EXACT INTEGRAL EQUATION FOR PION-PION SCATTERING INVOLVING ONLY PHYSICAL REGION PARTIAL WAVES

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We derive an exact relation for $\pi\pi$ scattering which yields the real parts of the $\pi\pi$ partial wave amplitudes $a_l^{(I)}(s)$ in the region $4m_\pi^2 \leqslant s \leqslant 60m_\pi^2$, given i) the $I = 0$ and the $I = 2$ S wave scattering lengths and ii) the Im $a_l^{(I)}(s)$ for $4m_\pi^2 \leqslant s \leqslant \infty$, where $\sqrt{s}$ denotes the centre-of-mass energy, l the angular momentum and I the isotopic spin. It also provides i) a system of integral equations to determine the $a_l^{(I)}(s)$ for $4m_\pi^2 \leqslant s \leqslant 16m_\pi^2$, $16m_\pi^2 \leqslant s \leqslant \infty$, and ii) *necessary and sufficient* conditions for crossing symmetry expressed in terms of the physical region partial wave amplitudes only.

Let $F^{(I)}(s,t)$ denote the $\pi\pi$ scattering amplitude with isotopic spin I in the s channel, normalized such that

$$\frac{d\sigma^{(I)}}{d\Omega}(s,t) = \left|\frac{F^{(I)}(s,t)}{\sqrt{s}}\right|^2, \tag{1}$$

where s and t denote respectively the squares of the centre-of-mass energy and the centre-of-mass momentum transfer. The partial wave amplitudes $a_l^{(I)}(s)$ are defined by

$$F^{(I)}(s,t) = \frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)[2a_l^{(I)}(s)]P_l\left(1+\frac{t}{2k^2}\right), \tag{2}$$

where k is the centre-of-mass momentum, and "the identical particle factor" 2 in the square bracket is introduced because of the identity of the pions to obtain the simple unitarity relation,

$$\mathrm{Im}\, a_l^{(I)}(s) = |a_l^{(I)}(s)|^2, \qquad 4 \leqslant s \leqslant 16 \tag{3}$$

where we use units such that the pion mass is unity. The optical theorem reads

$$\sigma_{\mathrm{tot}}^{(I)}(s) = \frac{8\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\,\mathrm{Im}\, a_l^{(I)}(s). \tag{4}$$

The S wave scattering lengths $a_0^{(0)}$ and $a_0^{(2)}$ are defined by

$$a_0^{(I)} \equiv \lim_{k\to 0}\frac{a_0^{(I)}(s)}{k} = \tfrac{1}{4}F^{(I)}(4,0). \tag{5}$$

The $F^{(I)}(s,t)$ satisfy the crossing relations written in matrix notation,

$$\underset{\sim}{F}(s,t) = C_{st}\,\underset{\sim}{F}(t,s) = C_{tu}\,\underset{\sim}{F}(s,u) = C_{su}\,\underset{\sim}{F}(u,t), \tag{6}$$

where

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$$\underset{\sim}{F}(s,t) \equiv \begin{bmatrix} F^{(0)}(s,t) \\ F^{(1)}(s,t) \\ F^{(2)}(s,t) \end{bmatrix}, \tag{7}$$

$$C_{st} = \begin{bmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{bmatrix}, \quad C_{tu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_{su} = \begin{bmatrix} 1/3 & -1 & 5/3 \\ -1/3 & 1/2 & 5/6 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}, \tag{8}$$

and

$$u = (4 - s - t). \tag{9}$$

Calculation of Re $a_l^{(I)}(s)$, $4 \leq s \leq 60$. Jin and Martin [1] have established from axiomatic field theory fixed t dispersion relations with two subtractions for $F^{(I)}(s,t)$ for $|t| < 4$. We may write them in matrix notation as

$$\underset{\sim}{F}(s,t) = C_{st}[\underset{\sim}{C}(t) + (s-u)\underset{\sim}{D}(t)] + \frac{1}{\pi}\int_4^\infty \frac{ds'}{s'^2}\left(\frac{s^2}{s'-s} + \frac{u^2}{s'-u}C_{su}\right)\underset{\sim}{A}(s',t), \tag{10}$$

