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Regge Models and Dispersion Relations for Partial Waves

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Abstract

The asymptotic behaviour of partial wave amplitudes is calculated supposing various Regge models for the total scattering amplitude $A(s, t, u)$. The high energy partial wave behaviour obtained is combined with the validity of partial wave dispersion relations. It is shown that consistency of these assumptions can only be achieved by demanding

1) a definite asymptotic behaviour of the discontinuity of the left hand cut of partial wave amplitudes,
2) the validity of partial wave sum rules of similar kind as the well-known finite energy sum rules for the total amplitude.

All steps of the derivation shall first be demonstated for elastic scattering of identical scalar particles. Then within the helicity formalism the results are generalized for particles with arbitrary spin and different masses. Finally the question is studied whether the sum rules can be employed to determine unknown CDD-pole parameters in an N/D approach for the $I = J = 1/2$ state in πN scattering. It is shown that the sum rules of highest order are able to do that.

1. Introduction

It is the well-known aim of the analytic S-matrix theory to determine scattering amplitudes uniquely by means of basic principles as unitarity, crossing symmetry, analyticity of first and second kind, duality etc. Mainly two approaches can be distinguished.

1) One tries to construct the total scattering amplitude $A(s, t, u)$ satisfying at least some of the basic assumptions. The various dual models proceed along this way. Here the main problem, namely the incorporation of unitarity, should be solved by modifying the amplitude in the tree approximation (dual models with Mandelstam analyticity [*1*]) or by a perturbation-like approach [*2*]. Both ways lead to certain troubles which have not been removed till now.

2) Partial wave amplitudes $A_l(s)$ are calculated generally by combining unitarity equations with dispersion relations. The results are usually checked against positivity and crossing symmetry properties employing additional conditions in the form of

sum rules, inequalities etc. In this way free parameters (input-potential, CDD-pole parameters in the case of an N/D approach [3], subtraction constants etc.) can be estimated up to a certain degree. Equations for partial waves are very useful for phenomenological investigations, e.g. for testing consistency of phase shift analysis, for the determination of scattering lengths, effective range parameters. However the whole program can be realized only approximately, because of the very large number of additional conditions, the rather unknown redundances between them, and the enormous computer capacity required [4].

In both approaches unitarity and asymptotic behaviour are treated on a quite different basis. In the first approach Regge behaviour for the total amplitude is taken into account in the model ad hoc, and this in a crossing symmetric manner. Unitarity has to be included later on. In the second approach, where partial waves are considered, one proceeds actually in the opposite way. Here one usually tries to statisfy unitarity directly by determining the partial wave amplitudes from corresponding integral equations. Inelasticity is treated phenomenologically in practical calculations. The inclusion of a definite asymptotic behaviour remains as a second step, and only a few authors turned to it up till now [5].

The present paper deals with this last question. We investigate the conditions which result from combining partial wave dispersion relations and Regge asymptotic partial wave behaviour. To calculate the high energy behaviour of partial waves we suppose that the total scattering amplitudes are dominated by Regge terms which in principle are in agreement with recent high energy data.

The following Regge contributions can be considered.

1. Regge poles with intercept $\alpha(0) = 1 + \varepsilon$, $\varepsilon \gtrsim 0$. This leads to geometrical scaling [6]. The total cross section rises logarithmically at least in a restricted energy region, and this because of
$$s^\varepsilon \simeq 1 + \varepsilon \ln s$$
2. Regge dipoles and triple poles, respectively. Geometrical scaling is fulfilled and total cross section shows logarithmic rise [7].
3. Sum of ordinary Regge poles and cuts. This gives asymptotically constant total cross section. The rising behaviour as it comes out at present accelerator energies is explained as an intermediate range effect [8].
4. Pairs of complex conjugate Regge poles. The total cross section is oscillating [9].

In the following we exclude Regge poles with intercept higher than one because of violation of the Froissart bound. Furthermore we do not consider complex trajectories although it would not lead to principle difficulties.

To calculate the partial wave asymptotic behaviour the Regge terms mentioned are projected supposing a strong enhancement of forward and backward scattering. One gets the asymptotic behaviour in the form of an asymptotic expansion of inverse powers of the logarithm of energy and of the energy itself. Consistency of this asymptotic behaviour with the validity of partial wave dispersion relations means that the latter ones have to produce the correct asymptotic behaviour of the real part of $A_l(s)$ if the partial wave discontinuity (according to that Regge behaviour) is used above a certain cutoff s_c. It turns outs that this is only possible, if firstly the asymptotic left-hand discontinuity of the partial wave amplitude is chosen in a definite way, and if secondly an infinite set of sum rules is fulfilled. These sum rules are similar to the finite energy sum rules for the total amplitudes [10] (containing, however, left-hand discontinuities as well). Partial wave sum rules could be employed for testing models against Regge asymptotic behaviour. It is quite possible that they can be used to restrict the freedom in the choice of trajectories in dual models with Mandelstam analyticity. Furthermore the sum rules give an average of the left-hand discontinuity including the region where the partial wave expansions of the crossed channels are not defined. Last not least they determine free parameters (e.g. CDD-pole parameters [3]) when the asymptotic behaviour is assumed to be known.

In chapter 2 the sum rules are derived in the scalar case generalizing also an earlier treatment where only single Regge pole contributions have been regarded [11]. This is extended to spinning particles with different masses using the helicity formalism in chapter 3. Chapter 4 demonstrates a practical calculation in the case of πN scattering. It is shown that the sum rules of highest order enable us to determine CDD-pole parameters in the $I = J = 1/2$ state.

2. Regge Asymptotic Behaviour and Dispersion Relations for Partial Waves in the Scalar Case

2.1. High Energy Behaviour of Partial Wave Amplitudes

We consider two particle elastic scattering of scalar identical particles (of mass $m = 1$). The scattering amplitude can be expanded in the form

$$A(s, t, u) = \sum_{l=0}^{\infty} (2l + 1)\, A_l(s)\, P_l(z) \tag{2.1}$$

where

$$A_l(s) = \frac{1}{2} \int_{-1}^{+1} dz\, P_l(z)\, A(s, t, u) \tag{2.2}$$

represents the l-th partial wave. The Mandelstam variables s, t, u are defined as usually. z is the cosine of the scattering angle in the c.m.s. of the s channel

$$z = 1 + \frac{2t}{s-4} = -1 - \frac{2u}{s-4}, \qquad s + t + u = 4 .$$

In agreement with experimental results and theoretic reasoning we assume that forward and backward scattering at high energies are strongly enhanced. Therefore we require for large physical s

$$|A(s, t, u)| \leqq O(s^{-N}) \qquad \text{if} \qquad |z| < 1 - \varepsilon(s) . \tag{2.3}$$

N is a sufficiently high positive number and $\varepsilon(s)$ has a small positive value with $\varepsilon(s) \to +0$ für $s \to +\infty$.[1])

In the standard approach the asymptotic behaviour of the total amplitude is described by a series of Regge poles and cuts if t or u are fixed, respectively. However, we adopt an ansatz of the same kind also for the asymptotic limit of large s at fixed scattering angle. That will be correct up to terms of orders $O(s^{-N})$ if the Regge terms as well as the background themselves are demanded to fulfil the relation (2.3).

Thus schematically we have

$$A(s, t, u) \underset{\substack{s\to+\infty \\ z \text{ fixed}}}{\to} \sum (t\text{-channel Regge terms}) + (t \leftrightarrow u) \tag{2.4}$$

What we call here "t, u-channel Regge terms" is explained in the following. As discussed in the introduction we consider three types of contributions due to exchanges in the

[1]) In order to determine N one may also refer to the well-known quark conting rules.

t-channel and u-channel (the background term will be omitted; it is assumed that it can be shifted sufficiently far to the left of the l-plane by Mandelstam's procedure).

1) Regge poles,
2) multifold Regge poles (to second order) [7], and
3) Regge cuts.

These models (at least appropriate combinations of them) explain rising total cross sections (at least in the intermediate energy region). We discuss them separately. For simplicity, only the t-channel contributions are regarded (the scalar problem is completely crossing symmetric).

1) Regge pole terms

We have for a single pole contribution

$$A_{\text{p.}} \xrightarrow[z \text{ fixed}]{s \to +\infty} \beta(\alpha(t), t) \frac{1 + \sigma \exp(-i\pi\alpha(t))}{\sin \pi\alpha(t)} s^{\alpha(t)} \tag{2.5}$$

where σ denotes the signature. The function β, which is essentially the Regge pole residue, is assumed to factorize with terms killing all nonsense singularities of the signature factor. As mentioned above the whole term should behave like (2.3). Let us discuss the case of Regge trajectories $\alpha(t)$ which rise linearly in t in some region $0 \geqq t \geqq -T$ ($T > 0$, sufficiently large) and tends for $t < -T$ monotonically to a finite limit $\alpha(-\infty) < \alpha(-T)$. Here the restriction to linear behaviour is not an essential one. One can show that our approach is generalizable to the case of nonlinear trajectories, too. Then a Regge pole contribution (2.5) behaves like

$$s^{\alpha(-T)} f(t) \qquad (s \to +\infty,\ z \text{ fixed})$$

and if for large negative t, $|f(t)| \leqq O(t^{-M})$ with $M \geqq N + \alpha(-T)$ then the Regge pole term (2.5) fulfils obviously condition (2.3).

2) Multifold Regge pole terms

Multifold poles of partial waves in the l-plane yield asymptotic contributions to the total amplitude of the following form [7]

$$A_{\text{M.p.}} \xrightarrow[z \text{ fixed}]{s \to +\infty} \frac{d^n}{dl^n} \left[\beta(l, t) \frac{1 + \sigma \exp(-i\pi l)}{\sin \pi l} s^l \right]_{l = \alpha(t)}, \qquad n = 1, 2. \tag{2.6}$$

Multifold Pomeron poles of higher degree than $n = 2$ are excluded because of the Froissart bound. Jenkovsky and Wall [7] have shown that a dipole Pomeron ($n = 1$, $\sigma = +1$, $\alpha(t)$ linear with $\alpha_0 = 1$) displays the main features of high energy pp data such as rising total cross section, dip structure in the differential cross section. Moreover this model shows geometrical scaling [6].
A multifold pole contribution (2.6) can be written as a sum of terms some of which have the ordinary Regge pole form (2.5). In the following we regard only dipoles for simplicity. The calculations in the case of triple poles proceed analogously and yield additionally a $\ln^2 s$-term. We have

$$A_{\text{M.p.}} \xrightarrow[z \text{ fixed}]{s \to +\infty} \beta_1(\alpha(t), t)\, \zeta(\alpha(t))\, s^{\alpha(t)} + \hat\beta_2(\alpha(t), t)\, r(\alpha(t))\, s^{\alpha(t)} + \beta(\alpha(t), t)\, \zeta(\alpha(t))\, s^{\alpha(t)} \cdot \ln s \tag{2.7}$$

where

$$\zeta(\alpha) \equiv \frac{1 + \sigma \exp(-i\pi\alpha)}{\sin \pi\alpha}$$

$$r(\alpha) \equiv \operatorname{Re} \zeta(\alpha) = \frac{1 + \sigma \cos \pi\alpha}{\sin \pi\alpha}$$

$$\beta_1(\alpha, t) = \frac{\partial}{\partial\alpha} \beta(\alpha, t)$$

$$\hat\beta_2(\alpha, t) = -\frac{\pi\sigma}{\sin \pi\alpha} \beta(\alpha, t).$$

The second term in equ. (2.7) contains only the real part of the signature factor $\zeta(\alpha)$. We reinterprete this term as a sum of two Regge poles with opposite signatures. One gets

$$\hat\beta_2 r(\alpha)\, s^\alpha = \beta_2 \left(\frac{1 + \sigma \exp(-i\pi\alpha)}{\sin \pi\alpha} + \frac{1 - \sigma \exp(-i\pi\alpha)}{\sin \pi\alpha} \right) s^\alpha \tag{2.8}$$

where

$$\beta_2(\alpha, t) \equiv -\sigma \frac{\pi}{2} r(\alpha)\, \beta(\alpha, t).$$

In order to satisfy condition (2.3) the individual terms of expression (2.7) should have the same properties as assumed for Regge poles, such as trajectories linearly rising in the range $[-T, 0]$, but $|\alpha(-\infty)| < \infty$, and a corresponding behaviour of the β-functions for negative t.

