and Intermediate Energies, Bonn, Germany, 1970, University of Florence report (unpublished).

[2]Y. Nambu, *Symmetries and Quark Models*, edited by R. Chand (Gordon and Breach, New York, 1970), p. 269; Y. Nambu, University of Chicago Report No. EFI 70-07 (unpublished); Y. Miyamoto, Progr. Theoret. Phys. (Kyoto) 42, 1166 (1969); L. Susskind, Phys. Rev. D 1, 1182 (1969); S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters 29B, 679 (1969); H. B. Nielsen, NORDITA report, 1969 (unpublished).

[3]For each conserved quantity we can make a similar construction. Further discussion on this point will be made elsewhere.

[4]Equation (11a) is equivalent, up to a divergence, to the Pauli-type interaction

$$L'_{\rm em}(\zeta) = \tfrac{1}{2}\chi(\zeta)\sigma_{\mu\nu}(Z)F_{\mu\nu}(Z), \quad \sigma_{\mu\nu}(Z) = \frac{\partial(Z_\mu, Z_\nu)}{\partial(\tau, \xi)},$$

where $\sigma_{\mu\nu}$ represents the surface element of the "rubber membrane" embedded in the external space. Dirac used this form with χ = const in his discussion of magnetic monopoles [P.A.M. Dirac, Phys. Rev. 74, 818 (1948)].

[5]It is a nonlocal substitution in the internal space. Therefore we may treat it only in perturbation theory.

[6]The normalization of $\bar{H}_0$ is chosen in such a way that the canonical momentum $p_{0\mu}$ coincides with the physical momentum p_μ and the vertex (20) gives a trajectory slope of unity.

[7]L. N. Chang, P. G. O. Freund, and Y. Nambu, Phys. Rev. Letters 24, 628 (1970). The relation (25) is obtained from a steepest-descent estimate. See Appendix B.

[8]Equation (35) is to be considered as representing the smoothed out average of cross sections around N.

[9]S. Drell and T. M. Yan, Phys. Rev. Letters 24, 181 (1970).

[10]E. D. Bloom *et al.*, report presented to the Fifteenth International Conference on High-Energy Physics, Kiev, U.S.S.R., 1970 [SLAC Report No. SLAC-PUB-796 (unpublished)].

[11]J. Ballam *et al.*, Phys. Rev. Letters 21, 1544 (1968); 23, 498 (1969).

[12]If we make the ansatz $W(\xi_i) = W(0)$ for all the quarks in the baryon, we get the preceding results for e-p scattering.

[13]A. Hacinliyan (private communication); Fong-Ching Chen, Phys. Letters 34B, 625 (1971).

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Inverse-Amplitude (K-Matrix) Method for Unitarizing Veneziano Amplitudes, with Application to $\pi\pi$ Scattering*

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The so-called K-matrix method for unitarizing Veneziano amplitudes is analyzed and shown to consist of proposing models for the differences between inverses of physical partial-wave amplitudes and the inverses of Veneziano partial-wave amplitudes. A precise formulation of the method is given, and a detailed analysis is made of the implications of analyticity, elastic and inelastic unitarity, crossing symmetry, Pomeranchukon exchange, and Regge asymptotic behavior. The difficulty of avoiding ghosts within such a framework is discussed, and the positions and residues of several ghosts present in a unitary model proposed earlier by Lovelace are given. It is argued that ghosts are likely to be present in a substantial number of the models for hadronic scattering amplitudes which exist in the literature. From our detailed discussion of the aforementioned topics, we are led to propose a set of equations for inverse partial-wave amplitudes whose solutions may afford an accurate description of $\pi\pi$ elastic scattering for center-of-mass energies up to 1 GeV or more. A numerical procedure for obtaining approximate solutions to the equations is proposed, but the actual computation of solutions is deferred to a later work.

I. INTRODUCTION

Although formulas of the type proposed by Veneziano[1] have many features of scattering amplitudes, they fail to satisfy the relations between real and imaginary parts of amplitudes implied by unitarity. As Lovelace[2] has pointed out, one way to obtain unitary amplitudes from the Veneziano formula is to interpret Veneziano amplitudes as elements of a K matrix.[3] However, a strict K-matrix interpretation of Veneziano amplitudes is not tenable, and the loose interpretation proposed by Lovelace was acknowledged[2] to be inconsistent with crossing symmetry.

Several authors[4] have attempted to improve agreement between crossing symmetry and unitarized amplitudes resulting from so-called K-matrix methods by modifying the original formulas of Lovelace. These efforts have met with varying degrees of success. However, no precise formulation of a K-matrix method for unitarizing Veneziano amplitudes has appeared heretofore in the literature, and the absence of such a formulation has precluded any systematic study of the strengths and weaknesses of such a method. Thus the first goal of our work is to develop a precise formulation of the so-called K-matrix method for unitarizing Veneziano amplitudes. The formulation which we shall propose is sufficiently general to encompass most of the work in the literature. For the sake of definiteness and because the reaction is of inherent interest, we concentrate attention on the $\pi\pi$ elastic scattering amplitudes.

Early in the course of our analysis, it will become apparent that the so-called K-matrix method for unitarizing Veneziano amplitudes actually consists of studying and proposing models for the inverses of partial-wave amplitudes. Therefore, both accuracy and clarity would be served by calling this method the "inverse-amplitude method" for unitarizing Veneziano amplitudes. Henceforth we shall do so.

After presenting our formulation of the inverse-amplitude method for unitarizing Veneziano amplitudes, we proceed to a systematic discussion of elastic and inelastic unitarity, analyticity, crossing symmetry, Pomeranchukon exchange, and asymptotic behavior. Our investigation yields a substantial number of new results, some of which concern ambiguities and practical difficulties that are inherent to the inverse-amplitude method.[5]

After completing our general discussion, we propose a set of equations whose solutions may afford an accurate description of $\pi\pi$ elastic scattering below about 1 GeV. A method for obtaining approximate solutions to these equations is proposed, but the attempt to solve these equations is deferred to a forthcoming paper.

II. FORMALISM

Consider the following Veneziano representation for $\pi\pi$ elastic scattering amplitudes[6]:

$$A^0 = \tfrac{1}{2}F(t,u) - \tfrac{3}{2}[F(s,t)+F(s,u)],$$
$$A^1 = F(s,u) - F(s,t), \qquad (2.1)$$
$$A^2 = -F(t,u),$$

where the superscript on A^I denotes s-channel isospin,

$$F(x,y) \equiv \beta\frac{\Gamma(1-\alpha(x))\Gamma(1-\alpha(y))}{\Gamma(1-\alpha(x)-\alpha(y))} + \text{secondary terms},$$

$$\alpha(x) \equiv a + bx,$$

where a and b are real. We shall use units wherein $m_\pi = \hbar = c = 1$ (except where MeV is explicitly stated), and we use the convenient energy-squared variable

$$\nu \equiv |\vec{k}_{\text{c.m.}}|^2 = \tfrac{1}{4}(s-4).$$

We normalize the A^I such that if they were unitary, their partial waves would satisfy the representation

$$A^{(l)I}(\nu) = R_l^I(\nu)(1+1/\nu)^{1/2}e^{i\delta_l^I}\sin\delta_l^I \qquad (2.2)$$

for $\nu > 0$, where R_l^I is the ratio of elastic to total partial-wave cross sections, and the phase shifts δ_l^I are real. The existence of the representation (2.2) is equivalent to the unitarity relation[7]

$$\text{Re}A^{(l)I} = \{\text{Im}A^{(l)I}[R_l^I(1+1/\nu)^{1/2} - \text{Im}A^{(l)I}]\}^{1/2}, \qquad (2.3)$$

with δ_l^I defined as

$$\delta_l^I = \tfrac{1}{2}\sin^{-1}\{2\,\text{Re}A^{(l)I}[R_l^I(1+1/\nu)^{1/2}]^{-1}\}. \qquad (2.4)$$

As a final remark on notation, we shall denote partial-wave projections of the Veneziano amplitudes (2.1) by $V^{(l)I}$.

The Veneziano partial waves $V^{(l)I}$ are real when $\nu > 0$ (except that $V^{(l)I}$ contains poles for $I=0$ and 1, and such poles can be regarded as δ functions in $\text{Im}V^{(l)I}$). In order to obtain unitary amplitudes from the $V^{(l)I}$, Lovelace has proposed[2] that the $V^{(l)I}$ be interpreted as partial waves $K^{(l)I}$ of a K matrix:

$$K^{(l)I}(\nu) = V^{(l)I}(\nu). \qquad (2.5)$$

The usefulness of this interpretation lies in the fact that amplitudes generated from a K matrix are automatically unitary. Therefore, let us consider the K-matrix formalism.[3]

For physical values of s, t, and u, the K matrix is unambiguously defined[3] as an inner product between standing-wave states, and the relation between the K matrix and the S matrix is free of ambiguity. To define the K matrix for unphysical values of s, t, and u, an analytic continuation is necessary, and there are two standard ways to define the continuation.[3] One definition leads to a K matrix which is discontinuous at every threshold, while the other definition results in a continuous K matrix. Since the $V^{(l)I}$ are analytic at threshold, we shall restrict attention to the second definition, according to which $A^{(l)I}$ and $K^{(l)I}$ are related by

$$A^{(l)I} = \frac{K^{(l)I}}{1+\rho K^{(l)I}}, \qquad (2.6)$$

where ρ is a completely unambiguous analytic matrix function of the energy. In a space of two-particle states, ρ can be diagonalized in such a way that its elements above threshold are proportional to the phase space for the outgoing particles. With amplitudes normalized in accordance with (2.1), the element of ρ between $\pi\pi$ states is unambiguously given by

$$\rho = -i\left(\frac{\nu}{\nu+1}\right)^{1/2}. \qquad (2.7)$$

Since $\rho(-1)=\infty$ and $V^{(l)I}(-1)\neq 0$, we see that $A^{(l)I}$ constructed in accordance with Eqs. (2.5)–(2.7) all vanish at $\nu=-1$. Such behavior constitutes a gross violation of crossing symmetry and is not acceptable. This difficulty of all amplitudes vanishing at $\nu=-1$ could be circumvented by assuming that $K^{(l)I}=(\nu+1)^{1/2}V^{(l)I}$, but the resulting $A^{(l)I}$ would have resonance widths which differed from those of Veneziano by the factor $(\nu+1)^{1/2}$. Then finite-energy sum rules would not be satisfied, and the unitarization would destroy the main virtue of the Veneziano model.