where the $A^{(I)}(s',t)$ are absorptive parts defined by

$$A^{(I)}(s',t) = \sqrt{\frac{4s'}{s'-4}}\sum_{l=0}^{\infty}(2l+1)[2\ \mathrm{Im}\ a_l^{(I)}(s')]P_l\left(1+\frac{2t}{s'-4}\right), \tag{11}$$

within the large Lehmann-Martin ellipse [2], and the subtraction constants $\underset{\sim}{C}(t)$, $\underset{\sim}{D}(t)$ are of the form

$$\underset{\sim}{C}(t) = \begin{bmatrix} C^{(0)}(t) \\ 0 \\ C^{(2)}(t) \end{bmatrix}, \qquad \underset{\sim}{D}(t) = \begin{bmatrix} 0 \\ d^{(1)}(t) \\ 0 \end{bmatrix}, \tag{12}$$

due to crossing symmetry. The main remark we shall make here is that the t dependence of the subtraction constants $\underset{\sim}{C}(t)$ and $\underset{\sim}{D}(t)$ can be extracted using crossing symmetry at a fixed value of s, say $s = 0$, so as to obtain a relation involving only experimentally accessible quantities. Thus, substituting eq. (10) into the equation

$$\underset{\sim}{F}(0,t) = C_{st}\underset{\sim}{F}(t,0), \tag{13}$$

to express $\underset{\sim}{C}(t)$ and $\underset{\sim}{D}(t)$ in terms of $\underset{\sim}{C}(0)$ and $\underset{\sim}{D}(0)$ and eliminating the latter in favour of the S wave scattering lengths given by eq. (5), we obtain

$$\underset{\sim}{F}(s,t) = g_1(s,t)\,g_0 + \int_4^\infty ds'[g_2(s,t,s')\underset{\sim}{A}(s',0) + g_3(s,t,s')\underset{\sim}{A}(s',t)], \tag{14}$$

where

$$g_1(s,t) \equiv s(1-C_{su}) + t(C_{st}-C_{su}) + 4C_{su}, \tag{15}$$

$$g_2(s,t,s') \equiv C_{st}\left(\frac{1+C_{tu}}{2} + \frac{2s+t-4}{t-4}\,\frac{1-C_{tu}}{2}\right) \times \frac{1}{\pi}\cdot\frac{1}{s'^2}\left[\frac{t^2}{s'-t} + \frac{(4-t)^2 C_{su}}{s'-(4-t)} - \frac{4t+4(4-t)C_{su}}{s'-4}\right], \tag{16}$$

$$g_3(s,t,s') \equiv \frac{1}{\pi s'^2}\left[\frac{s^2}{s'-s} + \frac{u^2}{s'-u}C_{su} - \frac{(4-t)^2}{s'-(4-t)}\left\{\frac{C_{su}+1}{2} + \frac{2s+t-4}{t-4}\,\frac{C_{su}-1}{2}\right\}\right], \tag{17}$$

and

$$g_0 \equiv \begin{bmatrix} a_0^{(0)} \\ 0 \\ a_0^{(2)} \end{bmatrix} . \tag{18}$$

Because of the Jin-Martin result [1] and the Froissart bound [3] for $t < 0$, eq. (14) holds for all real values of $t < 4$. Now, the $a_l^{(I)}(s)$ for $s \geq 4$ are calculated from eq. (14) to yield the relation

$$2a_l^{(I)}(s)\frac{\sqrt{s}}{k} = \frac{1+(-)^{l+I}}{2}\Big[\int_0^1 \mathrm{d}x P_l(x) \sum_{I'=0,2} g_1^{II'}(s, \frac{4-s}{2}(1-x))\, a_0^{(I')}$$

$$+ 4\int_0^1 \mathrm{d}x P_l(x)\int_4^\infty \mathrm{d}s' \sqrt{\frac{s'}{s'-4}} \sum_{l'=0}^{\infty}(2l'+1)\sum_{I'=0,1,2}\Big\{g_2^{II'}(s, \frac{4-s}{2}(1-x), s')\,\mathrm{Im}\, a_{l'}^{(I')}(s')$$

$$+ g_3^{II'}(s, \frac{4-s}{2}(1-x), s')\,\mathrm{Im}\, a_{l'}^{(I')}(s')\, P_{l'}\Big(1+\frac{(4-s)(1-x)}{(s'-4)}\Big)\Big\}\Big] , \tag{19}$$