3) Regge cut contributions

The contribution of an l-plane cut an be written as

$$A_{\text{c.}} \xrightarrow[z \text{ fixed}]{s \to +\infty} \int_{\bar\alpha}^{\alpha(t)} (2l + 1)\, \Delta(l, t) \frac{1 + \sigma \exp(-i\pi l)}{\sin \pi l} s^l \, dl. \tag{2.9}$$

Δ represents the discontinuity of the t-channel partial wave amplitude on the Regge cut up to a factor $2i$. $\bar\alpha$ is a sufficiently large negative number (in correspondence with the position of the background term). $\hat\alpha(t)$ denotes the branch point and is assumed to exhibit the same properties as the Regge pole trajectories. For definiteness let us assume

$$\bar\alpha < \alpha(-\infty) < \alpha(-T).$$

To verify the behaviour (2.3) the discontinuity Δ should behave like

$$|\Delta(l, t)| \leqq O(t^{-M}) \quad \text{for} \quad t < -T$$
$$\text{and} \quad \bar\alpha \leqq l \leqq \alpha(t)$$

(M a large positive number).
Cut contributions are generally calculated from special models (eikonal, absorption model, K matrix formalism etc.). However, the simplest one usually employed is due to Mandelstam, Gribov et al. [12].

They studied logarithmic branch points of the well-known form

$$\Delta(l, t) = \frac{1}{\pi} R(l, t) \left(\alpha(t) - l\right)^{m-2} \frac{1}{2i} \operatorname{disc}\left(\ln\left(\alpha(t) - l\right)\right), \qquad m = 2, 3, 4, \ldots \tag{2.10}$$

The exponent m denotes the m-fold exchange of a trajectory $\alpha_e(t)$

$$\alpha(t) = m\alpha_e\left(\frac{t}{m^2}\right) - m + 1 .$$

Restricting the original assumption that the function $R(l, t)$ should be regular near the branchpoint, $R(l, t)$ will be demanded to be regular in the whole integration region and $t \in [-T, 0)$. Then we may expand $R(l, t)$ into a Taylor expansion with respect to t and l in that region.

After having summarized the class of Regge models under consideration we are now in the position to derive the high energy behaviour of partial wave amplitudes. In accordance with condition (2.3) the partial wave projection (2.2) can be rewritten as

$$A_l(s) \xrightarrow[s\to+\infty]{} \frac{1}{s-4}\int_{-T}^{0} dt\, A(s, t, u)\, P_l\left(1 + \frac{2t}{s-4}\right) + \frac{(-1)^l}{s-4}\int_{-U}^{0} du\, A(s, t, u)\, P_l\left(1 + \frac{2u}{s-4}\right) \tag{2.11}$$

where the first (second) term describes the strong enhanced forward (backward) contribution. Here the lower limits $-T$ and $-U$, respectively, are introduced with sufficiently large but finite values instead of s dependent ones. Therefore expression (2.11) together with the ansatz (2.4)–(2.9) will describe the asymptotics of $A_l(s)$ correctly up to terms of order $O(s^{-L})$, L sufficiently large. We remark that the t-channel Regge terms dominate in the first integral, whereas the u-channel contributions dominate the backward scattering integral. Because of $t - u$ crossing we can sum up both terms of equ. (2.11) and perform the following calculations.
At first we compute the projection integral according to (2.11) for simple and multifold Regge poles (equ. (2.5) and (2.6), respectively) which can be written in a unified manner as demonstrated before. After that we shall deal with the Regge cut contributions. We have

$$A_l(s) \xrightarrow[s\to+\infty]{} \frac{1 + (-1)^l}{s-4} \sum_{\substack{\text{pole terms}\\ n=0,1,2}} (\ln s)^n \int_{-T}^{0} dt\, B_{\text{p.}}(t)\, P_l\left(1 + \frac{2t}{s-4}\right) s^{\alpha(t)} \tag{2.12}$$

with

$$B_{\text{p.}}(t) \equiv \beta\left(\alpha(t), t\right) \frac{1 + \sigma \exp\left(-i\pi\alpha(t)\right)}{\sin \pi\alpha(t)} .$$

It is

$$n = \begin{cases} 0 & \text{for simple poles} \\ 1 & \text{for dipoles} \\ 2 & \text{for triple poles} \end{cases}$$

In order to determine the integral in equ. (2.12) we expand the integrand. Care is necessary because the Taylor expansion around $t = 0$ is convergent only in the region $[-4, 4]$ ($B_{\text{p.}}(t)$ possesses a branchpoint at $t = 4$). Therefore we calculate the indefinite integral as function of the upper limit. To determine it at $t = 0$ we expand the integrand at $t = -\tau = 0$, but to get the value at $t = -T$ we expand at $t = -\tau$ with $\tau \geqq (T - 4)/2$.

The Taylor expansion around $t = -\tau$ reads

$$B_{\text{p.}}(t) = \sum_{\mu=0}^{\infty} B_{\text{p.}}^{(\mu)}(-\tau) \frac{(t+\tau)^\mu}{\mu!}$$

with

$$B_{\text{p.}}^{(\mu)}(-\tau) = \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} \beta^{(\mu-\nu)}(-\tau)\, r^{(\nu)}(-\tau) - i\sigma\beta^{(\mu)}(-\tau) \tag{2.13}$$

and

$$r(t) = \frac{1 + \sigma \cos \pi\alpha(t)}{\sin \pi\alpha(t)} \tag{2.14}$$

where $r(t)$ is the real part of the signature factor.
The Legendre polynomials in equ. (2.12) yield powers in t connected always with powers in s. We expand them, too

$$P_l\left(1 + \frac{2t}{s-4}\right) = \sum_{\lambda=0}^{l} \frac{2^\lambda}{(s-4)^\lambda} \frac{P_l^{(\lambda)}\left(1 - \frac{2\tau}{s-4}\right)}{\lambda!} (t+\tau)^\lambda .$$

Thus we are led to indefinite integrals of the form

$$F_K(\tau, x) = \int dt (t+\tau)^K s^{\alpha(t)} . \tag{2.15}$$

As outlined before the trajectories of all Regge contributions are assumed to be linearly rising in $[-T, 0]$

$$\alpha(t) = \alpha_0 + \alpha' t , \qquad \alpha' > 0 . \tag{2.16}$$

This yields

$$F_K(\tau, x) = s^{\alpha(x)} \left(\sum_{i=0}^{K} (-1)^i \binom{K}{i} i!\, \frac{(x+\tau)^{K-i}}{(\alpha' \ln s)^{i-1}}\right). \tag{2.17}$$

We conclude then

$$\left.\begin{aligned} A_l(s) &\xrightarrow[s\to+\infty]{} \left(1 + (-1)^l\right) \sum_{\substack{\text{pole terms}\\ n=0,1,2}} \sum_{\lambda=0}^{l} \frac{(\ln s)^n}{(s-4)^{\lambda+1}} \\ &\quad \times \sum_{\mu=0}^{\infty} \frac{2^\lambda}{\lambda!} P_l^{(\lambda)}(1) \frac{B_p^{(\mu)}(0)}{\mu!} F_{\lambda+\mu}(0, 0) + \Omega\left(s^{\alpha_L(-T)-1}\right) \\ &= \left(1 + (-1)^l\right) \sum_{\substack{\text{pole terms}\\ n=0,1,2}} \sum_{\substack{0\leqq\lambda\leqq l\\ \mu\geqq 0}} \left[(-1)^{\lambda+\mu} \binom{\lambda+\mu}{\lambda} \frac{2^\lambda P_l^{(\lambda)}(1)\, \beta^{(\mu)}(0)}{(\alpha')^{\lambda+\mu+1}}\right] \\ &\quad \times \frac{s^{\alpha_0}(\ln s)^{n-\lambda-\mu-1}}{(s-4)^{\lambda+1}} \left(-i\sigma + \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\lambda+\mu+\nu}{\nu} \frac{r^{(\nu)}(0)}{(\alpha')^\nu} (\ln s)^{-\nu}\right) \\ &\quad + \Omega\left(s^{\alpha_L(-T)-1}\right) \end{aligned}\right\} \tag{2.18}$$

where we used equs. (2.13) and (2.14). The Ω-term in equ. (2.18) is due to the indefinite integral calculated at the limit $t = -T$ and consists of an expansion of the $F_K(\tau, -T)$ terms with $\tau \geqq (T - 4)/2$. Thus it represents a function proportional to $s^{\alpha_L(-T)-1}$ up to logarithmic factors. α_L denotes that trajectory which is leading at $t = -T$.

Now we turn to the consideration of Regge cuts. The partial wave projection of their contributions leads to

$$A_l(s) \xrightarrow[s\to+\infty]{} \frac{1+(-1)^l}{s-4} \sum_{\left[\substack{\text{cut terms}\\ m=2,3,\dots}\right.} \int_{-T}^{0} dt \int_{\alpha(-T)}^{\alpha(t)} dl'(2l'+1)\, R(l',t)\, \big(\alpha(t)-l'\big)^{m-2}$$

$$\times \frac{[1+\sigma \exp(-i\pi l')]}{\sin \pi l'}\, s^{l'} P_l\left(1+\frac{2t}{s-4}\right) + \Omega(s^{\alpha_L(-T)-1}) \tag{2.19}$$

where equ. (2.10) has been employed. The cut integral corresponding to (2.9) has been restricted to the integration region $[\alpha(-T), \alpha(t)]$. The remaining part with upper limit $\alpha(-T)$ behaves like $s^{\alpha_L(-T)-1}$($\alpha_L(-T)$ is the leading branchpoint at $t=-T$).
Interchanging the order of integrations and substituting $l' = \alpha_0 + \alpha' t' = \alpha(t')$ we get

$$A_l(s) \xrightarrow[s\to+\infty]{} \frac{1+(-1)^l}{s-4} \sum_{\left[\substack{\text{cut terms}\\ m=2,3,\dots}\right.} \int_{-T}^{0} dt'\, B_c(t')\, s^{\alpha(t')} \tag{2.20}$$

where we have abbreviated

$$B_c(t') = \frac{1+\sigma \exp\big(-i\pi\alpha(t')\big)}{\sin \pi\alpha(t')}$$

$$\times \left[(\alpha')^{m-1}\big(2\alpha(t')+1\big) \cdot \int_{t'}^{0} dt\, R\big(\alpha(t'),t\big)\,(t-t')^{m-2}\, P_l\left(1+\frac{2t}{s-4}\right)\right]. \tag{2.21}$$