For the preceding reasons, we share the view of Lovelace[2] and subsequent workers[4] that a literal K-matrix interpretation of Veneziano amplitudes is not tenable. However, these authors have demonstrated that it is possible to unitarize Veneziano amplitudes by a method which utilizes an equation bearing a *superficial* resemblance to Eq. (2.6). In order to obtain a precise formulation of the method which has been used in practice, let us consider the set of functions $\rho_l^I(\nu)$ defined by

$$\rho_l^I \equiv \frac{1}{A^{(l)I}} - \frac{1}{V^{(l)I}}. \qquad (2.8)$$

Since $A^{(l)I}$ and $V^{(l)I}$ are real analytic functions, it follows that ρ_l^I is analytic except for branch cuts generated by those in $A^{(l)I}$ and $V^{(l)I}$, except for possible poles generated by zeros in $A^{(l)I}$ and/or $V^{(l)I}$. Furthermore, ρ_l^I has the reflection property, $\rho_l^I(\nu^*)=[\rho_l^I(\nu)]^*$.

It is a well-known consequence of the unitarity relations (2.2)–(2.4) that $A^{(l)I}$ satisfies elastic unitarity if and only if $\mathrm{Im}(1/A^{(l)I})=-(1+1/\nu)^{-1/2}$ for $\nu>0$ [except that $1/A^{(l)I}$ *contains poles* wherever $\delta_l^I=n\pi$, and such poles can be regarded as δ functions in $\mathrm{Im}(1/A^{(l)I})$]. Since $1/V^{(l)I}$ is real for $\nu>0$, it follows from the definition (2.8) that $A^{(l)I}$ satisfies elastic unitarity if and only if

$$\mathrm{Im}\,\rho_l^I = -(1+1/\nu)^{-1/2} \qquad (2.9)$$

for $\nu>0$ (we choose *not* to regard poles in ρ_l^I as δ functions in $\mathrm{Im}\,\rho_l^I$). Since the definition (2.8) implies

$$A^{(l)I} = \frac{V^{(l)I}}{1+\rho_l^I V^{(l)I}}, \qquad (2.10)$$

we conclude that unitary $A^{(l)I}$ can be obtained by inserting any ρ_l^I with the right cut (2.9) into Eq. (2.10). Conversely, *Eqs. (2.9) and (2.10) contain no information except a statement of elastic unitarity* unless $\mathrm{Re}\rho_l^I$ is specified, since arbitrary $A^{(l)I}$ can be generated from Eq. (2.10) by defining ρ_l^I in accordance with Eq. (2.8).

The inverse-amplitude method for unitarizing Veneziano amplitudes consists of analyzing and proposing models for the functions ρ_l^I defined by Eq. (2.8).[8] We shall examine in detail the implications of analyticity, crossing symmetry, Pomeranchukon exchange, and various aspects of asymptotic behavior for the functions ρ_l^I, but we shall postpone these discussions until we have considered the problem of inelasticity.

III. INELASTICITY

Although $\pi\pi$ scattering is almost entirely elastic below 1 GeV, it seems likely that $\pi\pi\to K\bar{K}$ becomes important in the $I=0$ S wave immediately above the $K\bar{K}$ threshold at 1 GeV, and inelasticity may become important in other partial waves above 1 GeV.

One way to couple the $\pi\pi$ and $K\bar{K}$ channels is to construct Veneziano formulas for $\pi\pi\to K\bar{K}$ and $K\bar{K}\to K\bar{K}$, and to regard (2.8) and (2.10) as matrix relations. A coupled-channel unitarization of this type has been carried out by Lovelace,[2] who assumed that ρ_l^I is a diagonal matrix whose $\pi\pi$ and $K\bar{K}$ elements are given by a particular pair of simple functions independent of l and I. In this way Lovelace obtained a set of unitary amplitudes wherein the $I=0$ $\pi\pi$ S wave resonates twice below 1 GeV, despite the fact that $V^{(0)0}$ contains only one pole below 1.25 GeV.[9] While there is some evidence for a second $I=0$ $\pi\pi$ resonance below 1 GeV,[10] we do not regard the evidence as sufficiently firm to be conclusive. If such a resonance does not exist, it would simply mean that Lovelace's model for the matrix ρ_l^I is not correct, since any coupled-channel S matrix can be represented by $A^{(l)I}$ of the form (2.10) if ρ_l^I is defined by (2.8).

Although the effects of inelasticity on $\pi\pi$ elastic amplitudes $A^{(l)I}$ can be dealt with by regarding (2.8) and (2.10) as matrix relations, it is clearly not necessary to do so, since arbitrary $A^{(l)I}$ can be expressed in the form (2.10). If (2.8) and (2.10) are regarded as scalar relations for $\pi\pi$ elastic amplitudes $A^{(l)I}$, the inelastic unitarity condition (2.3) will be satisfied if and only if the right cut of ρ_l^I is given by

$$\mathrm{Im}\,\rho_l^I = -[R_l^I(1+1/\nu)^{1/2}]^{-1}. \qquad (3.1)$$

(Again we choose not to regard poles in ρ_l^I as δ functions in $\mathrm{Im}\,\rho_l^I$.) Since $A^{(l)I}=0$ whenever $\delta_l^I=n\pi$ for $\nu>0$, a second resonance in $A^{(0)0}$ below 1 GeV would imply that ρ_0^0 has a pole[11] at the energy where $\delta_0^0=\pi$ or where $\delta_0^0=2\pi$ (or at both energies if neither coincides with a zero in $V^{(0)0}$). In order to limit our efforts, we shall restrict the remainder of our discussion to the case where (2.8) and (2.10) are regarded as scalar relations, and the right cut of ρ_l^I is given by Eq. (3.1).

IV. LEFT CUTS OF ρ_l^I

The nearest left cut of $A^{(l)I}$ begins at $\nu=-1$, which corresponds to the elastic threshold for crossed-channel reactions. There are also branch points further to the left in $A^{(l)I}$ corresponding to inelastic thresholds in the crossed channels. The nearest left cut of $V^{(l)I}$ begins at $-1-\nu_\rho$, and is generated by exchange of the ρ tower of resonances. There are also branch points further to the left in $V^{(l)I}$, generated by exchanges of higher towers of resonances. Thus it follows from the definition (2.8) that ρ_l^I has a branch point at $\nu=-1$, and additional branch points further to the left. If $A^{(l)I}$ is small near $\nu=-1$, as is usually assumed to be the case,[12] then $\operatorname{Im}\rho_l^I$ can be quite substantial along the near part of the left cut, since $\operatorname{Im}\rho_l^I=-\operatorname{Im}A^{(l)I}/|A^{(l)I}|^2$ for $(-1-\nu_\rho)<\nu<-1$.

In principle, analyticity and crossing symmetry enable one to express $A^{(l)I}(\nu)$ for $\nu<-1$ in terms of crossed-channel amplitudes $A^{I'}(\nu',\cos\theta')$ with $\nu'>-1$. In practice, one is usually forced to rely upon Legendre series expansions of the crossed-channel amplitudes in order to make the analytic continuation in $\cos\theta'$. One then obtains the formula

$$A^{(l)I}(\nu)=-\frac{1}{\nu}\int_{-1}^{-\nu-1}d\nu' P_l\left(1+2\frac{\nu'+1}{\nu}\right)\sum_{I'l'}\alpha_{II'}(2l'+1)[\operatorname{Re}A^{(l')I'}(\nu')-2i\operatorname{Im}A^{(l')I'}(\nu')]P_{l'}\left(1+2\frac{\nu+1}{\nu'}\right),\quad(4.1)$$

where

$$\alpha_{II'}\equiv\begin{pmatrix}\frac{1}{3} & 1 & \frac{5}{3}\\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6}\\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{pmatrix}.$$

The Legendre series for $\operatorname{Im}A^{I'}(\nu',\cos\theta')$ is convergent for values of ν' within the range of integration if $\nu>-9$, so Eq. (4.1) is valid for $\operatorname{Im}A^{(l)I}(\nu)$ if $\nu>-9$.[7] However, the Legendre series for $\operatorname{Re}A^{I'}(\nu',\cos\theta')$ diverges over part of the range of integration for all $\nu<-1$. Since $\operatorname{Im}\rho_l^I$ depends on $\operatorname{Re}A^{(l)I}$ as well as on $\operatorname{Im}A^{(l)I}$, it follows that Eq. (4.1) does not enable one to express $\operatorname{Im}\rho_l^I$ along any part of the left cut solely in terms of $A^{(l)I}(\nu)$ with $\nu>0$ and the known functions $V^{(l)I}$. While this failure of Eq. (4.1) to provide a representation for $\operatorname{Im}\rho_l^I$ along the left cut is unfortunate, it certainly does not preclude the possibility of using dispersion relations to construct the ρ_l^I. We shall return later to a further discussion of this point.

V. POLES IN ρ_l^I

If either $A^{(l)I}$ or $V^{(l)I}$ vanishes like $(\nu-\bar\nu)^n$ at some point $\bar\nu$, it follows from the definition (2.8) that ρ_l^I has an nth-order pole at $\bar\nu$, unless the values and the first n derivatives of $A^{(l)I}$ and $V^{(l)I}$ coincide at $\bar\nu$.

Along the real axis, $A^{(l)I}$ vanishes linearly at every point above threshold where $\delta_l^I=m\pi$. For $I=0$ and 1, $V^{(l)I}$ vanishes linearly at some point between each pair of adjacent resonance poles. The derivative at each such zero is closely related to the residues of the adjacent poles.

The Adler self-consistency condition suggests that $A^{(0)0}$ and $A^{(0)2}$ vanish linearly slightly below threshold.[12] If one keeps only the leading term of the Veneziano series (2) and incorporates the Adler zero by using the Lovelace values $a=0.483$, $b=0.017$, then $V^{(0)0}$ vanishes at $\nu=0.88$, and $V^{(0)2}$ vanishes at $\nu=-0.50$. The first derivative of each $V^{(0)I}$ is approximately given below threshold by a sum rule whose integrand corresponds to pure $I=1$ exchange, so each $V^{(0)I}$ has a reasonable value for its derivative at the zero below threshold despite the absence of a Pomeranchuk singularity in the Veneziano amplitudes (see Secs. VII and VIII).