where the $g_i^{II'}$, $i = 1,2,3$, are the matrix elements of the g_i defined in eqs. (15)-(17). For the validity of eq. (19) we have only to check that the partial wave expansion for $A(s',t)$, eq. (11), used to obtain eq. (19), is valid for $s' = (4, \infty)$, $t = (2-\frac{1}{2}s, 0)$. Martin [2] has shown that the expansion (11) converges for $s' = (4,\infty)$ for $t = (-28, 4)$. Hence, eq. (19) enables us to calculate Re $a_l^{(I)}(s)$ for $4 \leq s \leq 60$ using the experimental values of the S wave scattering lengths $a_0^{(0)}$, $a_0^{(2)}$ and the experimental values of the Im $a_l^{(I)}(s')$, for $s' \geq 4$. These values can then be checked against the experimental values of Re $a_l^{(I)}(s)$ from a phase-shift analysis, thus providing a test of analyticity, crossing and unitarity (used in phase-shift analysis).

We may remark here that the amplitude (14) has s-u crossing symmetry built in and hence complete crossing symmetry would be satisfied if eq. (14) as well as the relation

$$\underset{\sim}{F}(s,t) = C_{tu}\, \underset{\sim}{F}(s,u)$$

is satisfied. The above relation together with eq. (14) yield the crossing conditions

$$\int_4^\infty \mathrm{d}s' [\{g_2(s,t,s') - C_{tu} g_2(s,u,s')\}\, \underset{\sim}{A}(s',0) + \{g_3(s,t,s')\, \underset{\sim}{A}(s',t) - C_{tu}\, g_3(s,u,s')\, \underset{\sim}{A}(s',u)\}] = 0 , \tag{20}$$

valid for $|t|$ and $|u| < 4$. These relations are equivalent to, but somewhat simpler than the crossing conditions of Wanders [4] and Roskies [5]. Eqs. (14) and (20) together, are *necessary and sufficient* conditions for complete crossing symmetry; both of them can be checked directly against experiment as explained.

Integral equations for $a_l^{(I)}(s)$ *for* $4 \leq s \leq 16$. Eq. (19) expresses Re $a_l^{(I)}(s)$ for $4 \leq s \leq 60$ in terms of $a_0^{(0)}$, $a_0^{(2)}$ and the Im $a_{l'}^{(I')}(s')$ for $l' = (0,\infty)$, $I' = (0,1,2)$, and $s' \geq 4$. When this value is substituted into the unitarity equation

$$\mathrm{Im}\, a_l^{(I)}(s) = [\mathrm{Im}\, a_l^{(I)}(s)]^2 + [\mathrm{Re}\, a_l^{(I)}(s)]^2 , \qquad 4 \leq s \leq 16 , \tag{21}$$

we obtain a set of coupled non-linear singular integral equations for Im $a_l^{(I)}(s)$ in the interval $4 \leq s \leq 16$, with the driving terms involving $a_0^{(0)}$, $a_0^{(2)}$ and the Im $a_l^{(I)}(s)$ for $s \geq 16$. These integral equations resemble in a mathematical sense those studied recently by Atkinson [6] and Kupsch [7]; our equations are, however, different because we do not assume the Mandelstam representation. It would be interesting if the solution to our equations could be shown to be unique, because it would provide a "constructive proof" of the result that a knowledge of $a_0^{(0)}$, $a_0^{(2)}$, and the Im $a_l^{(I)}(s)$ for $s \geq 16$, fixes entirely the scattering amplitude *.

* This result is very similar to a result obtained in the framework of the Mandelstam representation by Martin [8], except for the fact that we need to know also the S wave scattering lengths in the present work.

Concluding remarks

i) Equation (19) is not equivalent to a partial wave dispersion relation; for example, cut-plane analyticity of $a_l(s)$ needed for the dispersion relations is not proved at present from the assumptions of axiomatic field theory [2] and does *not* follow from our equations because of the limited domain of convergence of the partial wave expansion for $A(s', t)$.

ii) Further practical and theoretical applications of the present equations and their comparison with other pion-pion equations are discussed in a parallel work by Basdevant, Le Guillou and Navelet.

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