Thus we achieved a form similar to that of Regge poles (2.12). $R\big(\alpha(t'),t\big)$ should be a regular function of both t' and t in a large region (cf. discussion below equ. (2.10)). Then it can be expanded as a double power series. Furthermore R is assumed to cancel the poles of the signature factor for negative t'. Therefore we get a power series in t' also for the function $B_c(t')$ in the same way as for the corresponding $B_{p,\lambda}(t)$-function (equs. (2.13), (2.14)). With the calculation of integrals of the type (2.15) in accordance with (2.17) we are finally led to the asymptotic expansion

$$A_l(s) \xrightarrow[s\to+\infty]{} \big(1+(-1)^l\big) \sum_{\left[\substack{\text{cut terms}\\ m=2,3,4}\right.} \sum_{\substack{0\le\lambda\le l\\ \mu,\varrho\ge 0}} [\chi_{\lambda\mu\varrho}]\, \frac{s^{\alpha_0}(\ln s)^{-m-\lambda-\mu-\varrho}}{(s-4)^{\lambda+1}}$$

$$\times \left(-i\sigma + \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\lambda+\mu+\varrho+m-1+\nu}{\nu} \frac{r^{(\nu)}(0)}{(\alpha')^\nu} (\ln s)^{-\nu}\right) + \Omega\,(s^{\alpha_L(-T)-1}) \tag{2.22}$$

where the numbers $\chi_{\lambda\mu\varrho}$ are in detail

$$\chi_{\lambda\mu\varrho} \equiv (-1)^{\lambda+\mu+\varrho}\, \frac{(\lambda+\mu+\varrho+m-1)!}{\lambda!}\, \frac{2^\lambda P_l^{(\lambda)}(1)\, r_{\mu\varrho} Z_{\lambda+\mu}}{(\alpha')^{\lambda+\mu+\varrho+1}}$$

with $r_{\mu\varrho}$ the expansion coefficients of the double power series of R in t and t', and Z_K denotes

$$Z_K \equiv \sum_{\nu=0}^{m-2} \binom{m-2}{\nu} \frac{(-1)^\nu}{K+\nu+1}.$$

We conclude that the asymptotic behaviour of partial wave amplitudes can be represented as a definite expansion with respect to powers of both s and $\ln s$. Except dipole and triple pole exchanges ($n = 1, 2$) there are no terms with pure power behaviour in s. Simple and multifold poles as well as cut contributions lead to similar expressions (compare equs. (2.18) and (2.22)).
The pure Pomeron pole contribution to the partial waves is of special interest $\big(\alpha_P(0) = 1$, $\sigma_P = 1$, $n = 0\big)$. Its leading terms ($\lambda = 0$, $\mu = 0, 1$) are given by

$$A_l^P(s) \xrightarrow[s\to+\infty]{} \big(1+(-1)^l\big) \frac{\beta_P(0)}{\alpha_P'} \left(-\frac{i}{\ln s} + \frac{\pi}{2}\frac{1}{\ln^2 s} + i\,\frac{\beta_P'(0)}{\alpha_P'\beta_P(0)}\frac{1}{\ln^2 s} + 0\,(\ln^{-3} s)\right) \tag{2.23}$$

i.e. the imaginary part dominates as we expected.
In the following only the terms of definite powers in s are of interest. The factors in front of them including the derivatives of the residue functions etc. are not important because of the linear character of dispersion relations. We denote these typical terms of the asymptotic expansions (2.18) and (2.22) as follows

$$h^n_{\omega\tau}(s) \xrightarrow[s\to+\infty]{} \frac{s^{\alpha_0-1-\omega}}{(\ln s)^{\tau-n+1}} \left\{-i\sigma + \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\tau+\nu}{\nu} \frac{r^{(\nu)}(0)}{(\alpha')^\nu} (\ln s)^{-\nu}\right\} \tag{2.24}$$

with

$$n = \begin{cases} 0 & \text{for simple poles and cuts} \\ 1 & \text{for dipoles} \\ 2 & \text{for triple poles} \end{cases}$$

$$\tau \geqq \begin{cases} 0 & \text{for all pole terms} \\ m-1 & \text{for cuts } (m = 2, 3, \dots) \end{cases}$$

$$\omega \geqq 0$$

The power $s^{-\omega}$ corresponds to the powers of $s-4$ in the denominator of (2.18) and (2.22), respectively.
In the partial wave dispersion relations to be studied the asymptotic behaviour for $s \to -\infty$ is also needed. In order to get this behaviour we assume the absence of an essential singularity at infinity. Then we may derive the asymptotics for $s \to -\infty$ by symmetry considerations on the basis of expressions (2.12) and (2.20), respectively, assumed to be valid independently of the direction how s tends to infinity. For the moment let us regard simple pole and cut contributions ($n = 0$). Expanding the Legendre polynomial the terms of pure power behaviour in s are of the form

$$\int_{-T}^{0} dt\, \bar{B}(t)\, \frac{s^{\alpha(t)} + \sigma(-s)^{\alpha(t)}}{s^{\omega+1}}. \tag{2.25}$$

All these terms have a definite symmetry. They are even or odd functions in s according to the signature σ as well as to the power ω. Corresponding to this behaviour the imaginary part of partial wave amplitudes (or of the functions $h^n_{\omega\tau}(s)$, respectively) has to be chosen for $s \to -\infty$ in order to compute dispersion integrals belonging to left-hand cuts. For the multifold pole case ($n = 1, 2$) the expression (2.12) itself does not exhibit such a symmetry behaviour. Therefore we decompose all terms with a definite power in s into two terms of opposite symmetry, i.e. we introduce instead of (2.12)

$$\frac{1}{2}\int_{-T}^{0} dt\, \bar{B}(t)\, \frac{s^{\alpha(t)} + \sigma(-s)^{\alpha(t)}}{s^{\omega+1}} \left(\ln^n s \pm \sigma \ln^n(-s)\right). \tag{2.26}$$

For $n = 0$ this reduces to (2.25).
Transmitting these considerations to the expression (2.24) we introduce

$$h^{n\pm}_{\omega\tau}(s) \xrightarrow[s\to+\infty]{} \frac{1}{2}\frac{s^{\alpha_0-1-\omega}}{(\ln s)^{\tau+1}} \left\{-i\sigma + \sum_{\nu=0}^{\infty}(-1)^\nu \binom{\tau+\nu}{\nu} \frac{r^{(\nu)}(0)}{(\alpha')^\nu}(\ln s)^{-\nu}\right\} \{\ln^n s \pm \sigma \ln^n(-s)\} \tag{2.27}$$

with

$$h^n_{\omega\tau}(s) = h^{n+}_{\omega\tau}(s) + h^{n-}_{\omega\tau}(s).$$

For the imaginary parts of (2.27) we require

$$\operatorname{Im} h^{n\pm}_{\omega\tau}(-s + i\varepsilon) = \pm(-1)^\omega \operatorname{Im} h^{n\pm}_{\omega\tau}(s + i\varepsilon). \tag{2.28}$$

This result will be extensively used in the next section.

2.2. Determination of Very Short Range Forces and Derivation of Partial Wave Sum Rules

We study partial wave dispersion relations separated at a certain s_c into a lower energy and an asymptotic part, where the Regge behaviour (equs. (2.18), (2.22)) derived above is assumed to be valid. The asymptotic behaviour suggests an once subtracted dispersion relation. Choosing the subtraction point at $s = 0$ and omitting pole terms for simplicity we have

$$A_l(s) = a_l + \frac{s}{\pi}\int_{-s_c}^{-s_L} ds' \frac{\operatorname{Im} A_l(s'+i\varepsilon)}{s'(s'-s)} + \frac{s}{\pi}\int_{s_R}^{s_c} ds' \frac{\operatorname{Im} A_l(s'+i\varepsilon)}{s'(s'-s)}$$

$$+ \frac{s}{\pi}\int_{s_c}^{\infty} ds' \frac{\operatorname{Im} A_l(-s'+i\varepsilon)}{s'(s'+s)} + \frac{s}{\pi}\int_{s_c}^{\infty} ds' \frac{\operatorname{Im} A_l(s'+i\varepsilon)}{s'(s'-s)}. \tag{2.29}$$

Into the infinite integrals we insert the imaginary parts of the expansions (2.18) and (2.22), respectively. We ask for the behaviour of expression (2.29) in the asymptotic region $s \to +\infty$. Consistency of the dispersion relations with the behaviour (2.18) and (2.22) means that both sides of equ. (2.29) must yield the same asymptotic expansions. The occurence of additional powers in s on the right-hand side then leads to the validity of sum rules following from the requirement that the coefficient of each of these powers must vanish.
The integrals with finite boundaries yield pure power behaviour in s. Therefore only the infinite integrals in (2.29) can reproduce the correct asymptotic behaviour of the partial wave amplitudes $A_l(s)$. Let us study the asymptotics of the infinite integrals for the even and odd components $h^{n\pm}_{\omega\tau}(s)$ separately. At first the integrals are calculated for single logarithmic powers.

One gets expressions like

$$\frac{s}{\pi} P\int_{s_c}^{\infty} \frac{dx}{x^\alpha(x \pm s)\ln^\beta x} \xrightarrow[s\to+\infty]{} s^{1-\alpha}\sum_i \frac{a_i^{(\pm)}}{(\ln s)^{\beta+i}} + \sum_j \frac{b_j^{(\pm)}(s_c)}{s^j}. \tag{2.30}$$

The integrals are tabulated in detail in the appendix. The coefficients $a_i^{(\pm)}$ and $b_j^{(\pm)}$ are well-known. The latter depend on the above introduced "cutoff" s_c. The first sum on the right-hand side of (2.30) is responsible for producing the asymptotic behaviour of Re $A_l(s)$. It can be proved that the correct behaviour result only of the imaginary parts of the components $h^{n\pm}_{\omega\tau}$ along the right and left hand cuts obey the definite symmetry mentioned above (cf. equ. (2.28)). Thus we get a necessary condition for the consistency of dispersion relations with the high energy behaviour requiring that the far left-hand cut discontinuity has to possess a definite form. Therefore we have a prescription to determine very short range forces.
Furthermore both the finite integrals in (2.29) and the infinite integrals due to equ. (2.30) (and in the possible case of pole terms, too) produce an expansion of pure powers in s, which has to be cancelled. The cancellation of each power in s gives a definite sum rule

$$a_l\delta_{0r} - \frac{1}{\pi}\int_{-s_c}^{-s_L} ds'\, s'^{r-1} \operatorname{Im} A_l(s') - \frac{1}{\pi}\int_{s_R}^{s_c} ds'\, s'^{r-1} \operatorname{Im} A_l(s') = \gamma^{(r)}; \qquad r = 0, 1, 2, \ldots \tag{2.31}$$

The constants $\gamma^{(r)}$ are due to the pure powers in s in equ. (2.30). Therefore they depend upon s_c and Regge parameters. The value s_c is chosen so that $(\ln s_c)^{-1} \ll 1$. For practical purposes $s_c \simeq 10^4$ will be sufficiently large as to be discussed in more detail later on.
In the following we confirm all these statements and calculate the $\gamma^{(r)}$ by studying in detail dipole and Regge cut contributions. The ordinary pole contributions (Pomeron pole with $\alpha_P(0) = 1$, $\sigma_P = +1$ and Regge poles with $\alpha(0) < 1$, $\sigma = \pm 1$) have been considered already in paper [11].

1) Dipole Pomeron contribution $\left(n = 1,\ \alpha_P(0) = 1,\ \sigma_P = +1\right)$

Taking into account $\ln(-s) = \ln s - i\pi$ the leading terms of the functions $h^{n-1\pm}_{\omega\tau}(s)$ can be expressed in the following way

$$h^{\pm}_{\omega\tau}(s) \xrightarrow[s\to+\infty]{} \frac{1}{2s^\omega(\ln s)^\tau}\left[\left\{\begin{bmatrix}\tau\\1\end{bmatrix}\frac{\pi}{\ln s} + \begin{bmatrix}1\\0\end{bmatrix}(\tau+1)(\tau+2)(\tau+3)\frac{\pi^3}{12}\frac{1}{\ln^3 s} + \cdots\right\}\right.$$

$$\left. + i\left\{\begin{bmatrix}-1\\0\end{bmatrix} + \begin{bmatrix}-1\\+1\end{bmatrix}(\tau+1)\frac{\pi^2}{2}\frac{1}{\ln^2 s} + \begin{bmatrix}-1\\+1\end{bmatrix}(\tau+1)(\tau+2)(\tau+3)\frac{\pi^4}{24}\frac{1}{\ln^4 s} + \cdots\right\}\right] \tag{2.32}$$

where

$$r_P(0) = 0,\quad r_P'(0) = -\frac{\pi}{2}\alpha_P',\quad r_P''(0) = 0,\quad r_P'''(0) = -\frac{\pi^3}{4}(\alpha_P')^3, \ldots$$

have been used. In the square brackets the upper value belongs to h^+ and the lower to h^-, respectively. Now one can show power by power that the asymptotic dispersion integrals written for $h^\pm$ with imaginary parts corresponding to (2.32) and (2.28) yield the real parts in equ. (2.32), indeed, and additionally the pure power terms in s mentioned above. We get in the case of even functions h^+ (compare with equ. (A.2)) in the limit $s \to +\infty$

$$\frac{s}{\pi} P \int_{s_c}^{\infty} ds' \left(\frac{(-1)^\omega}{s'+s} + \frac{1}{s'-s}\right) \frac{\operatorname{Im} h^+_{\omega\tau}(s')}{s'}$$

$$= \frac{1}{2} \frac{1}{s^\omega (\ln s)^\tau} \left[\tau \frac{\pi}{\ln s} + (\tau+1)(\tau+2)(\tau+3) \frac{\pi^3}{12} \frac{1}{\ln^3 s} + \cdots\right]$$

$$- \sum_{\substack{p=0\\ p\neq\omega}}^{\infty} \left(1 - (-1)^{p+\omega}\right) \frac{1}{s^p} \left\{\frac{1}{2\pi} \frac{s_c^{\,p-\omega}}{(p-\omega)(\ln s_c)^\tau} \left(1 + O(\ln^{-1} s_c)\right)\right\}. \tag{2.33}$$