Another type of zero in $A^{(l)I}$ and $V^{(l)I}$ results from the fact that for $l\geq1$, both $A^{(l)I}$ and $V^{(l)I}$ vanish like ν^l as $\nu\to0$. The lth derivative at threshold is proportional to the scattering length,[13] and all scattering lengths for $l\geq1$ are given by rapidly convergent dispersion integrals.[14]

Next we consider the possibility of complex zeros. It is straightforward to prove theorems which relate the asymptotic behavior of analytic functions to the number of zeros on the physical sheet.[15,16] On the basis of certain such theorems and a study of the functions $A^{(l)I}$ and $V^{(l)I}$, it has been conjectured[16] that $A^{(l)I}$ with $I=0$ and 2 contain infinitely many complex zeros on the physical sheet, and that $V^{(l)I}$ with $I=0$ and 2 contain infinitely many complex zeros at angles near $\theta=\pm\frac{1}{2}\pi$ on the physical sheet, with a unique accumulation point at infinity. For the leading term of the Veneziano series (2.1) and the values $a=0.483$, $b=0.017$, the locations of the nearest complex zeros in $V^{(0)I}$ and $V^{(2)I}$ for $I=0$ and 2 have been determined[16] by explicit computation of these functions at a set of closely spaced mesh points spanning a region of the ν plane centered about the origin. A similar computation of $V^{(1)1}$ over the regions $|\nu|<50$, $0<\theta<\pi$, and $50\leq|\nu|<150$, $0<\theta<\frac{3}{4}\pi$, reveals no zeros within these regions. However, for $\text{Im}\nu > 0$ near $\tilde{\nu} = -8.10$, $V^{(1)1}$ behaves like $\beta(-0.111 + 0.682i)(\nu - \tilde{\nu})$ so $1/V^{(1)1}$ contains a pole[17] at $\tilde{\nu}$ with residue equal to $-0.23\beta^{-1}$, where $\beta^{-1} \cong 2.0$ for $\Gamma(\rho) \cong 125$ MeV.[18]

Any crossing-symmetric unitarization of the $V^{(l)I}$ should modify to some extent the amplitudes and/or their derivatives everywhere, as should Pomeranchukon exchange. Thus the zeros and/or derivatives of $A^{(l)I}$ should not coincide exactly with those of $V^{(l)I}$, and poles should exist in ρ_l^I wherever $A^{(l)I}$ or $V^{(l)I}$ vanishes. On the other hand, it is clear that to whatever extent the zeros of $A^{(l)I}$ and $V^{(l)I}$ coincide, the poles in ρ_l^I will occur in closely spaced pairs. To whatever extent the relevant derivatives of $A^{(l)I}$ and $V^{(l)I}$ are equal at their respective zeros, the residues will be equal but opposite. This potential pairing of poles with attendant cancellations obviously enhances the tendency for the contributions of distant poles to be nearly constant. However, the extent of such cancellations is not known at present.

None of the models previously proposed[4] for the ρ_l^I have contained any poles (except for some along the negative real axis, where they were intended to mimic left cuts). Thus we conclude that a major dynamical assumption inherent to all such models is that the zeros of $A^{(l)I}$ coincide in one-to-one fashion with those of $V^{(l)I}$, and that the first n derivatives of $A^{(l)I}$ coincide with those of $V^{(l)I}$ at each zero, where n is the order of the lowest nonvanishing derivative of $V^{(l)I}$ at the position of the zero. In all such models, the $A^{(l)I}$ have the same resonance structure as the $V^{(l)I}$ (subject to the resonance poles having moved onto the second sheet). In addition, the $A^{(l)I}$ are approximately equal to the $V^{(l)I}$ near threshold, since every $V^{(l)I}$ vanishes either at or slightly below threshold[12] (assuming the usual incorporation of the Adler zero). However, even when ρ_l^I contains no poles, $A^{(l)I}$ is quite sensitive to the details of ρ_l^I if ν is not near a zero of $V^{(l)I}$. Thus for example the $I = 2$ S wave $A^{(0)2}$ is quite sensitive above threshold to the value of ρ_0^2, since $V^{(0)2}$ contains no zeros above threshold.

Next we shall discuss a matter of great practical importance, namely, the possibility that specific models for ρ_l^I may imply unphysical singularities in $A^{(l)I}$.

VI. GHOSTS

If there exists a solution to the equation

$$1 + \rho_l^I V^{(l)I} = 0 \tag{6.1}$$

at some point where $V^{(l)I} \neq 0$, then it follows from Eq. (2.10) that $A^{(l)I}$ has a pole at that point. A pole in $A^{(l)I}$ on the interval $-1 < \nu < 0$ would correspond to a bound state, but a pole anywhere else on the physical sheet would have no physical interpretation and would be a "ghost." Since there exist no bound states in the $\pi\pi$ system, we conclude that $\rho_l^I \neq -1/V^{(l)I}$ at every point on the physical sheet where $V^{(l)I} \neq 0$.

Unfortunately, no search for ghosts has heretofore been carried out over any part of the physical sheet for any model yet proposed for the ρ_l^I, and there is no reason to believe that any of the models is free of ghosts.

The Lovelace model for ρ_l^I is

$$\rho_l^I = -\frac{2\nu + 1}{\pi} \int_0^\infty d\nu' \, \frac{(1 + 1/\nu')^{-1/2}}{(\nu' + \nu + 1)(\nu' - \nu)} \,. \tag{6.2}$$

Assuming that ρ_l^I is given by Eq. (6.2), the present author has searched for ghosts over the region $-100 \leq \text{Re}\nu \leq 100$, $0 < \text{Im}\nu \leq 100$. Keeping only the leading term in the Veneziano series (2.1) and using the Lovelace values $a = 0.483$, $b = 0.017$, $\beta = 0.62$,[19] one finds two ghosts in $A^{(0)0}$ and one ghost in $A^{(0)2}$ in the aforementioned region. The ghosts in $A^{(0)0}$ are at $\nu = -7.3 + 68i$ with residue $-0.057 - 17i$, and at $\nu = 21 + 0.20i$ with residue $0.15 - 0.11i$.[9] The ghost in $A^{(0)2}$ is at $\nu = -4.6 + 5.8i$ with residue $-2.7 - 5.9i$. Of course there are also ghosts at conjugate points with conjugate residues. The contributions of the ghosts in $A^{(0)0}$ all vary quite slowly near threshold, but the pair of ghosts in $A^{(0)2}$ contributes a term to $A^{(0)2}$ which is roughly given by $0.78 - 0.23\nu$ near threshold. Thus the energy dependence of $A^{(0)2}$ is strongly influenced by a ghost in Lovelace's model, and the energy dependence of δ_0^2 is therefore not reliable.[20]

A plausibility argument can be made that great care must be taken if the generation of ghosts is to be avoided when using the inverse-amplitude method. One begins by noting that the unitarity relation (2.3) is a highly stringent condition for a real analytic function to satisfy. If one imposes unitarity on some function by "brute force" without simultaneously modifying the left cut in an appropriate way, additional singularities are likely to be required in order to make the real part of the function satisfy Eq. (2.3). In the inverse-amplitude method unitarity is imposed by brute force, and great care must be taken in constructing the left cuts if ghosts are to be avoided. The only *a priori* guide we have for constructing left cuts in such a way as to avoid ghosts stems from the fact that in nature, left cuts are related to right cuts by crossing symmetry. However, crossing symmetry is certainly not a sufficient condition to guarantee the absence of ghosts. In every calculation resulting in exactly unitary strong-interaction amplitudes of which the present author is aware (including N/D calculations), unitarity has been imposed by brute force, and each such model should be suspected of containing ghosts until it is proven to be free of ghosts.

For example, in addition to the aforementioned ghosts in Lovelace's model, the $I=2$ $\pi\pi$ S wave of Brown and Goble[21] has been examined and found to contain an important ghost.[22] Also, the $\pi\pi$ S waves of Chew, Mandelstam, and Noyes[23] have been tested for ghosts by Chung, who found that the ratios N/D did not satisfy the dispersion relation assumed for A when $\lambda \geq 0.1$.[24]

It is highly unfortunate that so few of the authors who have published exactly unitary amplitudes have searched for ghosts, since it is impossible to judge the significance of the results of a model unless one knows whether the model contains significant ghosts.

In Sec. VII, we shall discuss certain equations whose derivation is based on crossing symmetry alone, and other equations whose derivation depends on analyticity as well as crossing symmetry. If some set of amplitudes satisfies the equations based only on crossing symmetry but violates an equation based on analyticity as well as on crossing symmetry, the presence of ghosts is a strong possibility.

VII. CROSSING SYMMETRY

We are now ready to discuss some of the implications of crossing symmetry for the $\pi\pi$ elastic amplitudes $A^{(l)I}$. Since our unitarization procedure applies to partial waves rather than the full amplitudes, we shall restrict attention to those consequences of crossing symmetry which apply directly to partial waves. We shall further limit our discussion by restricting attention to the S waves and P wave.

It has recently been shown by Roskies[25] that crossing symmetry together with analyticity over the Mandelstam triangle implies an infinite set of equations which interrelate weighted integrals of partial waves over the interval $-1 \leq \nu \leq 0$. The equations which involve only S waves are

$$\int_{-1}^{0} d\nu\, \nu A^{(0)0} = \tfrac{5}{2}\int_{-1}^{0} d\nu\, \nu A^{(0)2}, \tag{7.1}$$

$$\int_{-1}^{0} d\nu\,(3\nu+2)\nu A^{(0)0} = -2\int_{-1}^{0} d\nu\,(3\nu+2)\nu A^{(0)2}. \tag{7.2}$$

There are three independent integral crossing equations which involve only the S waves and P wave. The first of these is

$$\int_{-1}^{0} d\nu\,(3\nu+2)\nu A^{(0)0} = 2\int_{-1}^{0} d\nu\, \nu^2 A^{(1)1} \tag{7.3}$$

which, when combined with Eq. (7.2), implies that

$$\int_{-1}^{0} d\nu\,(3\nu+2)\nu A^{(0)2} = -\int_{-1}^{0} d\nu\, \nu^2 A^{(1)1}. \tag{7.4}$$

The remaining integral crossing relations which involve only the S waves and P wave are

$$\int_{-1}^{0} d\nu\,(10\nu^2+12\nu+3)\nu(2A^{(0)0}-5A^{(0)2}) = -6\int_{-1}^{0} d\nu\,(5\nu+4)\nu^2 A^{(1)1}, \tag{7.5}$$

$$\int_{-1}^{0} d\nu\,(35\nu^3+60\nu^2+30\nu+4)\nu(2A^{(0)0}-5A^{(0)2}) = 15\int_{-1}^{0} d\nu\,(21\nu^2+30\nu+10)\nu^2 A^{(1)1}. \tag{7.6}$$

