neglecting the Pomeranchukon contribution, we obtain from (17)

$$r_{\mathrm{rms}} \equiv [6F_\pi'(0)]^{1/2} \approx [12\alpha' \ln(\alpha'\mu^2)^{-1}]^{1/2} = 1.4\times 10^{-13}\ \mathrm{cm}. \tag{19}$$

(3) The ρ meson occurs in the form factor as a second-order pole plus a simple pole. Near $t=m_\rho^2$, the second-order pole dominates, and is given by (13) in the zero-width approximation. Putting the width simply by replacing m_ρ by $m_\rho - i\Gamma_\rho/2$, we obtain, near $t=m_\rho^2$,

$$|F_\pi(t)|^2 \approx [6\pi\alpha'^2]^{-2}[(t-m_\rho^2)^2 + (m_\rho\Gamma_\rho)^2]^{-2}. \tag{20}$$

Hence

$$|F_\pi(m_\rho^2)|^2 = (6\pi)^{-2}(\alpha' m_\rho\Gamma_\rho)^{-4}, \tag{21}$$

which is extremely sensitive to $m_\rho\Gamma_\rho$, whose value should ideally be taken from $\pi\pi$ resonance scattering. Using $m_\rho\Gamma_\rho \approx 0.1\ (\mathrm{MeV})^2$, we obtain $|F_\pi(m_\rho^2)|^2 \approx 30$, which lies within experimental bounds.[9] From a practical point of view, (20) can hardly be distinguished from a Breit-Wigner formula. If we simulate it with an equivalent Breit-Wigner formula of the same height and width, we obtain an effective form factor near $t=m_\rho^2$:

$$|F_{\mathrm{eff}}(t)|^2 \approx C[(t-m_\rho^2)^2 + (m_\rho\Gamma_{\mathrm{eff}})^2]^{-1}, \tag{22}$$

$$C = (6\pi)^{-2}(\alpha' m_\rho\Gamma_\rho)^{-4}(m_\rho\Gamma_{\mathrm{eff}})^2, \tag{23}$$

$$\Gamma_{\mathrm{eff}} = (2^{1/2}-1)^{1/2}\Gamma_\rho = 0.65\Gamma_\rho. \tag{24}$$

This may explain why e^+e^- colliding-beam experiments apparently give a smaller ρ width. In fact, taking $\Gamma_\rho = 140$ MeV, we find $\Gamma_{\mathrm{eff}} = 91$ MeV, which is consistent with experiments.[9]

We wish to make the following comments:

(1) Our value for r_π is about 50% greater than recently quoted experiments.[10] We do not regard this as a serious defect since we expect the Pomeranchukon to contribute for small values of t and preliminary estimates indicate this will reduce r_π. In any event the success of Eq. (19) is that it yields the correct order of magnitude.

(2) The complications due to spin for the nucleon form factor are severe but it appears that our model predicts that these form factors behave asymptotically like $t^{\alpha_N(0)3/2} \sim t^{-2}$ for $\alpha_N(0) \simeq -\frac{1}{2}$. We will give details in a separate publication.

(3) The appearance of double poles is characteristic of our model for the electromagnetic interaction. Double poles appear in the Compton amplitude also and we have not fully resolved all questions about them. We will return to this question in the future but feel that our treatment in Eqs. (19) and (20) is worth reporting.

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[1]Our metric is (1, -1, -1, -1). Our normalization is such that $S_{fi} = \delta_{fi} + i(2\pi)^4\delta^4(p_f - p_i)A$, and the sum over single-particle states is $\int d^3p/(2\pi)^3 2p_0$.

[2]J. B. Bronzan, I. S. Gerstein, B. Lee, and F. E. Low, Phys. Rev. 157, 1448 (1967).

[3]H. Harari, Phys. Rev. Letters 20, 1395 (1968).

[4]G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[5]N. N. Khuri, Phys. Rev. 185, 1876 (1969).

[6]C. Lovelace, Phys. Letters 28B, 265 (1968).

[7]S. Nussinov and J. Rosner (to be published) have obtained the same asymptotic behavior using an approach quite different from ours.

[8]K. Huang, Phys. Rev. Letters 23, 903 (1969).

[9]V. L. Auslander et al., Phys. Letters 25B, 433 (1967); J. E. Augustin et al., Phys. Letters 28B, 508 (1969).

[10]C. Mistretta et al., Phys. Rev. Letters 20, 1523 (1968).