Here we are allowed to neglect all lower powers in $\ln s_c$ because s_c is assumed to be sufficiently large.
Analogously the odd functions h^- are treated. However, contrary to h^+ and to the treatment of simple Regge pole contributions [11] the first term of the right-hand side of equ. (A.2) will not be cancelled. It must be kept to produce the asymptotic behaviour of $\operatorname{Re} h^-$ which surpasses the imaginary part by one $\ln s$ power.
In the h^- case we get

$$\frac{s}{\pi} P \int_{s_c}^{\infty} ds' \left(\frac{-(-1)^\omega}{s'+s} + \frac{1}{s'-s}\right) \frac{\operatorname{Im} h^-_{\omega\tau}(s')}{s'} = \operatorname{Re} h^-_{\omega\tau}(s)$$

$$+ \sum_{\substack{p=0\\ p\neq 0}}^{\infty} \left(1 + (-1)^{p+\omega}\right) \frac{1}{s^p} \left\{\frac{\pi}{4} (\tau+1) \frac{s_c^{\,p-\omega}}{(p-\omega)(\ln s_c)^{\tau+2}} \left(1 + O(\ln^{-1} s_c)\right)\right\} - \frac{1}{s^\omega} \frac{\pi/2}{(\ln s_c)^{\tau+1}} \tag{2.34}$$

The last term on the right-hand side is due to the uncompensated first contribution in (A.2) and will give an important contribution to the constants $\gamma^{(r)}$ in equ. (2.31).
With respect to the s_c dependence the leading contributions to the $\gamma^{(r)}$ are given by $(r = 0, 1)$

$$\gamma^{(0)} \sim \ln^{-1} s_c\left(1 + O(\ln^{-1} s_c)\right)$$
$$\gamma^{(1)} \sim s_c\left(1 + O(\ln^{-1} s_c)\right). \tag{2.35}$$

Here we have omitted all proportionality factors depending on the Regge "residue" and slope parameters (compare with equ. (2.18)). Our result shows the following. In the $\gamma^{(r)}$ the powers $\omega = 0$ and $\tau = 0$ give the essential contributions. This means that on the one side it is sufficient to regard only the leading s power in (2.18), whereas on the other side the proportionality factors in equ. (2.35) do not depend on the derivatives of the "residue" function $\beta(\alpha(t), t)$.
If the total amplitude is dominated by a dipole Pomeron, then there is indeed according to equs. (2.7) and (2.8) additionally a term like a simple Pomeron pole ($n = 0$) but with negative signature. Contributions of that kind were not considered in [11]. Since the real part of the signature factor $r(t)$ does not exist at $t = 0$, we cannot expand $r(t)$ as in (2.13) and (2.14). Therefore we include the denominator $\sin \pi\alpha(t)$ in the "residue" β and expand real and imaginary part of the numerator of the signature factor separately.

It leads to a modification in (2.18). Denoting

$$1 - \cos \pi\alpha(t) \equiv R(t) \qquad \sin \pi\alpha(t) \equiv I(t)$$

we get for the "Pomeron contribution with negative signature" the following expression (compare with equ. (2.24))

$$h_{\omega\tau}(s) \xrightarrow[s\to+\infty]{} \frac{1}{s^\omega (\ln s)^{\tau+1}} \left\{\sum_{\nu=0}^{\infty} (-1)^\nu \binom{\tau+\nu}{\nu} \frac{R^{(\nu)}(0) + iI^{(\nu)}(0)}{(\alpha')^\nu} \frac{1}{(\ln s)^\nu}\right\}$$

with $\omega, \tau = 0, 1, 2, \ldots$ There are no additional powers in $\ln s$ ($n = 0$). Therefore $h_{\omega\tau}(s)$ is identical with the corresponding function with odd symmetry $h^-_{\omega\tau}$, i.e.

$$h_{\omega\tau}(s) \equiv h^-_{\omega\tau}(s) \xrightarrow[s\to+\infty]{} \frac{1}{s^\omega (\ln s)^{\tau+1}}$$

$$\times \left\{i \left[(\tau+1)\frac{\pi}{\ln s} - \frac{1}{6}(\tau+1)(\tau+2)(\tau+3)\frac{\pi^3}{\ln^3 s} + \cdots\right]\right.$$

$$\left. + \left[2 - \frac{1}{2}(\tau+1)(\tau+2)\frac{\pi^2}{\ln^2 s} + \cdots\right]\right\}. \tag{2.36}$$

The dispersion integrals are treated as before using the symmetry behaviour (2.28). The calculation shows that the real part is again reproduced exactly. The power expansion in s finally reads

$$\sum_{\substack{p=0\\ p\neq\omega}}^{\infty} \left(1 + (-1)^{p+\omega}\right) \frac{1}{s^p} \left\{(\tau+1) \frac{s_c^{\,p-\omega}}{(p-\omega)(\ln s_c)^{\tau+2}} \left(1 + O(\ln^{-1} s_c)\right)\right\} - \frac{2}{s^\omega (\ln s_c)^{\tau+1}}$$

and the contributions to the $\gamma^{(r)}$ are determined to be

$$\gamma^{(0)} \sim \ln^{-1} s_c\left(1 + O(\ln^{-1} s_c)\right)$$
$$\gamma^{(1)} \sim \ln^{-1} s_c\left(1 + O(\ln^{-1} s_c)\right). \tag{2.37}$$

Thus the "negatively signatured Pomeron" ($n = 0$) will give a contribution to $\gamma^{(0)}$ of the same order as the $n = 1$ term of the dipole, whereas in $\gamma^{(1)}$ the $n = 1$ term dominates alone.

2) Dipole Regge contributions ($n = 1$, $0 < \alpha_0 < 1$, $\sigma = \pm 1$)

The more general case of trajectory intercepts $\alpha_0 < 1$ is treated in the same way as before. Only the evaluation of the asymptotic integrals leads to more complicated froms. We have

$$h^\pm_{\omega\tau}(s) \xrightarrow[s\to+\infty]{} \frac{1}{2} \frac{s^{\alpha_0-1}}{s^\omega (\ln s)^\tau} \left[\left\{\begin{bmatrix}1+\sigma\\1-\sigma\end{bmatrix} r(0) - \left(\begin{bmatrix}1+\sigma\\1-\sigma\end{bmatrix}(\tau+1)\frac{r'(0)}{\alpha'} + \begin{bmatrix}+1\\-1\end{bmatrix}\pi\right)\frac{1}{\ln s} + \cdots\right\}\right.$$

$$\left. + i\left\{-\sigma\begin{bmatrix}1+\sigma\\1-\sigma\end{bmatrix} + \begin{bmatrix}-1\\+1\end{bmatrix}\sigma r(0)\frac{\pi}{\ln s} + \cdots\right\}\right]. \tag{2.38}$$

According to equ. (2.28) and equs. (A.3), (A.4) we get in the case of negative signature the following results. For $\sigma = -1$ and the even function $h^+_{\omega\tau}$

$$\frac{s}{\pi} P \int_{s_c}^{\infty} ds' \left(\frac{(-1)^\omega}{s'+s} + \frac{1}{s'-s}\right) \frac{\operatorname{Im} h^+_{\omega\tau}(s')}{s'} = -\frac{1}{2} \frac{s^{\alpha_0 - 1} r(0)}{s^\omega (\ln s)^{\tau+1}}$$

$$\times \left(\zeta(1, \alpha_0) + \zeta(1, 1-\alpha_0) - \zeta\left(1, \frac{1+\alpha_0}{2}\right) - \zeta\left(1, \frac{2-\alpha_0}{2}\right) + \zeta(1, \alpha_0) - \zeta(1, 1-\alpha_0)\right) + \cdots$$

$$-\sum_{i=0}^{\infty} \left(1 - (-1)^{p+\omega}\right) \frac{1}{s^p} \left\{\frac{r(0)}{2} \frac{s_c^{p+\alpha_0-1-\omega}}{(p+\alpha_0-1-\omega)(\ln s_c)^{\tau+1}} \left(1 + O(\ln^{-1} s_c)\right)\right\}$$

$$= -\frac{\pi}{2} \frac{s^{\alpha_0 - 1}}{s^\omega (\ln s)^{\tau+1}} \left(\frac{1-\cos \pi\alpha_0}{\sin \pi\alpha_0}\right)\left(\frac{1+\cos \pi\alpha_0}{\sin \pi\alpha_0}\right) + \cdots = \operatorname{Re} h^+_{\omega\tau}(s) + \cdots. \quad (2.39)$$

In the last two equations we have dropped the power expansion in s. To derive (2.39) we used properties of the generalized zeta function

$$\zeta(1, \alpha_0) + \zeta(1, 1-\alpha_0) - \zeta\left(1, \frac{1+\alpha_0}{2}\right) - \zeta\left(1, \frac{2-\alpha_0}{2}\right) = \frac{\pi}{\sin \pi\alpha_0} \quad (2.40)$$

$$\zeta(1, \alpha_0) - \zeta(1, 1-\alpha_0) = \pi \cot \pi\alpha_0 . \quad (2.41)$$

In the same way the calculations for h^- with $\sigma = -1$ and $h^\pm$ with $\sigma = +1$ can be done. In each case the symmetry behaviour (2.28) is confirmed. Apart from the correct reproduction of the real part of the asymptotic expansion of $A_l(s)$ we are interested mainly in the pure power expansion in s to calculate the $\gamma^{(r)}$. Therefore we write down for the remaining cases only the corresponding power expansions. For $\sigma = -1$ and the odd function $h^-_{\omega\tau}$ we get

$$+\sum_{p=0}^{\infty} \left(1 + (-1)^{p+\omega}\right) \frac{1}{s^p} \left\{\frac{1}{\pi} \frac{s_c^{p+\alpha_0-1-\omega}}{(p+\alpha_0-1-\omega)(\ln s_c)^\tau} \left(1 + O(\ln^{-1} s_c)\right)\right\}$$

for $\sigma = +1$, $h^+_{\omega\tau}$

$$-\sum_{p=0}^{\infty} \left(1 - (-1)^{p+\omega}\right) \frac{1}{s^p} \left\{\frac{1}{\pi} \frac{s_c^{p+\alpha_0-1-\omega}}{(p+\alpha_0-1-\omega)(\ln s_c)^\tau} \left(1 + O(\ln^{-1} s_c)\right)\right\}$$

for $\sigma = +1$, $h^-_{\omega\tau}$

$$+\sum_{p=0}^{\infty} \left(1 + (-1)^{p+\omega}\right) \frac{1}{s^p} \left\{\frac{r(0)}{2} \frac{s_c^{p+\alpha_0-1-\omega}}{(p+\alpha_0-1-\omega)(\ln s_c)^{\tau+1}} \left(1 + O(\ln^{-1} s_c)\right)\right\}.$$

This yields finally the leading contributions to the $\gamma^{(r)}$. Für $\sigma = -1$ we find

$$\gamma^{(0)} \sim s_c^{\alpha_0 - 1}\left(1 + O(\ln^{-1} s_c)\right)$$
$$\gamma^{(1)} \sim s_c^{\alpha_0} \ln^{-1} s_c \left(1 + O(\ln^{-1} s_c)\right)$$

and for $\sigma = +1$ (2.42)

$$\gamma^{(0)} \sim s_c^{\alpha_0 - 1} \ln^{-1} s_c \left(1 + O(\ln^{-1} s_c)\right)$$
$$\gamma^{(1)} \sim s_c^{\alpha_0}\left(1 + O(\ln^{-1} s_c)\right)$$

3) m-fold Pomeron exchange cut ($n = 0$, $\alpha_P(0) = 1$, $\sigma_P = +1$)

The calculations proceed in this case in the same way as before. Because of $n = 0$ similar to the case of simple Regge poles one of the functions of definite symmetry (2.27) is identical with $h_{\omega\tau}(s)$ itself. So the discussion requires a smaller expense.
The leading terms of $h^+_{\omega\tau} \equiv h_{\omega\tau}$ are

$$h_{\omega\tau}(s) \xrightarrow[s \to +\infty]{} \frac{1}{s^\omega (\ln s)^{\tau+1}} \left[-i + \frac{\pi}{2}(\tau+1)\frac{1}{\ln s} + \frac{\pi^3}{24}(\tau+1)(\tau+2)(\tau+3)\frac{1}{\ln^3 s} + \cdots\right]. \quad (2.43)$$