To gain insight into the content of Eqs. (7.1)–(7.4), let us suppose that the S waves and P wave are linear:

$$A^{(0)I}(\nu) = a_I + b_I\nu, \quad I=0,2$$
$$A^{(1)1}(\nu) = a_1\nu. \tag{7.7}$$

When Eqs. (7.1)–(7.4) are imposed on the linear amplitudes (7.7), it follows that

$$a_0 - \tfrac{2}{3}b_0 = \tfrac{5}{2}(a_2 - \tfrac{2}{3}b_2), \tag{7.8}$$

$$b_0 = -2b_2, \tag{7.9}$$

$$b_0 = 6a_1, \tag{7.10}$$

$$b_2 = -3a_1. \tag{7.11}$$

If the full amplitudes $A^I(\nu, \cos\theta)$ were linear in ν and $\cos\theta$ over the Mandelstam triangle, they would contain only S waves and a P wave of the form (7.7):

$$A^I(\nu,\cos\theta) = a_I + b_I\nu, \quad \text{for} \quad I=0,2$$
$$A^1(\nu,\cos\theta) = 3a_1\nu\cos\theta. \tag{7.12}$$

For the linear amplitudes (7.12), it is trivial to verify that Eqs. (7.1)–(7.4) have precisely the same content as the Chew-Mandelstam crossing relations[26]

$$A^0|_{\text{sym. pt.}} = \tfrac{5}{2}A^2|_{\text{sym. pt.}},$$
$$\left.\frac{\partial A^0}{\partial\nu}\right|_{\text{sym. pt.}} = -2\left.\frac{\partial A^2}{\partial\nu}\right|_{\text{sym. pt.}} = -3\left.\frac{\partial A^1}{\partial\cos\theta}\right|_{\text{sym. pt.}}, \tag{7.13}$$

where the "symmetry point" is characterized by $\nu=-\frac{2}{3}$, $\cos\theta=0$. Thus Eqs. (7.1)–(7.4) primarily constrain the zeroth and first derivatives of the amplitudes below threshold. It is straightforward to verify that the left- and right-hand sides of Eqs. (7.5) and (7.6) are independent of constant[27] and linear terms in the amplitudes, and that Eq. (7.6) is independent of quadratic terms as well.

Assuming crossing symmetry, the validity of the Froissart-Gribov representation for partial waves with $l \geq 2$, and the positivity of absorptive parts of partial waves with definite isospin, several authors have derived inequalities involving $\pi\pi$ partial waves on the interval $-1 \leq \nu \leq 0$. Let us denote the S wave of the $\pi^0\pi^0 \to \pi^0\pi^0$ amplitude by f_0:

$$f_0(\nu) \equiv \tfrac{1}{3}[A^{(0)0}(\nu) + 2A^{(0)2}(\nu)].$$

It has been deduced from the above assumptions that[28]

$$\frac{df_0}{d\nu} < 0, \quad \text{for} \quad -1 \leq \nu \leq -0.68 \tag{7.14}$$

$$\frac{df_0}{d\nu} > 0, \quad \text{for} \quad -0.57 \leq \nu \leq 0 \tag{7.15}$$

$$\frac{d^2 f_0}{d\nu^2} > 0, \quad \text{for} \quad -1 \leq \nu \leq -0.57 \tag{7.16}$$

$$f_0(-0.95) < f_0(-0.20), \tag{7.17}$$

$$f_0(-0.25) < f_0(-0.95), \tag{7.18}$$

$$f_0(-1) > 2\int_{-1/2}^{0} d\nu\, f_0(\nu). \tag{7.19}$$

If $A^{(0)0}$ and $A^{(0)2}$ were linear over the interval $-1 \leq \nu \leq 0$ and satisfied Eq. (7.2) [or alternatively, Eq. (7.9)], then f_0 would be constant over the interval $-1 \leq \nu \leq 0$, and the relations (7.14)–(7.19) would be marginally satisfied as equalities. Thus the constraints which the inequalities (7.14)–(7.19) add to Eq. (7.2) concern only quadratic and higher-order terms.

It has been deduced from the same assumptions which lead to (7.14)–(7.19) that[29]

$$\int_{-1}^{0} d\nu\, A^{(0)0} \leq \int_{-1}^{0} d\nu\, A^{(0)2} + \tfrac{3}{2}A^{(0)2}(-1). \tag{7.20}$$

If $A^{(0)0}$ and $A^{(0)2}$ were linear over the interval $-1 \leq \nu \leq 0$ and satisfied Eqs. (7.1) and (7.2) [or alternatively, Eqs. (7.8) and (7.9)], then the inequality (7.20) would be marginally satisfied as an equality. Thus the constraint which the inequality (7.20) adds to Eqs. (7.1) and (7.2) concerns only quadratic and higher-order terms.

It is also possible to derive inequalities for D and F waves and then use crossing symmetry to deduce consequences for the S waves.[30] However, these inequalities are rather complicated and obviously constrain only quadratic and higher-order terms, and we choose not to enumerate them here.

When combined with analyticity and basic assumptions of Regge theory, crossing symmetry implies that the s-channel amplitudes which are pure $I=1$ and $I=2$ in the t channel satisfy unsubtracted forward dispersion relations.[31] The equation for the amplitude with $I=1$ in the t channel implies that[32]

$$2a_0 - 5a_2 = \frac{1}{\pi}\int_0^\infty \frac{d\nu}{\nu(\nu+1)} \operatorname{Im}[2A_F^0(\nu) + 3A_F^1(\nu) - 5A_F^2(\nu)], \tag{7.21}$$

where A_F^I denotes the forward amplitude and $a_I \equiv A_F^I(0)$ is the S-wave scattering length for $I=0, 2$. The P-wave scattering length[13] a_1 is also given by an $I=1$ sum rule, namely[14]

$$a_1 = \frac{1}{18\pi}\int_0^\infty \frac{d\nu}{(\nu+1)^2} \operatorname{Im}[2A_F^0(\nu) + 3A_F^1(\nu) - 5A_F^2(\nu)] + \frac{1}{3\pi}\int_0^\infty d\nu\, \frac{2\nu+1}{\nu^2(\nu+1)^2} \operatorname{Im} A_F^1(\nu). \tag{7.22}$$

Note that the integral in Eq. (7.21) and the first integral in Eq. (7.22) converge rapidly in the sense that the unitarity bound on partial waves guarantees that the integral over each partial wave $A^{(l)I}$ converges like $\int_0^\infty d\nu/\nu^2$. Consequently, the integral in Eq. (7.21) and the first integral in Eq. (7.22) can be reliably determined from low-energy phase shifts, intermediate-energy resonances, and the asymptotic contribution of t-channel ρ exchange, which corresponds to the high-energy towers of Veneziano s-channel resonances. The second integral in Eq. (7.22) converges even more rapidly, and is dominated by the ρ resonance.

If one keeps only the leading term in the Veneziano amplitudes and inserts the corresponding δ-function absorptive parts into the integrand of Eq. (7.21) and the first of the two integrands in Eq. (7.22), one finds that the ϵ and ρ resonances each contribute about one third of the total values of the integrals, while the remaining third comes from resonances above 1 GeV (which, through duality, contain the asymptotic contribution of ρ Regge exchange).

Because there are no known resonances with $I=2$, it has often been assumed that $I=2$ contributions to Eqs. (7.21) and (7.22) are negligible. However, it is important to realize that the crossing matrix elements present in the integrands of Eq. (7.21) and (7.22) favor $I=2$ contributions over $I=0$ contributions by a factor of $-\frac{5}{2}$. Since $\sin^2 39° \cong \frac{2}{5}$, an $I=2$ phase shift of 39° would contribute just as strongly to the integrand as would an $I=0$ phase shift of 90°. Several of the models which have been proposed for $\pi\pi$ S waves have had δ_0^2 as large as $-39°$ in the ρ region,[33] so the $I=2$ contributions to the right-hand sides of Eqs. (7.21) and (7.22) would be quite substantial in these models. It is an important defect of these models[33] that the values for $(2a_0 - 5a_2)$ and a_1 have not reflected the large negative contributions which $\operatorname{Im} A^{(0)2}$ would make to the right-hand sides of Eqs. (7.21) and (7.22). Thus these models are substantially inconsistent with analyticity[34] and/or crossing symmetry, unless the net contributions from energies above 1 GeV are twice as large as has usually been supposed.

The forward dispersion relation with $I=2$ in the t channel implies that[35]

$$2a_0+a_2=\frac{1}{\pi}\int_0^\infty d\nu\,\frac{(2\nu+1)}{\nu(\nu+1)}\operatorname{Im}[2A_F^0(\nu)-3A_F^1(\nu)+A_F^2(\nu)]\,. \tag{7.23}$$

Since the coefficient of the absorptive part only decreases like ν^{-1}, the integral over any single partial wave will diverge (unless $\operatorname{Im}A^{(l)I}$ tends to zero as $\nu\to\infty$ faster than $\nu^{-\epsilon}$ for some $\epsilon>0$). Therefore the value of the integral in Eq. (7.23) is quite sensitive to variations in any partial wave, and it cannot be reliably determined without detailed knowledge of the amplitudes in the intermediate and high-energy regions. For example, the $g(1670)$ contributes 1.0 to the right-hand side of Eq. (7.23) for $2a_0+a_2$, despite its high mass and small partial width $\Gamma(g\to 2\pi)=40$ MeV.[36] Since we will be interested in values for $2a_0+a_2$ of the order of 0.3, *a variation of only 10 MeV in* $\Gamma(g\to 2\pi)$ *would result in a variation of nearly 100% in* $2a_0+a_2$. Because of this extreme sensitivity of the integral in Eq. (7.23) to the intermediate and high-energy regions, Eq. (7.23) will not be of any practical use in determining $2a_0+a_0$ within the foreseeable future.[37]

VIII. POMERANCHUKON EXCHANGE AND ASYMPTOTIC BEHAVIOR OF ρ_l^I

It is well known that Veneziano amplitudes contain no Pomeranchuk trajectory. The leading trajectories in the Veneziano $\pi\pi$ amplitudes (2.1) are the degenerate ρ and f_0 trajectories, so that the only s-channel amplitude which has the correct asymptotic behavior is the amplitude with $I=1$ in the t channel, namely,

$$2A^0+3A^1-5A^2\,.$$

This is precisely the combination of amplitudes which appears in the integrand of Eq. (7.21) and the first integrand of Eq. (7.22), and that is why low-energy phase shifts determine the difference between a realistic value for the integral and the pure Veneziano value. If one knows the low-energy phase shifts, intermediate-energy resonances, and asymptotic contributions of ρ Regge exchange, then realistic values for the integrals can be obtained without any explicit knowledge of the Pomeranchuk trajectory.

If one accepts the philosophy of Freund[38] and Harari,[39] then low-energy nonresonant absorptive parts correspond to Pomeranchukon exchange, within the context of duality. Since any unitarization of Veneziano amplitudes generates nonresonant absorptive parts in the low-energy region, e.g., with $I=2$, any unitarization procedure generates effects corresponding to Pomeranchukon exchange in the low-energy region.

Unfortunately, it is much more difficult to generate effects of Pomeranchukon exchange in the asymptotic region. In fact it is easy to show that for a large class of ρ_l^I, the $A^{(l)I}$ generated by Eq. (2.10) will not contain any effects of Pomeranchukon exchange in the asymptotic region. To see this, one need only note that along a ray displaced slightly from the positive real axis, every $V^{(l)I}$ tends asymptotically to zero like[40]

$$V^{(l)I}(\nu)\sim(\nu^{1-a}\ln\nu)^{-1}\,. \tag{8.1}$$

Thus it follows from Eq. (2.10) that $A^{(l)I}$ will tend asymptotically to zero as fast as (or faster than) $V^{(l)I}$ unless ρ_l^I grows asymptotically like

$$\rho_l^I\sim\nu^{1-a}\ln\nu \tag{8.2}$$

as $\nu\to+\infty$. (If ρ_l^I grows less rapidly than $\nu^{1-a}\ln\nu$, then $\operatorname{Im}A^{(l)I}$ will in fact[41] approach $\operatorname{Im}V^{(l)I}$, in the sense of local averages, as it tends to zero.) We conclude that if ρ_l^I violates the asymptotic condition (8.2), then $A^{(l)I}$ will contain no effects of Pomeranchukon exchange in the asymptotic region, and the asymptotic phase shifts will be so unphysical as to be devoid of value. Unfortunately, it is a common weakness of all models yet proposed for ρ_l^I that ρ_l^I grows less rapidly than the rate (8.2).

For $I=0$ and 2, it has been conjectured and made plausible[42] that $V^{(l)I}$ grows faster than any power as $\nu\to-\infty$. This suggests (but does not prove) that physical $A^{(l)I}$ with $I=0$ and 2 also contain essential singularities at infinity. However, putting aside the question of essential singularities, it is evident from the definition (2.8) that if $V^{(l)I}$ and $A^{(l)I}$ with $I=0$ and 2 increase without limit *at any rate whatsoever* as $\nu\to-\infty$, then

$$\lim_{\nu\to-\infty}\rho_l^0=\lim_{\nu\to-\infty}\rho_l^2=0\,. \tag{8.3}$$

Unfortunately, it is a common feature of all models yet proposed for ρ_l^I that ρ_l^0 and ρ_l^2 violate Eq. (8.3).

Recently Park and Desai[43] have shown that for any $\epsilon>0$,

$$\lim_{|\nu|\to\infty}\nu^{1-a-\epsilon}V^{(l)1}=0 \tag{8.4}$$

along any ray with $\operatorname{Im}\nu\neq 0$. It is obvious that unitarity with crossing symmetry (and also Pomeranchukon exchange) implies that the left cut of $A^{(l)1}$ is different from the left cut of $V^{(l)1}$. Therefore, one would expect that along any ray with $\operatorname{Im}\nu\neq 0$, for any $\epsilon>0$ and $\Omega>0$, there exists a ν with $|\nu|>\Omega$ such that

$$\left|\frac{\rho_l^1(\nu)}{\nu^{1-a-\epsilon}}\right| > 1\,, \tag{8.5}$$

where the value of $|\nu|$ in (8.5) may depend on the ray as well as on the values of ϵ and Ω. Unfortunately, every model yet proposed for ρ_l^1 fails to satisfy the asymptotic condition summarized by (8.5).

It should be remarked that if $A^{(l)I}$ were equal to $V^{(l)I}$ so that $\rho_l^I = 0$, then the asymptotic condition (8.3) would be satisfied, whereas the asymptotic condition (8.5) would not. Thus one would expect the asymptotic condition summarized by (8.5) to be less significant for the low-energy amplitudes than the asymptotic condition (8.3).

IX. A LOW-ENERGY MODEL FOR ρ_l^I

We are now ready to propose a set of equations which may afford an accurate description of $\pi\pi$ elastic scattering below about 1 GeV. The equations for the S waves are

$$\rho_0^I(\nu) = \rho_0^I(\nu_s) + (\nu-\nu_s)\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} d\nu' \frac{\operatorname{Im}\rho_0^I}{(\nu'-\nu_s)(\nu'-\nu)} + \frac{(dA^{(0)I}/d\nu|_{z_I})^{-1}}{(z_I-\nu_s)(\nu-z_I)} - \frac{(dV^{(0)I}/d\nu|_{\tilde z_I})^{-1}}{(\tilde z_I-\nu_s)(\nu-\tilde z_I)} + \delta_{0I}\left[\frac{(dA^{(0)0}/d\nu|_{z_\pi})^{-1}}{(z_\pi-\nu_s)(\nu-z_\pi)} - \frac{(dV^{(0)0}/d\nu|_{\tilde z_\pi})^{-1}}{(\tilde z_\pi-\nu_s)(\nu-\tilde z_\pi)}\right]\right\}, \tag{9.1}$$

where ν_s is the point at which a subtraction has been made, z_I is the point slightly below threshold where $A^{(0)I}$ vanishes because of the Adler zero,[44] $\tilde z_I$ is the corresponding point where $V^{(0)I}$ vanishes, δ_{0I} is the Kronecker delta, $\tilde z_\pi$ is the point above the ϵ resonance where δ_0^0 reaches π,[45] and $\tilde z_\pi$ is the corresponding zero[46] in $V^{(0)0}$. For the P wave, we write

$$\rho_1^1(\nu) = \rho_1^1(\nu_s) + (\nu-\nu_s)\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} d\nu' \frac{\operatorname{Im}\rho_1^1}{(\nu'-\nu_s)(\nu'-\nu)} - \frac{\eta}{(\bar\nu-\nu_s)(\nu-\bar\nu)} + \frac{(dV^{(1)1}/d\nu|_0)^{-1} - (a_1)^{-1}}{\nu_s\nu} + \frac{(dA^{(1)1}/d\nu|_{\nu_\pi})^{-1}}{(\nu_\pi-\nu_s)(\nu-\nu_\pi)} - \frac{(dV^{(1)1}/d\nu|_{\tilde\nu_\pi})^{-1}}{(\tilde\nu_\pi-\nu_s)(\nu-\tilde\nu_\pi)}\right\}, \tag{9.2}$$

where ν_s is the point at which the subtraction has been made, $\bar\nu$ and η are the position and residue, respectively, of the aforementioned pole in $1/V^{(1)1}$ on the negative real axis, ν_π is the point above the ρ resonance where δ_1^1 reaches π, and $\tilde\nu_\pi$ is the corresponding zero in $V^{(1)1}$.

It is obvious that the parameters z_π, $dA^{(0)0}/d\nu|_{z_\pi}$, ν_π, and $dA^{(1)1}/d\nu|_{\nu_\pi}$ are closely related to the masses and widths of the ϵ and ρ resonances, respectively. Of the two parameters z_I, only one is left independent by the crossing relation (7.1) of Roskies. The remaining z_I may be fixed by imposing a value for the ratio a_0/a_2 (or alternatively, by imposing the Adler zero in some specific way[44]). Of the parameters $dA^{(0)I}/d\nu|_{z_I}$ and a_1 which represent first derivatives near threshold, only one is left independent by the crossing relations (7.2) and (7.3) of Roskies, say a_1. However, a_1 is given by Eq. (7.22) in terms of integrals over absorptive parts. The only remaining discrete parameters in Eqs. (9.1) and (9.2) are the subtraction constants $\rho_0^I(\nu_s)$ and $\rho_1^1(\nu_s)$. These are constrained by the crossing relations (7.5) and (7.6) of Roskies, and also by Eq. (7.21). Thus it appears that the equations of constraint presented in Sec. VII are sufficient to determine all the discrete parameters in Eqs. (9.1) and (9.2) except for those parameters which correspond to M_ϵ, Γ_ϵ, M_ρ, Γ_ρ, and a_0/a_2.[47]

We have already remarked that if $A^{(l)I}$ is small near $\nu=-1$ (as is usually assumed to be the case[12]), then $\operatorname{Im}\rho_l^I$ can be quite substantial along the near part of the left cut. Thus a careful treatment of the left cut is clearly to be desired.

To obtain $\operatorname{Im}\rho_l^I$ for $\nu<-1$, we begin by noting that $A^{(l)I}$ is given for all ν in terms of ρ_l^I by Eq. (2.10). In addition, crossing symmetry implies that $\operatorname{Im}A^{(l)I}$ is given for $-9<\nu<-1$ in terms of $\operatorname{Im}A^{(l')I'}$ with $\nu'>0$ by Eq. (4.1). Since we have assumed that the $\operatorname{Re}\rho_l^I$ with $l=0$ and 1 are given for all ν in terms of $\operatorname{Im}\rho_l^I$ by Eqs. (9.1) and (9.2), the $\operatorname{Im}\rho_l^I$ for $\nu<-1$ are heavily constrained by the requirement that the $A^{(l)I}$ generated by Eq. (2.10) have left cuts consistent with Eq. (4.1).

In order to construct approximate solutions to Eqs. (2.10), (4.1), and (9.1) and (9.2), one could represent each $\operatorname{Im}\rho_l^I$ for $-\Lambda<\nu<-1$ by a flexible, multiparameter trial function which has the correct type of branch point at $\nu=-1$, and which depends linearly on each of its variable parameters. By choosing Λ to be sufficiently large, $\operatorname{Im}\rho_l^I$ could be ignored (set equal to zero) for $\nu<-\Lambda$.[48] Then for any specific value of ν, the integrals over the left cuts in Eqs. (9.1) and (9.2) would be definite linear functions of the trial-function parameters, with coefficients which could be computed. The integrals over the right cuts could also be computed, since $\mathrm{Im}\,\rho_l^I$ is given for $\nu>0$ by Eq. (3.1) in terms of R_l^I (which, however, must be given). Thus the right-hand sides of Eqs. (9.1) and (9.2) for ρ_l^I could be evaluated in terms of the trial-function parameters and the discrete parameters which are explicitly present.