Considering the symmetry behaviour (2.28) the asymptotic integrals yield the correct real part of $h_{\omega\tau}$ and the following power expansion in s

$$-\sum_{\substack{p=0 \\ p \neq \omega}}^{\infty} \left(1 - (-1)^{p+\omega}\right) \frac{1}{s^p} \left\{\frac{1}{\pi} \frac{s_c^{p-\omega}}{(p-\omega)(\ln s_c)^{\tau+1}} \left(1 + O(\ln^{-1} s_c)\right)\right\}.$$

Therefore the leading contributions to the $\gamma^{(r)}$ of the highest order sum rules ($r = 0, 1$) are ($\tau = m - 1$, $m = 2, 3, \ldots$)

$$\left.\begin{aligned} \gamma^{(0)} &\sim \frac{1}{s_c (\ln s_c)^m}\left(1 + O(\ln^{-1} s_c)\right) \\ \gamma^{(1)} &\sim \frac{s_c}{(\ln s_c)^m}\left(1 + O(\ln^{-1} s_c)\right). \end{aligned}\right\} \quad (2.44)$$

4) m-fold Reggeon exchange cut ($n = 0$, $0 < \alpha_0 < 1$, $\sigma = \pm 1$)

Because of equ. (2.27) we have

$$h^\mp_{\omega\tau} \equiv 0 \quad \text{if} \quad \sigma = \mp 1 .$$

Thus the signature determines the symmetry behaviour. Correspondingly we introduce the notation $h^\sigma_{\omega\tau}(\equiv h_{\omega\tau})$. Then equ. (2.28) takes the form

$$\operatorname{Im} h^\sigma_{\omega\tau}(-s + i\varepsilon) = \sigma(-1)^\omega \operatorname{Im} h^\sigma_{\omega\tau}(s + i\varepsilon).$$

The leading terms are

$$h^\sigma_{\omega\tau}(s) \xrightarrow[s \to +\infty]{} \frac{s^{\alpha_0 - 1}}{s^\omega (\ln s)^{\tau+1}} \left[-i\sigma + r(0) - \frac{r'(0)}{\alpha'}(\tau+1)\frac{1}{\ln s} + \cdots\right]. \quad (2.45)$$

The asymptotic dispersion integrals then produce the correct real part and as before a power expansion in s

$$\frac{s}{\pi} P \int_{s_c}^{\infty} ds' \left(\frac{\sigma(-1)^\omega}{s'+s} + \frac{1}{s'-s}\right) \frac{\operatorname{Im} h^\sigma_{\omega\tau}(s')}{s'}$$

$$= \operatorname{Re} h^\sigma_{\omega\tau}(s) - \sum_{p=0}^{\infty} \left(\sigma - (-1)^{p+\omega}\right) \frac{1}{s^p} \left\{\frac{1}{\pi} \frac{s_c^{p+\alpha_0-1-\omega}}{(p+\alpha_0-1-\omega)(\ln s_c)^{\tau+1}} \left(1 + O(\ln^{-1} s_c)\right)\right\}.$$

Finally we get the corresponding leading contributions to the $\gamma^{(r)}(r = m - 1, m = 2, 3, \ldots)$ i.e. for

positive signature $\sigma = +1$	negative signature $\sigma = -1$
$\gamma^{(0)} \sim \frac{s_c^{\alpha_0 - 2}}{(\ln s_c)^m}$	$\gamma^{(0)} \sim \frac{s_c^{\alpha_0 - 1}}{(\ln s_c)^m}$
$\gamma^{(1)} \sim \frac{s_c^{\alpha_0}}{(\ln s_c)^m}$	$\gamma^{(1)} \sim \frac{s_c^{\alpha_0 - 1}}{(\ln s_c)^m}$

(2.46)

In Table I we tabulated all contributions to the first two $\gamma^{(r)}$ including the leading Regge pole contributions evaluated previously in [11].

Table I

Leading contributions to the first $\gamma^{(r)}$ constants (scalar case)

	Regge poles [11]	Pomeron / Reggeon dipole contribution			"Pomeron pole with	m-fold exchange cut Pomeron / Reggeon		
	$n = 0$	$n = 1$ $\sigma = +1$	$\sigma = +1$	$\sigma = -1$	$n = 0$ $\sigma = -1$"***)	$n = 0$ $\sigma = +1$	$\sigma = +1$	$\sigma = -1$
$\gamma^{(0)}$	$\sim \frac{s_c^{\alpha_0 - 1}}{\ln s_c}$ *)	$\sim \frac{1}{\ln s_c}$	$\sim \frac{s_c^{\alpha_0 - 1}}{\ln s_c}$	$\sim s_c^{\alpha_0 - 1}$	$\sim \frac{1}{\ln s_c}$	$\sim \frac{1}{s_c (\ln s_c)^m}$	$\frac{s_c^{\alpha_0 - 2}}{(\ln s_c)^m}$	$\frac{s_c^{\alpha_0 - 1}}{(\ln s_c)^m}$
$\gamma^{(1)}$	$\sim \frac{s_c}{\ln s_c}$ **)	$\sim s_c$	$\sim s_c^{\alpha_0}$	$\sim \frac{s_c^{\alpha_0}}{\ln s_c}$	$\sim \frac{1}{\ln s_c}$	$\sim \frac{s_c}{(\ln s_c)^m}$	$\frac{s_c^{\alpha_0}}{(\ln s_c)^m}$	$\frac{s_c^{\alpha_0 - 1}}{(\ln s_c)^m}$

*) Regge pole contribution with negative signatured trajectory (ϱ-trajectory), $\alpha_0 < 1$
**) Pomeron pole contribution $\sigma_P = +1$
***) This term arises in connection with the dipole Pomeron

As one should naively expect the dipole Pomeron will really dominate in the sum rules (2.31) if s_c is sufficiently large²). We remark that the additionally arising Pomeron pole like terms with negative signature will contribute to $\gamma^{(0)}$ in the same leading order as the corresponding dipole term with logarithmic power $n = 1$.
Summarizing we have classified the main contributions to the sum rules (2.31) due to different Regge models with real trajectories. In the last chapter we shall apply the sum rules to the realistic case of πN scattering. It will turn out that the results are highly independent on the special Regge ansatz. Therefore we shall regard in the following discussions only simple Regge poles which are better known from experimental fits being valid in the intermediate energy-regions.
Formally the sum rules (2.31) are similar to the finite energy sum rules for the total amplitudes [10]. But in our case it is generally impossible to express the finite left-hand cut integral via crossing symmetry by the right-hand cut contribution. This is the main disadvantage of these conditions, because there is not much known about the short range forces. We shall show how this difficulty can be surrounded in our application to πN scattering. One should remark that the equations (2.31) permit a duality interpretation for partial waves. The left-hand cuts are determined by t- and u-channel contributions together. Therefore the resonances of all three channels are simultaneously creating the s-channel high energy behaviour represented by the constands $\gamma^{(r)}$.

²) This is not a matter of course. As Table I shows among the ordinary Regge poles it is not the Pomeron pole which dominates in $\gamma^{(0)}$ but a Regge pole with negative signature (ϱ-trajectory). This means that the dominance in the total scattering amplitude is not tantamount with the dominance in the partial wave sumrules.

3. Helicity Partial Wave Amplitudes and the Formulation of Sum Rules

3.1. High Energy Behaviour of Helicity Partial Wave Amplitudes

To apply sum rules of the same kind as (2.31) to more realistic cases than the totally crossing symmetric scalar one (realized e.g. in the $\pi^0\pi^0$ elastic scattering) we generalize our investigations to the case of elastic scattering of particles with arbitrary spin. To study πN scattering later on we assume also different masses. It is convenient to use the helicity formalism.
The general assumptions (dominance of sharp forward and backward scattering, analyticity of Regge residue functions, linearly rising Regge trajectories, validity of dispersion relations) and the methods of derivation are essentially the same ones. Therefore only the main steps will be sketched. But contrary to the investigations in chapter 2 we shall restrict ourselves to contributions of single Regge-poles, although these will generally not give the leading influence in the corresponding constants $\gamma^{(r)}$ as shown below. However, we are justified for doing this because in the concrete case of πN scattering the first sum rules are not sensitive with respect to the special Regge parametrization, so that a pole approximation will suffice.
The description of scattering processes by helicity amplitudes has the advantage that the decomposition into invariant amplitudes can be avoided. Furthermore for studying Regge asymptotic behaviour t- und u-channel exchange contributions can be considered in a unified way. On the other hand care is necessary because of kinematic singularities and crossing relations. In our investigations conventions of Drechsler [15] and Cohen-Tannoudji et al. [16] will be employed. Properties of the rotation functions of the first kind are tabulated there, too.
The partial wave projection usually used can be written as in chapter 2 assuming dominance of enhanced forward and backward scattering (which is connected with a power behaviour (2.3) applied to the total s-channel helicity amplitude)

$$F^j_{\lambda_1'\lambda_2',\lambda_1\lambda_2}(s) \xrightarrow[s\to+\infty]{} \frac{1}{2}\left(\int_{-1}^{-1+\varepsilon} + \int_{1-\varepsilon}^{1}\right) dz\, f_{\lambda_1'\lambda_2',\lambda_1\lambda_2}(s, z)\, d^j_{\lambda\lambda'}(z) \quad (3.1)$$

with $\lambda = \lambda_1 - \lambda_2$, $\lambda' = \lambda_1' - \lambda_2'$. The quantities λ_i und λ_i' are the helicities of the incoming and outgoing particles, respectively. $d^j_{\lambda\lambda'}$ represents the rotation function [15]. The mass of one particle will be denoted by M. We use a normalization where the smaller mass of the other particle is put $m = 1$. As usually the Mandelstam variables s, t, u are defined by

$$s + t + u = 2M^2 + 2.$$

Moreover we have

$$z = 1 + \frac{t}{2q^2} = -1 - \frac{u - \frac{\Delta}{s}}{2q^2} \quad (3.2)$$

with

$$\Delta = (M^2 - 1)^2$$

$$q^2 = \frac{\left(s - (M+1)^2\right)\left(s - (M-1)^2\right)}{4s}$$

(s- and u-channel are assumed to be crossing symmetric to each other). The following Regge pole ansatz is applied

$$f_{\{\lambda\}}(s, z) \xrightarrow[s \to +\infty]{} \sum_{\substack{\text{poles} \\ t \text{ channel}}} \left[\left(\sqrt{-t}\right)^{|\lambda - \lambda'|} B^t_{\{\lambda\}}(t) \, \frac{1 + \sigma_t \exp(-i\pi\alpha)}{\sin \pi\alpha} \, s^{\alpha(t)} \right]$$
$$+ \sum_{\substack{\text{poles} \\ u \text{ channel}}} \left[\left(\sqrt{-u + \frac{\Delta}{s}} \right)^{|\lambda + \lambda'|} B^u_{\{\lambda\}}(u) \, \frac{1 + \sigma_u \exp\left(-i\pi(\alpha - v)\right)}{\sin \pi(\alpha - v)} \, s^{\alpha(u)} \right] \quad (3.3)$$

with

$$v = \begin{cases} 0 & \text{for boson-boson, fermion-fermion processes} \\ 1/2 & \text{for boson-fermion processes.} \end{cases}$$

All Regge terms themselves should satisfy the condition (2.3) and the trajectories are assumed to be linearly rising at least in a certain region as discussed in the scalar case. In the functions $B^{t,u}$ Regge residues as well as the crossing matrices [16] are involved. It is not important for our purposes to know them explicitly. However, it turns out that the functions $B^{t,u}$ are free of kinematic singularities for $t, u \leq 0$, if the residues of t- and u-channel Regge poles are chosen appropriately[3]). The remaining square root singularities factorize and have been written separately in equ. (3.3). Moreover one can choose a dip mechanism (nonsense-choosing) killing all unphysical poles of the signature factor in the regions $t, u \leq 0$.