Next we note that Eq. (2.10) implies that[49]

$$\mathrm{Re}\rho_l^I=\frac{1}{|V^{(l)I}|^2}\left\{-\mathrm{Re}V^{(l)I}\pm\left[-(\mathrm{Im}V^{(l)I})^2+|V^{(l)I}|^2\left(2\,\mathrm{Im}\,\rho_l^I\,\mathrm{Im}V^{(l)I}-(\mathrm{Im}\,\rho_l^I)^2|V^{(l)I}|^2+\frac{\mathrm{Im}V^{(l)I}-\mathrm{Im}\,\rho_l^I|V^{(l)I}|^2}{\mathrm{Im}A^{(l)I}}\right)\right]^{1/2}\right\}. \tag{9.3}$$

If we use Eq. (4.1) to determine $\mathrm{Im}A^{(l)I}$ for $\nu<-1$ in terms of $\mathrm{Im}A^{(l')I'}$ with $\nu'>0$, and require that Eqs. (9.1)–(9.3) be simultaneously satisfied over a set of closely spaced mesh points which span the near parts of the left and right cuts [say for $|\nu|\leq 12$, which corresponds to about (1 GeV)^2], then the values of the trial-function parameters are determined [assuming that R_l^I is given, and that the discrete parameters in Eqs. (9.1) and (9.2) are given or can be simultaneously determined by imposing the constraints presented in Sec. VII]. It is straightforward to establish that if ρ_l^I satisfies Eq. (9.3) while the $\mathrm{Im}A^{(l)I}$ on the right-hand side of Eq. (9.3) satisfies Eq. (4.1), then the $\mathrm{Im}A^{(l)I}$ generated by Eq. (2.10) will also satisfy Eq. (4.1).

A potential difficulty with constructing simultaneous solutions to Eqs. (4.1) and (9.1)–(9.3) stems from the fact that while Eq. (9.3) implies bounds on both $\mathrm{Im}\,\rho_l^I$ and $\mathrm{Re}\rho_l^I$, there are no bounds on $\mathrm{Re}\rho_l^I$ inherent to Eqs. (9.1) and (9.2). Thus the values of the subtraction constants $\rho_l^I(\nu_s)$ in Eqs. (9.1) and (9.2) are constrained by Eqs. (4.1) and (9.3). However, a little reflection suggests that these new constraints on the $\rho_l^I(\nu_s)$ should be consistent with (and roughly equivalent to) the constraints of analyticity and crossing expressed by Eqs. (7.5), (7.6), and (7.21). The reason is that the values and first derivatives of the S waves and P wave have already been fixed at the locations of their respective zeros near threshold by certain of the pole terms in Eqs. (9.1) and (9.2). Thus Eq. (7.21) only constrains second and higher derivatives between threshold and the locations of the zeros in $A^{(0)0}$ and $A^{(0)2}$, and we have already noted that Eqs. (7.5) and (7.6) only constrain second and higher derivatives on the interval $-1<\nu<0$. However, the second and higher derivatives of $A^{(l)I}$ should also be largely determined by the nearby singularities of $A^{(l)I}$, which are consistent with crossing symmetry and analyticity if Eq. (4.1) is satisfied (provided there are no ghosts).[50]

If we succeed in constructing unitary, crossing-symmetric S waves and P wave which are analytic and free of ghosts, then the inequalities (7.14)–(7.20) will be automatically satisfied. The reason for this is that one can incorporate such S waves and P wave into full amplitudes $A^I(\nu,\cos\theta)$ which satisfy all the assumptions upon which the inequalities are based. The higher partial waves of these full amplitudes are simply the $V^{(l)I}$ plus a correction term, where the correction term has no singularity except the left cut implied by exchange of the unitarized S waves and P wave. This point has been fully developed elsewhere,[47] so we shall not pursue it further here.

The principal dynamical assumption inherent to our model is that the ρ_l^I contain no poles except those explicitly present in Eqs. (9.1) and (9.2). We conclude from the discussion of Sec. V that except for the locations of these poles, the zeros of $A^{(l)I}$ will coincide with those of $V^{(l)I}$ in one-to-one fashion, and the first n derivatives of $A^{(l)I}$ will coincide with those of $V^{(l)I}$ at each zero, where n is the order of the lowest nonvanishing derivative of $V^{(l)I}$ at the position of the zero. This assumption is certainly not an empty one, for it has been shown[16] that $V^{(0)0}$ and $V^{(0)2}$ both contain at least two pairs of zeros at complex points on the physical sheet, and it has been conjectured[16] that $V^{(0)0}$ and $V^{(0)2}$ contain infinitely many such zeros. In addition, $V^{(0)0}$ and $V^{(1)1}$ contain infinitely many zeros on the positive real axis, since a zero must occur between each pair of adjacent resonance poles. Thus an immediate consequence of Eqs. (9.1) and (9.2) is that the resonance structures of the $A^{(l)I}$ are in semiquantitative agreement with those of the $V^{(l)I}$.[51]

If the pole structures implied by Eqs. (9.1) and (9.2) for the ρ_l^I are to be good approximations to nature, it is necessary that the $V^{(l)I}$ be good approximations to nature, at least near the points where they vanish. Thus the Regge parameters a and b and the coefficient β of the leading term in the Veneziano series must be given values which maximize agreement between the $V^{(l)I}$ and nature [except possibly near threshold, where Eqs. (9.1) and (9.2) provide considerable flexibility]. Of course, it may also be necessary to include secondary terms in the Veneziano series, if the agreement with nature is to be maximized.

We remarked in Sec. V that unitarity with crossing symmetry (as well as Pomeranchukon exchange) should imply that the zeros and/or relevant derivatives of $A^{(l)I}$ differ somewhat from those of $V^{(l)I}$. Thus we doubt the existence of *exactly* crossing-symmetric solutions to Eqs. (9.1) and (9.2). The Veneziano amplitudes are analytic and crossing-symmetric, and we have already noted that for $I=0$ and 2, the $V^{(l)I}$ appear to grow faster than any power as $\nu\to-\infty$. However, Eq. (9.1) possesses no solutions corresponding to an $A^{(0)I}$ which grows as rapidly as ν^{ϵ} as $\nu\to-\infty$. This can be seen as follows. If $A^{(0)I}$ grows as rapidly as ν^{ϵ} for some $\epsilon>0$ as $\nu\to-\infty$, then this growth together with the rapid growth of $V^{(0)I}$ implies that $\rho_0^I\to 0$ as $\nu\to-\infty$, and that the term involving the integral over the left cut in Eq. (9.1) tends to a constant as $\nu\to\pm\infty$. The integral over the right cut depends on R_l^I, which is not presently known. However, the rigorous bounds $0\leq R_l^I\leq 1$ imply that the term generated by the right cut in Eq. (9.1) grows at least logarithmically as $\nu\to-\infty$, so we conclude that Eq. (9.1) does not possess solutions ρ_0^I which tend to zero as $\nu\to-\infty$, in contradiction of the assumed growth of $A^{(0)I}$ and the rapid growth of $V^{(0)I}$ as $\nu\to-\infty$. We also conclude that if $A^{(0)I}$ does in fact grow as rapidly as ν^{ϵ} as $\nu\to-\infty$ for some $\epsilon>0$, then infinitely many pole terms must be added to the right-hand side of Eq. (9.1) in order for it to be valid.

Another probable deficiency of Eq. (9.1) is that it is incompatible with Pomeranchuk asymptotic behavior, unless $A^{(0)I}$ tends to zero at least as rapidly as $\nu^{-\epsilon}$ as $\nu\to-\infty$, or unless R_l^I tends to zero like $\nu^{a-1}/\ln\nu$ as $\nu\to\infty$. This can be seen by comparing the asymptotic condition (8.2) with the asymptotic behavior of the integrals in Eq. (9.1). The rigorous bounds $0\leq R_l^I\leq 1$ imply that the real part of the term involving the integral over the right cut in Eq. (9.1) does not grow more rapidly than $\ln\nu$ as $\nu\to+\infty$. The imaginary part of this term is simply $-[R_l^I(1+1/\nu)^{1/2}]^{-1}$, and behaves asymptotically like $1/R_l^I$. The term generated by the integral over the left cut in Eq. (9.1) cannot grow as rapidly as $\nu^{1-a}\ln\nu$ as $\nu\to+\infty$ unless $\mathrm{Im}\,\rho_0^I$ grows at least as rapidly as ν^{ϵ} as $\nu\to-\infty$. Therefore, we conclude that Eq. (9.1) is inconsistent with Pomeranchuk asymptotic behavior unless $A^{(0)I}$ tends to zero as fast as some negative power as $\nu\to-\infty$, or unless R_l^I tends to zero like $\nu^{a-1}/\ln\nu$ as $\nu\to-\infty$. If neither of these conditions is satisfied, then infinitely many poles must be added to the right-hand side of Eq. (9.1) if it is to be made consistent with Pomeranchuk asymptotic behavior.

Notwithstanding the aforementioned difficulites, it is not unreasonable to hope that Eqs. (9.1) and (9.2) afford a good approximation to nature over some appreciable range of energies above threshold. The ability of Eqs. (9.1) and (9.2) to describe the threshold region is assured by the presence of the adjustable parameters z_I, $dA^{(0)I}/d\nu|_{z_I}$, and a_1. The equations are compatible with the ϵ and ρ resonances and with the absence of resonances with $I=2$. The nearest poles at complex ν in $1/V^{(0)0}$ and $1/V^{(0)2}$ do not give rise to terms which vary rapidly near threshold,[16] and the net contributions of the infinitely many poles which have been conjectured to exist in $1/V^{(0)I}$ and $1/A^{(0)I}$ for $I=0$ and 2 may also vary slowly near threshold.[16] In addition, the pole terms in ρ_l^I should vary less rapidly near threshold than those in $1/V^{(l)I}$ or $1/A^{(l)I}$, since some degree of cancellation between the latter is expected. Thus the absence of poles at complex ν in Eqs. (9.1) and (9.2) does not rule out the possibility that the resulting ρ_l^I may be good approximations to nature over a fairly broad range of energies above threshold.

Assuming that ρ_0^I with $I=0$ and 2 should in fact tend to zero as $\nu\to-\infty$, we can constrain the $\mathrm{Im}\,\rho_0^I$ in the integral over the left cut in Eq. (9.1) to tend to zero as $\nu\to-\infty$, and it need not concern us that the resulting $\mathrm{Re}\rho_0^I$ grows without limit as $\nu\to-\infty$, provided that we use the resulting $\mathrm{Re}\rho_0^I$ only over a finite region about the origin. A reasonable estimate of the regions over which the $\mathrm{Re}\rho_l^I$ generated by Eqs. (9.1) and (9.2) are reliable would be circles about the origin within which Eqs. (4.1) and (9.1)–(9.3) are simultaneously satisfied.[52]

Although inelastic scattering may become appreciable above 1 GeV and may (through analyticity) influence the $A^{(l)I}$ below 1 GeV, inelastic effects have been explicitly taken into account by the presence of R_l^I (which, however, must be given) in Eq. (3.1) for $\mathrm{Im}\,\rho_l^I$. Thus there is no evident reason why Eqs. (9.1) and (9.2) cannot afford a good approximation to nature from threshold up to 1 GeV or more. However, the actual construction of approximate solutions to Eqs. (4.1) and (9.1)–(9.3) requires extensive computations, and we shall defer them to a later work.

In earlier work by the present author, an exactly crossing-symmetric method was proposed[53] for unitarizing Veneziano S and P waves. The amplitudes generated by that method contain no ghosts[54] and are almost exactly unitary below 1 GeV or more, but violate unitarity rather badly at energies above 5 GeV or so. The present simple model for $1/A^{(l)I}$ is quite complementary to the earlier model in the sense that amplitudes generated by the present method are exactly unitary (for any given R_l^I), but are only approximately crossing-symmetric, and may contain ghosts. Because of the complementarity of the two methods, it will be quite interesting to see the extent to which amplitudes generated by the two methods are in agreement with each other. If the amplitudes generated by the two methods are in close agreement with one another over some range of energies including threshold, the validity of each method would thereby be confirmed for that range of energy.

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[1]G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[2]C. Lovelace, CERN Report No. CERN-TH-1041, 1969 (unpublished).

[3]For a comprehensive discussion of the K matrix, see R. H. Dalitz and S. F. Tuan, Ann. Phys. (N.Y.) 10, 307 (1960).

[4]H. M. Lipinski, University of Wisconsin report, 1969 (unpublished), and revised version, 1970 (unpublished); K. Kang, Lett. Nuovo Cimento 3, 576 (1970); E. P. Tryon, Columbia University Report No. NYO-1932(2)-165 (unpublished); K. M. Carpenter and S. M. Negrine, Nuovo Cimento 1A, 13 (1971); David M. Scott, K. Tanaka, and R. Torgerson, Phys. Rev. D 2, 1301 (1970).

[5]For an early study of the inverses of $\pi\pi$ partial-wave amplitudes, see J. W. Moffat, Phys. Rev. 121, 926 (1961). For a recent study, see J. B. Carrotte and R. C. Johnson, Phys. Rev. D 2, 1945 (1970).

[6]J. Shapiro and J. Yellin, LRL Report No. UCRL-18500, 1968 (unpublished); C. Lovelace, Phys. Letters 28B, 264 (1968); J. Yellin, LRL Report No. UCRL-18637, 1968 (unpublished); J. Shapiro, Phys. Rev. 179, 1345 (1969); D. Sivers and J. Yellin, Ann. Phys. (N.Y.) 55, 107 (1969).

[7]G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960). The reader is cautioned that R_l^I in this reference is the inverse of our present R_l^I.

[8]This method characterizes all work wherein $A^{(l)I}$ is related to $V^{(l)I}$ by an equation of the form (2.10). Thus the works by Lovelace (Ref. 2), Lipinski (Ref. 4), Tryon (Ref. 4), and Scott *et al.* (Ref. 4) all utilize this method, though the authors do not state it explicitly or draw several important conclusions which follow immediately from the definition (2.8). The work by Kang (Ref. 4) is closely related to this method. The work by Carpenter and Negrine assumes the relation between $A^{(l)I}$ and $V^{(l)I}$ to be of a somewhat different form from Eq. (2.10), so that much of the analysis of the present paper must be modified to be applicable to the latter work.

[9]If one keeps only the leading term in the Veneziano amplitudes, then the pole in $V^{(0)0}$ in the second tower of resonances is in fact a tiny ghost for values of the Regge parameters near those proposed by Lovelace. The second pole in $V^{(0)0}$ with a physical interpretation occurs in the third tower, near 1.65 GeV.

[10]L. J. Gutay, J. H. Scharenguivel, N. Fuchs, J. Gaidos, D. H. Miller, F. T. Meiere, and B. Harms, Purdue University Report No. COO-1428-174, 1970 (unpublished); M. H. MacGregor, LRL Report No. UCRL-72495, 1970 (unpublished).

[11]Suppose a real analytic function $f(\nu)$ behaves like $\eta(\nu-\tilde{\nu})$ in the upper half plane near a point $\tilde{\nu}$ on the real axis. Then it is easy to verify that $1/f(\nu)$ contains a pole at $\tilde{\nu}$ if and only if $\mathrm{Re}\eta \neq 0$; the residue is simply $\mathrm{Re}(1/\eta)$. Since $\mathrm{Re}A^{(l)I}$ vanishes like $\sin\delta_l^I$ while $\mathrm{Im}A^{(l)I}$ vanishes like $\sin^2\delta_l^I$ as δ_l^I passes through $n\pi$ above threshold, it follows that $1/A^{(l)I}$ contains a pole wherever $\delta_l^I = n\pi$ above threshold.

[12]The S waves are usually assumed to vanish somewhere on the interval $-1<\nu<0$ because of the Adler zero [S. Adler, Phys. Rev. 137, B1022 (1965)]. For $l \geq 1$, each $A^{(l)I}$ vanishes like ν^l at $\nu = 0$.

[13]The $\pi\pi$ scattering lengths a_l^I are defined in general as $a_l^I \equiv \lim_{\nu\to 0}[\nu^{-l}A^{(l)I}(\nu)]$. For brevity, we shall denote the $I=0$ and 2 S-wave and $I=1$ P-wave scattering lengths by a_I.

[14]The P-wave scattering length is given by a dispersion relation which has been studied by M. G. Olsson, Phys. Rev. 162, 1338 (1967). For $l \geq 2$, the scattering lengths can be computed from the Froissart-Gribov representation for partial waves; see V. N. Gribov, Zh. Eksperim. i Teor. Fiz. 42, 1260 (1962) [Soviet Phys. JETP 15, 873 (1962)].

[15]Y. S. Jin and K. Kang, Phys. Rev. 152, 1227 (1966).

[16]E. P. Tryon, following paper, Phys. Rev. D 4, 1216 (1971).

[17]One might suppose it coincidental that both $\mathrm{Im}V^{(1)1}$ and $\mathrm{Re}V^{(1)1}$ vanish at the same point along the left cut when one uses $a=0.483$, $b=0.017$. However, the present author has also computed $V^{(1)1}$ along the left cut using $a=0.477$, $b=0.0175$, and found again that both $\mathrm{Im}V^{(1)1}$ and $\mathrm{Re}V^{(1)1}$ vanish at a common point, namely, at $\tilde{\nu} = -7.81$. Thus the existence of a zero in $V^{(1)1}$ along the left cut appears to be independent of the values chosen for a and b (at least to some extent).

[18]We determine β by requiring that the integral over the Veneziano ρ-resonance δ-function absorptive part be equal to the integral over a Breit-Wigner resonance absorptive part with correct threshold behavior.

[19]Lovelace does not state the value for β which he uses in his work (Ref. 2). However, the value of β can be inferred from the values of his S-wave scattering lengths together with the threshold value of his function ρ.

[20]In a simple model for ρ_0^2 proposed earlier by the present author (Ref. 4), a forward dispersion relation for $2a_0 - 5a_2$ was imposed on the S waves in addition to certain constraints based on crossing symmetry alone. Examination of the resulting $A^{(0)2}$ reveals that it contains a ghost at $\nu = -4.5 + 1.6i$ with residue $1.5 - 2.0i$, and of course a pole with conjugate residue at the conjugate

point. Near $\nu = 0$, these two ghosts contribute approximately $0.86 - 0.21\nu$ to $A^{(0)2}$, which is comparable to the contribution of the corresponding ghosts in Lovelace's $A^{(0)2}$. However, these ghosts in the $A^{(0)2}$ of Ref. 4 are much closer to the negative real axis than the corresponding ghosts of Lovelace, and hence are much more similar to a left cut in $A^{(0)2}$ than are the ghosts in Lovelace's $A^{(0)2}$. Thus the imposition of the forward dispersion relation for $2a_0 - 5a_2$, which depends on analyticity as well as on crossing symmetry, improves the analytic properties of $A^{(0)2}$.

[21]L. S. Brown and R. L. Goble, Phys. Rev. Letters 20, 346 (1968).

[22]L. S. Brown, private communication.

[23]G. F. Chew, S. Mandelstam, and H. P. Noyes, Phys. Rev. 119, 478 (1960).

[24]K. P. Chung, IBM Thomas J. Watson Research Center report, 1971 (unpublished). The failure of N/D to satisfy the dispersion relation for A must be due to zeros in D, which occur either at complex ν or along the negative real axis.

[25]R. Roskies, Nuovo Cimento 65A, 467 (1970).

[26]G. F. Chew and S. Mandelstam, Nuovo Cimento 19, 752 (1961).

[27]The right sides of Eqs. (7.5) and (7.6) would depend on a constant term in $A^{(1)1}(\nu)$; however, the centrifugal barrier implies that $A^{(1)1}(0) = 0$, so that any such constant term is necessarily zero.

[28]A. Martin, Nuovo Cimento 58A, 303 (1968), and references therein; A. K. Common, *ibid.* 53A, 946 (1968).

[29]G. Auberson, O. Piguet, and G. Wanders, Phys. Letters 28B, 41 (1968).

[30]O. Piguet and G. Wanders, Phys. Letters 30B, 418 (1969).

[31]G. F. Chew, S. Frautschi, and S. Mandelstam, Phys. Rev. 126, 1202 (1962).

[32]M. G. Olsson, Ref. 14; T. Akiba and K. Kang, Phys. Letters 25B, 35 (1967).