In the ansatz (3.3) and in the following we consider only terms of highest order in s, and this for each pole contribution. So we neglect e.g. the influence of daughter trajectories in the u-channel needed to avoid difficulties connected with kinematics of different masses. The neglect of lower powers in s can be justified as in the scalar case (conclusion below equ. (2.35)).

The ansatz (3.3) is inserted into (3.1). By transforming the integrations and introducing the lower limits $-T$, $-U$ (with T, U sufficiently large, but independent on s) we get

$$F^j_{\{\lambda\}}(s) \xrightarrow[s \to +\infty]{} \frac{(-1)^{j-\lambda'}}{4q^2} \, s^v \sum_{u \text{ poles}} \left[\int_{-U}^{0} du \left(\sqrt{-u}\right)^{|-\lambda-\lambda'|} B^u_{\{\lambda\}}(u) \, \frac{1 + \sigma_u \exp\left(-i\pi\bar{\alpha}(u)\right)}{\sin \pi\bar{\alpha}(u)} \right.$$
$$\left. \times \, s^{\bar{\alpha}(u)} \, d^j_{-\lambda\lambda'} \left(1 + \frac{u}{2q^2}\right) \right]$$
$$+ \frac{1}{4q^2} \sum_{t \text{ poles}} \left[\int_{-T}^{0} dt \left(\sqrt{-t}\right)^{|\lambda-\lambda'|} B^t_{\{\lambda\}}(t) \, \frac{1 + \sigma_t \exp\left(-i\pi\alpha(t)\right)}{\sin \pi\alpha(t)} \right.$$
$$\left. \times \, s^{\alpha(t)} \, d^j_{\lambda\lambda'} \left(1 + \frac{t}{2q^2}\right) \right] \quad (3.4)$$

with the shifted trajectory $\bar{\alpha}(u) \equiv \alpha(u) - v$.

[3]) In the boson-fermion case difficulties with a nonfactorizable $\sqrt{u}$ singularity of u-channel residues can be avoided by introducing parity doublets.

A Taylor expansion of the t and u dependent terms in (3.4) leads again to integrals of the type (2.15). Their evaluation yields the asymptotic expansion for the helicity partial wave amplitude (leading powers in s)

$$F^j(s) \xrightarrow[s \to +\infty]{} \sum_{t,u \text{ poles}} \sum_{\mu=0}^{\infty} \frac{(-1)^\mu (\mu + |\delta\lambda - \lambda'|)!}{\mu!} \, \frac{B^{(\mu)}(0)}{[\alpha'(0)]^{\mu + |\delta\lambda - \lambda'| + 1}}$$
$$\times \frac{s^{\alpha(0) - (|\delta\lambda - \lambda'|/2) - 1}}{(\ln s)^{\mu + |\delta\lambda - \lambda'| + 1}} \left[-i\sigma + \sum_{k=0}^{\infty} (-1)^k \binom{\mu + k + |\delta\lambda - \lambda'|}{k} \frac{r^{(k)}(0)}{(\alpha')^k} \frac{1}{(\ln s)^k} \right. \quad (3.5)$$

with

$$\delta = \begin{cases} +1 \\ -1 \end{cases} \text{for} \quad \begin{matrix} t- \\ u- \end{matrix} \Bigg\} \text{channel contributions.}$$

Helicity indices have been omitted. The coefficients $r^{(k)}(0)$ are the derivatives of the real part of the signature factor (compare with expression (2.14); in the u-channel case the trajectory is replaced by the shifted one $\bar{\alpha}$).

In the special case of πN scattering assuming dominance of the Pomeron pole it turns out that in leading order both spin-flip and nonflip amplitudes are imaginary. As a kinematic consequence helicity conservation is obtained

$$\left. \begin{aligned} F^j_{++}(s) &\xrightarrow[s \to +\infty]{} -i \, \frac{B^P_{++}(0)}{\alpha_P'(0)} \, \frac{1}{\ln s} + \cdots \\ F^j_{+-}(s) &\xrightarrow[s \to +\infty]{} -i \left(j + \frac{1}{2}\right) \frac{B^P_{+-}(0)}{\left(\alpha_P'(0)\right)^2} \, \frac{1}{\sqrt{s}\,(\ln s)^2} + \cdots \end{aligned} \right\} \quad (3.6)$$

3.2. Dispersion Relations for Helicity Partial Waves and Sum Rules

Let us introduce partial wave amplitudes with normality n

$$F^{jn}_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(s) = F^j_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(s) + n\eta_1\eta_2(-1)^{s_1 - s_2 - v} \, F^j_{-\lambda_1' - \lambda_2' \lambda_1\lambda_2}(s). \quad (3.7)$$

Here η_i and s_i represent the internal parity and the spin of the i-th particle, respectively. In the following we use reduced amplitudes

$$H^{jn}_{\{\lambda\}}(W) \equiv s^j(w) \, F^{jn}_{\{\lambda\}}(W), \qquad W \equiv \sqrt{s} \quad (3.8)$$

with the threshold factor

$$s^j(W) = \frac{W^{4j}}{(4sq^2)^{j - s_1 - s_2}}$$

chosen so that

1. the generalized Mac Dowell symmetry [17] is satisfied

$$H^{jn}_{\{\lambda\}}(-W) = (-1)^{\lambda - \lambda'} \, H^{j[(-1)^{2\lambda'} n]}_{\{\lambda\}}(W) \quad (3.9)$$

defining the analytic continuation of the amplitude from Re $W > 0$ to Re $W < 0$,
2. $H^{jn}_{\{\lambda\}}$ has no kinematic singularities in the W plane [16],
3. the asymptotic behaviour of $F^{jn}_{\{\lambda\}}$ is modified in a manner independently on j,
4. the dynamical threshold behaviour following from a generalized Froissart-Gribov representation is taken into account.

Assuming a Mandelstam representation for amplitudes without kinematic singularities the location of dynamical singularities of the partial wave amplitudes H^{jn} can be found. The result is the same as known from elastic πN scattering [18].
In the s plane we get the following dynamical cuts (if pole terms are omitted for simplicity)

1. $|s| = M^2 - 1$, a circle cut,
2. $-\infty \leqq s \leqq -M^2 + 1, \quad -M^2 + 1 \leqq s \leqq 0, \quad 0 \leqq s \leqq (M-1)^2$,
3. $(M+1)^2 \leqq s \leqq +\infty$ the physical cut.

According to these cuts dispersion relations are written for h^{jn} in the W plane. With reference to the behaviour (3.5) the number of subtractions chosen at $W = 0$ is

$$p = 4(s_1 + s_2) - \big||\lambda| - |\lambda'|\big| + 1. \tag{3.10}$$

Analogously to the scalar case (equ. (2.29)) the integrals are separated at certain sufficiently high W_c, namely at both the imaginary and the real axis. Property (3.9) as well as real analyticity lead to

$$\begin{aligned} H^{jn}(W) = {} & \sum_{n=0}^{p-1} a_n W^n + \frac{W^p}{\pi} \int_U^{iW_c} dW' \frac{\operatorname{disc} H^{jn}(W')}{W'^p(W'-W)} \\ & + \frac{W^p}{\pi} \int_{M+1}^{W_c} dW' \frac{\operatorname{Im} H^{jn}(W'+i\varepsilon)}{W'^p(W'-W)} + (-1)^{p+\lambda-\lambda'} \frac{W^p}{\pi} \int_{M+1}^{W_c} dW' \frac{\operatorname{Im} H^{j[(-1)^{2\lambda'} n]}(W'+i\varepsilon)}{W'^p(W'+W)} \\ & + \frac{W^p}{2\pi} \int_{s_c=W_c^2}^{\infty} ds' \frac{\Delta_R H^{jn}(\sqrt{s'}, W)}{s'^{(p+1)/2}(s'-W^2)} - (-1)^{-(p+1)/2} \frac{W^p}{2\pi} \int_{s_c=W_c^2}^{\infty} ds' \frac{\Delta_{JM} H^{jn}(\sqrt{s'}, W)}{s'^{(p+1)/2}(s'+W^2)} \end{aligned} \tag{3.11}$$

with

$$\begin{aligned} \Delta_R H^{jn}(\sqrt{s'}, W) \equiv {} & \operatorname{Im} H^{jn}(\sqrt{s'} + i\varepsilon)\,(\sqrt{s'} + W) + (-1)^{p+\lambda-\lambda'} \\ & \times \operatorname{Im} H^{j[(-1)^{2\lambda'} n]}(\sqrt{s'} + i\varepsilon)\,(\sqrt{s'} - W) \\ \Delta_{JM} H^{jn}(\sqrt{s'}, W) \equiv {} & -\operatorname{disc} H^{jn}(\sqrt{s'} + \varepsilon)\,(\sqrt{s'} + W) - (-1)^{p+1} \\ & \times \operatorname{disc} H^{jn}(-\sqrt{s'} - \varepsilon)\,(\sqrt{s'} - W). \end{aligned} \tag{3.12}$$

The first integral in equ. (3.11) carrying the notation U belongs to all finite unphysical cuts including the circle and possible pole terms. $\Delta_R H^{jn}$ is determined by the imaginary parts of the expansion (3.5). $\Delta_{JM} H^{jn}$ has to be chosen in such a way that both the infinite integrals of (3.11) will reproduce the correct asymptotic behaviour of $\operatorname{Re} H^{jn}(W)$ according to (3.5). The integrals with finite boundaries yield an expansion of powers in pure $W^{-\nu} (\nu > 0)$ which is absent in the expansion (3.5). To prove the existence of an appropriate $\Delta_{JM} H^{jn}$ — the determination of which is tantamount to an evaluation of very short range forces — it is convenient to regard only components of (3.5) (cf. expression (2.24)). Therefore we introduce the following auxiliary function rather than $H^{j}_{\{\lambda\}}(W)$

$$h_m(w) = \frac{W^{2\alpha(0)+n}}{(\ln W^2)^{m+1}} \left\{ -i\sigma + \sum_{k=0}^{\infty} (-1)^k \binom{m+k}{k} \frac{\tau^{(k)}(0)}{(\alpha')^k} \frac{1}{(\ln W^2)^k} \right\} \tag{3.13}$$

with the integers

$$m = \mu + |\delta\lambda - \eta\lambda'|, \qquad \mu = 0, 1, 2, \ldots$$
$$n = 4(s_1 + s_2) - |\delta\lambda - \eta\lambda'| - 2.$$

$\eta = +1, (-1)$ belongs to the spin-non flip (spin flip) part according to (3.7). The generalized Mac Dowell symmetry (3.9) leads to

$$h_m(-W) = (-1)^{\lambda - \eta\lambda'} h_m(W).$$

If we define the corresponding "discontinuities" $\Delta_R h_m(\sqrt{s'}, W)$ and $\Delta_{JM} h_m(\sqrt{s'}, W)$ then the symmetry assumption

$$\Delta_{JM} h_m(\sqrt{-s'}, W) = \sigma(-1)^{(n+i)/2} \Delta_R h_m(\sqrt{s'}, W) \tag{3.14}$$

yields the requested result similarly to the behaviour (2.28) in the scalar case. Here we have to put

$$i = \begin{cases} 2 & \text{— for boson-boson, fermion-fermion scattering, for boson-fermion scattering contributions with } \delta = +1 \text{ and } |\lambda - \eta\lambda'| = \big||\lambda| - |\lambda'|\big| \\ 3 & \text{— for boson-fermion scattering contributions with } \delta = +1 \text{ and } |\lambda - \eta\lambda'| = |\lambda| + |\lambda'| \\ & \quad \text{with } \delta = -1 \text{ and } |\lambda + \eta\lambda'| = |\lambda| + |\lambda'| \\ 4 & \text{— for boson-fermion scattering contributions with } \delta = -1 \text{ and } |\lambda + \eta\lambda'| = \big||\lambda| - |\lambda'|\big|. \end{cases}$$

In the boson-fermion case the u-channel contributions ($\delta = -1$) to the infinite integrals in (3.11) can be calculated analogously to the scalar case if one introduces formally the shifted trajectory intercept $\bar{\alpha}(0) = \alpha(0) - 1/2$.
By making use of (3.12), (3.13) and (3.14) one is able to show now that the infinite integrals of (3.11) calculated for the contributions $h_m(W)$ with arbitrary m yield $\operatorname{Re} h_m(W)$ and an additional expansion of pure powers in W. The proof proceeds in the same way as in the scalar case. Finally we demand the vanishing of all powers $W^{-\nu}$ (ν integer) and get the infinite set of sum rules

$$\begin{aligned} & \frac{1}{\pi} \int_U^{iW_c} dW' \operatorname{disc} H^{jn}(W')\, W'^r \\ & + \frac{1}{\pi} \int_{M+1}^{W_c} dW' \left[\operatorname{Im} H^{jn}(W' + i\varepsilon) - (-1)^{r+\lambda-\lambda'} \operatorname{Im} H^{j[(-1)^{2\lambda'} n]}(W' + i\varepsilon)\right] W'^r = a_{-r-1} + \gamma^{(r)} \end{aligned} \tag{3.15}$$

with $r = -p, -p+1, \ldots$ Here the constants $\gamma^{(r)}$ are descended from the infinite integrals and depend upon the cutoff W_c and Regge pole parameters. In Table II the S_c dependence of the leading contributions is summarized.
Thus we arrive at the following conclusion. In all cases of boson-boson and fermion-fermion scattering the first sum rule ($r = -p$) is dominated by a Regge pole exchange with negative signature (ϱ-meson pole). The sum rules for odd $p + r$ values vanish identically, because the amplitudes H^{jn} are analytic in $s = W^2$. For boson-fermion scattering different results can be realized depending on the value of $\lambda_{\min} = \operatorname{Min}(|\lambda|, |\lambda'|)$. However, in πN scattering ($\lambda_{\min} = 1/2$) both $\gamma^{(-p)}$ and $\gamma^{(-p+1)}$ are dominated by a negatively signa-

tured t channel exchange taking into account the empirical fact that the intercepts of baryon trajectories lie considerably lower than those of the leading meson trajectories. After having derived the sum rules (3.15) for arbitrary spin the question arises how to apply them. In these relations integrals along unphysical cuts appear. Using a sufficiently large number of sum rules one should be able to determine appropriately parametrized left-hand cut discontinuities. At this one has to insert empirical values in the physical cut region integral $(M + 1 \ldots W_c)$. A more difficult but from the theoretic point of view more interesting question consists in the simultaneous determination of short-range forces, subtraction constants and CDD pole parameters [3] in the framework of N/D equations [19]. This problem could be enlarged including the determination of long-range parameters. With respect to the pion-nucleon problem this would raise the question, whether it is possible to calculate the nucleon pole parameters, too. If this could be done the result would be a uniquely fixed partial wave amplitude, satisfying analyticity, Regge asymptotic behaviour in the sense of equ. (3.5), and unitarity. However, it turns out that the technical expense rises up enormously. Therefore a reduced program can only be solved within a realistic expense. In the next chapter we shall demonstrate how the partial problem of fixing CDD-pole parameters can be solved by applying empirical data.

Table II

Leading contributions to the first $\gamma^{(r)}$ constants (spin dependent case, only Regge poles)

	boson-boson, fermion-fermion, scattering	boson-fermion scattering: t-channel exchange	boson-fermion scattering: u-channel exchange
$\gamma^{(-p)}$	$\sim \frac{1-\sigma}{s_c^{1-\alpha(0)}(\ln s_c)^{\lVert\lambda\vert-\vert\lambda'\rVert+1}}$	$\sim \frac{1-\sigma}{s_c^{1-\alpha(0)}(\ln s_c)^{\lVert\lambda\vert-\vert\lambda'\rVert+1}}$	$\sim \frac{1+\sigma(-1)^{\lambda_{\min}+1/2}}{s_c^{\lambda_{\min}+1-\alpha(0)}(\ln s_c)^{\vert\lambda\vert+\vert\lambda'\vert+1}}$
$\gamma^{(-p-1)}$	$= 0$	$\sim \frac{1+\sigma(-1)^{\lambda_{\min}+1/2}}{s_c^{\lambda_{\min}-1/2-\alpha(0)}(\ln s_c)^{\vert\lambda\vert+\vert\lambda'\vert+1}}$	$\sim \frac{1+\sigma}{s_c^{1/2-\alpha(0)}(\ln s_c)^{\lVert\lambda\vert-\vert\lambda'\rVert+1}}$

$\lambda_{\min} = \mathrm{Min}\,(|\lambda|, |\lambda'|)$

4. Application of Partial Wave Sum Rules to Pion-Nucleon Scattering

4.1. Sum Rules for Pion-Nucleon Scattering

In this chapter we investigate the πN state with isospin $I = 1/2$ and total angular momentum $J = 1/2$. At first we discuss the modifications of the sum rules (3.15) being necessary in this special case. We remark that no pole terms have been taken into account up till now. Therefore the nucleon pole must be included in (3.15). Furthermore in the πN case the reduced amplitudes $H^{n}_{\{\lambda\}}(w)$ defined by (3.8) are not identical with the reduced amplitudes $h_{l+}, h_{(l+1)-}$ usually used. For the latter ones different threshold factors have to be applied which take into account a more detailed knowledge of the kinematic singularity behaviour than it could be treated in the general case of arbitrary spins. Consequently modified sum rules are valid for the functions $h_{l+}, h_{(l+1)-}$. Firstly no subtractions are necessary in partial wave dispersion relations. Secondly the concrete form of the constants $\gamma^{(r)}$ will be changed, but only with respect to powers in $\ln s_c$. The derivation is analogous to the general spin case. We state therefore only the results [20].

Similarly to (3.15) we consider now the relation

$$\frac{1}{\pi}\int_{U}^{iW_c} dW' \,\mathrm{disc}\, h(W')\, W'^{r-1}$$

$$+ \frac{1}{\pi}\int_{M+1}^{W_c} dW' \,\{\mathrm{Im}\, h(W') + (-1)^r\, \mathrm{Im}\, h(-w')\}\, W'^{r-1} + \frac{3}{8\pi}\, g^2_{\bar{N}N\pi}(-M)^{r-1} = \gamma^{(r)},$$

$$r = 1, 2, \ldots \tag{4.1}$$

where M (= 6.75) is the nucleon mass and $g_{\bar{N}N\pi}(\sim 13{,}5)$ represents the πN coupling constant. According to the Mac Dowell symmetry we have assumed that for $W \geqq M + 1$

$$h(W) = \frac{\eta(W)\exp\big(2i\delta(W)\big) - 1}{2i\varrho(W)} \equiv h_{0+}(W) = \frac{\eta_{0+}(W)\exp\big(2i\delta_{0+}(W)\big) - 1}{2i\varrho_{0+}(W)}$$

$$-h(-W) = -\frac{\eta(-W)\exp\big(2i\delta(-W)\big) - 1}{2i\varrho(-W)} \equiv h_{1-}(W) = \frac{\eta_{1-}(W)\exp\big(2i\delta_{1-}(W)\big) - 1}{2i\varrho_{1-}(W)} \tag{4.2}$$

$\delta(W)$ denotes the real phase shift and $\eta(W)$ the inelasticity. The function $\varrho(W)$ produces the correct threshold behaviour

$$\varrho(W) = \varrho_{0+}(W) = q\,\frac{(W+M)^2 - 1}{2W^2}$$

$$-\varrho(-W) = \varrho_{1-}(W) = q\,\frac{(W-M)^2 - 1}{2W^2}.$$

For the first two $\gamma^{(r)}$ the following expressions can be derived [20]

$$\gamma^{(1)} = -\frac{1}{\pi^2}\,\frac{C_+^{\varrho}(0)}{\big(1 - \alpha_\varrho(0)\big)\,\alpha_\varrho'(0)}\,(2M)^{1-\alpha_\varrho(0)}\,\frac{(W_c^2)^{\alpha_\varrho(0)-1}}{\ln(W_c^2)}$$

$$\gamma^{(2)} = -2M\gamma^{(1)} \tag{4.3}$$

where only the dominant contribution has been written down. Corresponding to the investigations of chapter 3 the ϱ trajectory (with negative signature) dominates in $\gamma^{(1)}$ and $\gamma^{(2)}$ ($\gamma^{(3)}$ and $\gamma^{(4)}$ are dominated by the Pomeron pole). In equ. (4.3) $C_+^{\varrho}(t)$ denotes the Regge pole residue of the A'-amplitude introduced by SINGH [21]. For numerical calculations parameter values of a fit of CHIU et al. [22] have been used. Within their notation and parameterization we have

$$C_+^{\varrho}(0) = \left(\frac{1}{E_0}\right)^{\alpha_\varrho(0)} C_0^{\varrho}\big(\alpha_\varrho(0) + 1\big)$$

with $E_0 = 1$ GeV (scale constant), $\alpha_\varrho(0) = .58$, $\alpha_\varrho'(0) = 1.02$ GeV^{-2}, $C_0^{\varrho} = 1.49$ mb GeV. This fit corresponds to a rather large slope of the Pomeron trajectory ($\alpha_P'(0) \sim 1/2$) and has the advantage that the value W^* where $\eta(W)$ and $\delta(W)$ start to grow monotonously to asymptotic limits is lower than for other fits. We have $W^* \lesssim 10^2$ and take therefore $W_c = 100$.

4.2. Formulation of the N/D Equations

To study the question whether the sum rules (4.1) enable us to determine CDD-pole parameters in the πN $I = J = 1/2$ state we employ a formalism of N/D equations similar to that of Frye and Warnock [19]. In these equations the cutoff W_c occurs introduced above. Therefore from the mathematical point of view we have to solve the same type of equations as in the strip approximation [23]. For practical purposes it is sufficient to use the method of matrix inversion [24]. In the region $M + 1 \leqq |W| \leqq W_c$ and if n_c CDD poles (W_i, C_i) are present the equations have the following form

$$\frac{2\eta(W)}{1+\eta(W)} \operatorname{Re} N(W) = \operatorname{Re} B(W) + \sum_{i=1}^{n_c} c_i \frac{W \operatorname{Re} B(W) - W_i \operatorname{Re} B(W_i)}{W - W_i}$$
$$+ \frac{1}{\pi}\left(P\int_{-W_c}^{-M-1} + P\int_{M+1}^{W_c}\right) dW' \frac{2\varrho(W') \operatorname{Re} N(W')}{(1+\eta(W'))\, W'(W'-W)} \{W' \operatorname{Re} B(W') - W \operatorname{Re} B(W)\} \tag{4.4}$$

$$\operatorname{Re} D(W) = 1 + W\left(\sum_{i=1}^{n_c} \frac{c_i}{W - W_i} - \frac{1}{\pi}\left(P\int_{-W_c}^{-M-1} + P\int_{M+1}^{W_c}\right) dW' \frac{2\varrho(W') \operatorname{Re} N(W')}{(1+\eta(W'))\, W'(W'-W)}\right). \tag{4.5}$$

According to

$$\operatorname{Im} D(W) = -\frac{2\varrho(W)}{1+\eta(W)} \operatorname{Re} N(W)$$

the output phase shift is given by

$$\tan \delta_{\text{out}}(W) = -\frac{\operatorname{Im} D(W)}{\operatorname{Re} D(W)} \tag{4.6}$$

which yet depends on the undetermined CDD pole parameters. The input function $\operatorname{Re} B(W)$ has to fulfil the following two conditions.

1. Along the physical cuts below W_c $\operatorname{Re} B(W)$ has to be identical with [25]

$$\operatorname{Re} B_{\text{emp}}(W) \equiv \frac{\eta(W) \sin 2\delta(W)}{2\varrho(W)} - \frac{1}{\pi}\left(P\int_{-W_c}^{-M-1} + P\int_{M+1}^{W_c}\right) dW' \frac{\eta(W') \sin^2 \delta(W')}{\varrho(W')(W'-W)} \tag{4.7}$$

where the right-hand side is calculated with empirical phase shifts.

2. The discontinuities of the unphysical cuts disc $h(W)$ in the finite energy region has to be consistent with the sum rules (equ. (4.1)) calculated with empirical phase shifts, too.

Only in this way it seems possible to get results comparable with experimental data. Such a potential could be constructed, for instance, by a pole approximation of long range forces (considering crossing relations) and of suitably adjusted short range forces the parameters of which could be determined in principle by the sum rules themselves. Here it turns out that short range forces must give a contribution large enough fo fulfil at least the first sum rules.