[33]In the model of Brown and Goble, Ref. 23, $|\delta_0^2|$ exceeds 40° above $E_{\text{c.m.}} = 700$ MeV. In the revised model of Lipinski, Ref. 4, δ_0^2 can be as large as $-65°$ at the ρ mass without violating any of the crossing conditions imposed in that work. In the model of Kang, Ref. 4, $|\delta_0^2|$ exceeds 40° above 850 MeV. In the model of Scott *et al.*, Ref. 4, $|\delta_0^2|$ exceeds 45° above 750 MeV. None of these models is manifestly free of ghosts, and the present author believes it likely that important ghosts exist in each of them. In the model of P. Curry, I. O. Moen, J. W. Moffat, and V. Snell, Phys. Rev. D 3, 1233 (1971), $|\delta_0^2|$ exceeds 40° above 800 MeV. This model appears to be free of ghosts, and the explanation of the large δ_0^2 above 800 MeV is not clear to the present author.

[34]The δ_0^2 of Brown and Goble, Ref. 23, has been shown to contain a ghost; L. S. Brown, private communication.

[35]Cf. D. Morgan and G. Shaw, Nucl. Phys. B10, 261 (1969).

[36]For typical values of the Regge parameters, the Veneziano formula (2.1) predicts $\Gamma(g \to 2\pi) \approx 40$ MeV. This value for $\Gamma(g \to 2\pi)$ is also suggested by a careful analysis of the results of several experiments, as was pointed out by E. P. Tryon, Phys. Rev. Letters 22, 110 (1969). The value receives further support from the experiment reported by W. D. Walker at the Conference on the $\pi\pi$ and $K\pi$ Interactions at Argonne National Laboratory, 1969 (unpublished). However, the actual value of $\Gamma(g \to 2\pi)$ plays no essential role in any of our present discussions.

[37]Recently, M. Kugler, Phys. Letters 31B, 379 (1970), has assumed that the right-hand sides of $I = 2$ dispersion relations like Eq. (7.23) can be saturated by resonances. However, the modest $I = 2$ S waves resulting from crossing-symmetric, ghost-free dispersive calculations carried out earlier by the present author [E. P. Tryon, Phys. Rev. Letters 20, 769 (1968); also Columbia University Report No. NYO-1932(2)-153, 1969 (unpublished)] contribute about 0.2 to the right-hand side of Eq. (7.23) when integrated from threshold to 1.5 GeV, while the somewhat larger (we believe too large) $I = 2$ S wave of Lovelace (Ref. 2) contributes about 0.35 when integrated over the same range. Thus, something close to the opposite of Kugler's assumption is true; namely, $I = 2$ dispersion relations like Eq. (7.23) can be saturated by the contributions of nonresonant $I = 2$ absorptive parts integrated from threshold to about 1.5 or 2.0 GeV. The proper conclusion to be drawn is that $I = 2$ dispersion relations like Eq. (7.23) are of no practical value in the determination of $\pi\pi$ threshold parameters, either now or in the foreseeable future. For an earlier discussion leading to the same conclusion, see E. P. Tryon, Ref. 36.

[38]P. G. O. Freund, Phys. Rev. Letters 20, 235 (1968).

[39]H. Harari, Phys. Rev. Letters 20, 1395 (1968).

[40]Cf. C. M. Carpenter and S. M. Negrine, Ref. 4.

[41]E. P. Tryon, Ref. 4.

[42]The existence of essential singularities at infinity in all the $V^{(t)I}$ was suggested without proof by F. Drago and S. Matsuda, Phys. Rev. 181, 2095 (1969). However, it has since been established by R. T. Park and B. R. Desai [Phys. Rev. D 2, 786 (1970)] that for $I = 1$, $V^{(t)I}$ does tend asymptotically to zero in all directions (except along the positive real axis). The $V^{(t)I}$ with $I = 0$ and 2 have recently been studied by the present author, and the conjectures which we cite are presented in detail in Ref. 16.

[43]R. T. Park and B. R. Desai, Ref. 42.

[44]The zero in $\pi\pi$ amplitudes resulting from the Adler self-consistency condition (Ref. 12) occurs at a point off the mass shell. Thus one can only deduce approximate consequences for on-shell amplitudes at the present time. In the work of L. J. Gutay *et al.*, Phys. Rev. Letters 23, 431 (1969), the Adler zero is used together with a simple model for continuation to the mass shell in order to infer the ratio $a_0/a_2 = -3.2 \pm 0.1$ from the data. (The quoted uncertainty is statistical in origin, and does not reflect theoretical uncertainties in the method of analysis.) Together with Eqs. (7.1) and (7.21), this value for a_0/a_2 implies that $A^{(0)0}$ vanishes near $\nu \approx -0.8$ and that $A^{(0)2}$ vanishes near $\nu \approx -0.6$.

[45]We assume that δ_0^0 does in fact rise to π. In order to fit the so-called "down-down" or "up-down" branch of phase-shift analyses of Chew-Low extrapolations of $\pi\pi$ cross sections from $\pi N \to \pi\pi N$ data [cf. *Proceedings of a Conference on $\pi\pi$ and $K\pi$ Interactions at Argonne National Laboratory, 1969,* edited by F. Loeffler and E. Malamud (Argonne National Laboratory, Argonne, Ill., 1969)] one must of course place z_π well above 1 GeV.

[46]If a and b are given values which result in a pole in $V^{(0)0}$ with residue of wrong sign, i.e., a "ghost" pole (Ref. 9), then $V^{(0)0}$ has yet another zero. To prevent the

ghost in $V^{(0)0}$ from generating a ghost in $A^{(0)0}$, one need only include an additional pole in ρ_0^0 to cancel the pole in $1/V^{(0)0}$ which results from the spurious zero in $V^{(0)0}$.

[47]In earlier work [Columbia University Report No. NYO-1932(2)-153, 1969 (unpublished)]; Columbia University Report No. NYO-1932(2)-196, 1971 (to be published in Phys. Letters B) by the present author, it was assumed that the functions $\Delta A^{(l)I} \equiv A^{(l)I} - V^{(l)I}$ can be approximated below about 1 GeV by solutions to partial-wave dispersion relations with left cuts generated by S- and P-wave exchange. The resulting $A^{(l)I} = V^{(l)I} + \Delta A^{(l)I}$ are analytic and free of ghosts, crossing-symmetric, approximately unitary below 1 GeV, and automatically satisfy the inequalities of Martin (Ref. 28). It was found that the five parameters, a_0/a_2, m_ϵ, Γ_ϵ, m_ρ, and Γ_ρ, were independent of one another for fairly broad ranges of their respective values. A survey of the literature reveals that in order to derive any useful relations among these five quantities, it has been necessary in practice to make at least one additional assumption. Examples of the additional assumptions which have been used are duality, the Adler sum rule [S. L. Adler, Phys. Rev. 140B, 736 (1965)], and a restriction on the asymmetry of the ϵ resonance (D. Morgan and G. Shaw, Ref. 35). Since the present paper is concerned with a unitarization of Veneziano amplitudes, duality of course would be a natural assumption. However, we wish to emphasize that analyticity, crossing symmetry, and unitarity have not yet been shown to imply any useful relations between the five aforementioned parameters. [For reasons presented in the text and in Ref. 37, we do not regard $I=2$ sum rules such as Eq. (7.23) as useful relations.]

[48]For $I=0$ and 2, we have already argued that ρ_0^I should in fact tend to zero as $\nu \to -\infty$. For $I=1$, $\mathrm{Im}\rho_1^1$ is not expected to tend to zero, but the integral over the left cut in Eq. (9.2) is expected to converge. Thus there should exist a Λ such that the integral over the left cut in Eq. (9.2) can be well approximated by setting $\mathrm{Im}\rho_1^1$ equal to zero for $\nu < -\Lambda$.

[49]Equation (9.3) is obtained by taking the imaginary parts of both sides of Eq. (2.10) and regarding the resulting equation as a quadratic equation for $\mathrm{Re}\rho_l^I$.

[50]The notion that second and higher derivatives of the $A^{(l)I}$ should be determined by nearby singularities is usually based on the premise that the $A^{(l)I}$ satisfy dispersion relations with no more than two subtractions required to ensure rapid convergence of the integrals. For $I=0$ and 2, it appears likely (Ref. 16) that the $V^{(l)I}$ do not satisfy dispersion relations with any finite number of subtractions, and the same might be true of physical $A^{(l)I}$. However, notwithstanding the likelihood that $V^{(l)I}$ with $I=0$ and 2 contain essential singularities at infinity, it has been established (Ref. 16) that the $V^{(0)I}$ and $V^{(1)1}$ can be approximated within 10% between threshold and 700 MeV by functions which *do* satisfy twice-subtracted dispersion relations, where the left cuts are generated by inserting the Veneziano δ-function absorptive parts of the first two (ρ and f_0) towers of resonances into the right-hand side of Eq. (4.1). Thus even if the $A^{(l)I}$ contain essential singularities at infinity, it is reasonable to suppose that their second and higher derivatives are largely determined on the interval $-1 < \nu < 0$ by the nearby singularities.

[51]If ρ_l^I contains no poles above some energy and grows less rapidly than $\nu^{1-a}\ln\nu$, then $\mathrm{Im}A^{(l)I}$ will asymptotically approach $\mathrm{Im}V^{(l)I}$ in the sense of local averages, and the masses of the resonances in $A^{(l)I}$ will tend to those in $V^{(l)I}$ (E. P. Tryon, Ref. 4). However, even without poles in ρ_l^I, the masses and widths of resonances in the low-energy region are sensitive to details in the behavior of ρ_l^I.

[52]For the Veneziano resonance structure, use of Eq. (4.1) results in $\mathrm{Im}A^{(l)I}$ which grow without limit as $\nu \to -\infty$, if $I=0$ or 2. Then ρ_l^I with $I=0$ and 2 tend to zero as $\nu \to -\infty$, and this is why Eqs. (4.1), (9.1), and (9.3) cannot be simultaneously satisfied to arbitrarily large negative ν.

[53]E. P. Tryon, Ref. 47.

[54]The $A^{(l)I}$ contain no poles off the real axis. In the asymptotic region, the $A^{(l)I}$ with $I=0$ and 1 contain the same poles as the $V^{(l)I}$, so any poles with residue of wrong sign in $V^{(l)I}$ appear also in $A^{(l)I}$. However, such poles should not affect the low-energy amplitudes appreciably.