However, for our purposes outlined above it is not necessary to know completely the input, i.e. to have an explicit knowledge of disc $h(W)$. Therefore we only assume that a function $\operatorname{Re} B(W)$ should exist satisfying the two conditions stated above. The

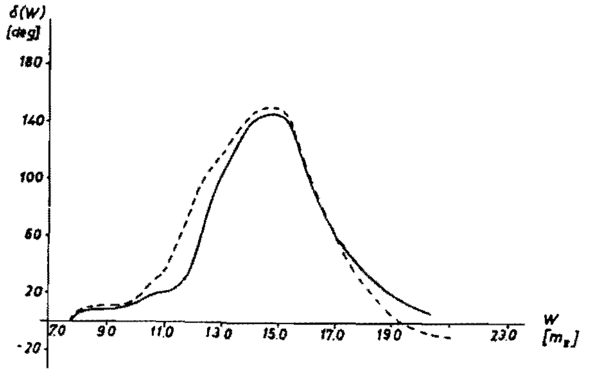

Fig. 1. S 11 output phase shift (the dotted curve denotes empirical values).

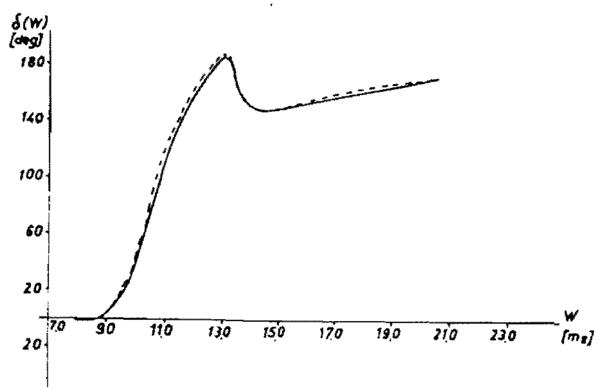

Fig. 2. P 11 output phase shift (the dotted curve denotes empirical values).

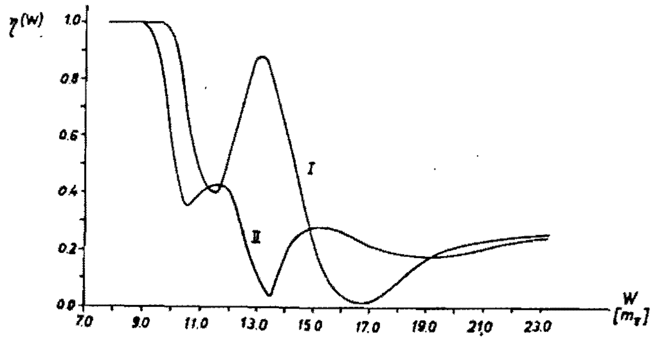

Fig. 3. Empirical inelasticities ($I \triangleq S$ 11-wave, II $\triangleq P$ 11-wave).

solve the N/D equations with $\operatorname{Re} B_{\text{emp}}(W)$ and calculate the integrals

$$\frac{1}{\pi}\int_{U}^{iW_c} dW' \operatorname{disc} h(W')\, W'^{r-1}$$

using the sum rules (4.1) with empirical phase shifts [26] (Figs. 1, 2, 3). Thus we can replace an approximative pole input by a model independent one.[4])
We remark that $\operatorname{Re} B_{\text{emp}}(W)$ contains the direct channel nucleon pole. To examine the case without this pole we have to subtract its contribution

$$-\frac{3}{8\pi}\,\frac{g^2_{\bar N N\pi}}{W+M}$$

from $\operatorname{Re} B_{\text{emp}}(W)$.

4.3. Numerical Results

We studied the N/D equations for the three cases
a) neither with direct channel nucleon pole nor with CDD pole,
b) with nucleon pole but without CDD pole,
c) both with nucleon pole and with one variable CDD pole (C_1, W_1).
In all cases the first two sum rules were tested by inserting the output phases δ_{out} (equ. 4.6) into the sum rules (4.1). The numerical calculations led to the following results.
For the cases a) and b) the sum rules cannot be fulfilled. The disagreement between the left-hand and right-hand sides of equ. (4.1) is extreme for case a), but becomes smaller for the case b). In the case c) we tried to find the points where the individual sum rules are satisfied, and this by variation of C_1 and W_1 (Fig. 4). The region shown in Fig. 4 is

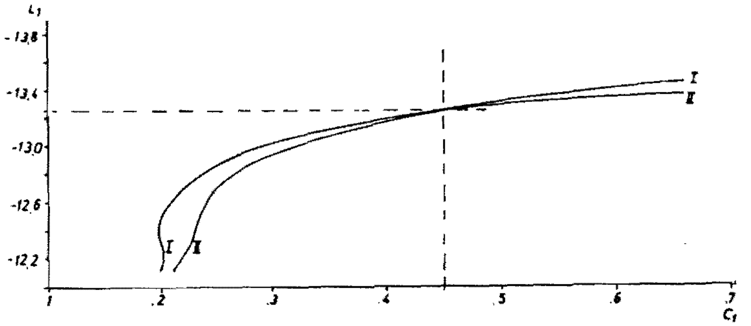

Fig. 4. Geometric locus where the sum rules I and II are satisfied; optimal CDD parameters $W_1 = -13.25$, $C_1 = 0.45$

the only one in a wide range where the sum rules I and II are fulfilled in a small distance from each other. We have not drawn other curves in other regions where only one of these sum rules is satisfied. Thus we see that the first sum rules are fulfilled simultaneously at one point ($W_1 = -13.25$, $C_1 = 0.45$). One can show that the third and

[4]) Of course, our input contains assumptions about the behaviour of the phase shifts in the energy region where no experimental phase shifts are available. Thus we use a smoothed interpolation between empirical values and those determined from the high energy behaviour of the amplitude $h(W)$.

fourth order sum rules are not satisfied at this point. But it turns out that these CDD pole parameter values are a certain "optimum" for the higher sum rules, too.
To reach better agreement we should introduce obviously more CDD poles. Furthermore we conclude that the CDD-pole position W_1 derived above agrees with that point where the empirical P_{11} phase shift equals π the second time. The parameters determined correspond to output phase shifts in a very good agreement with the experimental ones (Figs. 1 and 2). In table III the behaviour of the individual parts of the sum rules (4.1) is stated for the three cases considered. We conclude that in the two sum rules very big contributions must be cancelled to give the rather small $\gamma^{(r)}$ values. In this sense the determination of the CDD parameters does not depend sensitively on the chosen Regge fit. Thus we should expect that the consideration of terms which are responsible for rising cross sections (Regge cut and Regge multifold pole contributions) will not modify the main results derived above.
Thus we can finally conclude that the sum rules enable us to determine CDD pole parameters provided a correct input potential is used.

Table III

Values of the individual parts of the first two sum rules in the cases
a) neither direct channel nucleon pole nor CDD pole
b) with nucleon pole but without CDD pole
c) both with nucleon pole and fixed CDD pole ($W_1 = -13.25$, $C_1 = 0.45$)
(UPC, PC, NP, Σ, resp., denote symbolically the contributions of the unphysical cut, the physical cut, the nucleon pole, the whole left hand side of the r^{th} sum rule (4.1), respectively)

Case	r	$\gamma^{(r)}$	Σ	PC	NP	UPC
a)	1	−0.02	−11.36	−15.02	21.93	−18.27
b)		−0.02	0.82	−2.84	21.93	−18.27
c)		−0.02	−0.02	−3.68	21.93	−18.27
a)	2	0.3	117.9	248.0	−147.7	17.6
b)		0.3	−1.1	129.0	−147.7	17.6
c)		0.3	0.3	130.4	−147.7	17.6

5. Conclusion

It was shown that on the basis of definite assumptions finite energy sum rules for partial waves can be derived which make their dispersion representations consistent with Regge asymptotics. For the latter different models were used. We treated the scalar as well as the general spin case. In the special example of pion-nucleon scattering it was demonstrated that the sum rules of highest order ($r = 1, 2$) can be employed to determine CDD-pole parameters in the $I = J = 1/2$ state. Whether sum rules of lower order (and thus eventually Regge asymptotics) are appropriate to fix also the nucleon mass and the pion-nucleon coupling constant remained an open question. It may well be that lower order sum rules ($r = 3, 4, \ldots$) require the inclusion of further CDD poles.[5]) Thus we cannot draw actually definite conclusions about the bootstrap idea. This situation resembles to that of the various dual models, where two fundamental constants, an overall coupling constant and the intercept, are left undetermined by the unitarization procedure via loop techniques.

[5]) This situation might be quite different from the non-relativistic case where neither crossing symmetry nor CDD-poles (The both aspects are possibly connected) are involved.

Appendix

Asymptotic Behaviour of Dispersion Integrals

We state the behaviour of integrals of the type

$$I^{(\pm)}_{\alpha\beta}(s) = P \int_{s_c}^{\infty} \frac{dx}{x^\alpha (x \pm s) \ln^\beta x} \tag{A.1}$$

in the limit $s \to \infty$ [11]. The cases α integer and α noninteger are distinguished. β is assumed to be integer. Then the following asymptotic expansions can be derived

<u>$\alpha = 1, 2, 3, \ldots$</u>

$$\begin{aligned} I^{(\pm)}_{\alpha\beta}(s) = {} & \begin{bmatrix} (-1)^{\alpha+1} \\ -1 \end{bmatrix} \frac{1}{s^\alpha} L_\beta - \sum_{n=0}^{\infty} \frac{\Gamma(2n+1+\beta)}{\Gamma(\beta)} \\ & \times \frac{1}{s^\alpha (\ln s)^{2n+1+\beta}} \, 2\zeta(2n+2) \begin{bmatrix} (-1)^{\alpha+1} \left(1 - \frac{1}{2^{2n-1}}\right) \\ 1 \end{bmatrix} \\ & + \sum_{\substack{n=0 \\ n \neq \alpha-1}}^{\infty} \begin{bmatrix} (-1)^{n+1} \\ 1 \end{bmatrix} \frac{1}{s^{n+1}} \left(s_c^{\,n+1-\alpha} \sum_{p=0}^{\infty} \frac{\Gamma(p+\beta)}{\Gamma(\beta)} \frac{1}{(n+1-\alpha)^{p+1}} \frac{1}{(\ln s_c)^{\beta+p}} \right) \end{aligned} \tag{A.2}$$

with

$$L_\beta = \begin{cases} \ln \ln s - \ln \ln s_c & \text{for } \beta = 1 \\ \frac{1}{\beta - 1} \left(- \frac{1}{(\ln s)^{\beta-1}} + \frac{1}{(\ln s_c)^{\beta-1}} \right) & \text{for } \beta \neq 1 \end{cases}$$

<u>$k < \alpha < k + 1$, k integer</u>

$$\begin{aligned} I^{(\pm)}_{\alpha\beta}(s) = {} & - \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta)}{\Gamma(\beta)} \frac{1}{s^\alpha (\ln s)^{n+\beta}} Z_n^{\pm} \\ & + \sum_{n=0}^{\infty} \begin{bmatrix} (-1)^{n+1} \\ 1 \end{bmatrix} \frac{1}{s^{n+1}} \left(s_c^{\,n+1-\alpha} \sum_{p=0}^{\infty} \frac{\Gamma(p+\beta)}{\Gamma(\beta)} \frac{1}{(n+1-\alpha)^{p+1}} \frac{1}{(\ln s_c)^{\beta+p}} \right. \end{aligned} \tag{A.3}$$

with

$$\left. \begin{aligned} Z_n^+ = {} & (-1)^{k+1} \left(\zeta(n+1, 1-\alpha+k) - \frac{1}{2^n} \zeta\left(n+1, \frac{2-\alpha+k}{2}\right) \right. \\ & \left. + (-1)^n \zeta(n+1, \alpha-k) - \left(-\frac{1}{2}\right)^n \zeta\left(n+1, \frac{1+\alpha-k}{2}\right) \right) \\ Z_n^- = {} & \zeta(n+1, 1-\alpha+k) - (-1)^n \zeta(n+1, \alpha-k). \end{aligned} \right\} \tag{A.4}$$

The upper (lower) values in the angular brackets belong to $I^+_{\alpha\beta}(I^-_{\alpha\beta})$. $\zeta(z)$ denotes the Riemann zeta function whereas $\zeta(z, \alpha)$ is the generalized zeta function.

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