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$\rho\pi\pi$ Regge Residue Function from a New Class of Sum Rules, and a New Representation for $\pi\pi$ Amplitudes

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A new class of sum rules is developed within the context of the $\pi\pi$ system. One of these sum rules is used to extract the $\rho\pi\pi$ Regge residue function from the recent $\pi\pi$ data of Carroll *et al.*, for a wide range of momentum transfer. The residue has a zero near $t = -0.5$ GeV^2, in agreement with duality and with a new prediction which we make from the dual absorption model together with πp and $\bar{p}p$ data. The $\pi\pi$ charge-exchange amplitude is shown to be dominated by ρ exchange above 1.0 GeV. We also derive a new representation which expresses $\pi\pi$ amplitudes in terms of a single subtraction parameter and integrals over physical-region absorptive parts. The representation is valid over a substantial portion of the physical region, and constitutes a powerful tool for studying the $\pi\pi$ interaction. Finally, we show that a unitarized Veneziano model proposed earlier by the present author is an excellent approximation to nature below about 800 MeV, despite its neglect of Pomeranchon exchange and $K\bar{K}$ production.

I. INTRODUCTION AND SUMMARY

Standard assumptions of S-matrix theory enable us to express scattering amplitudes in terms of fixed-s dispersion relations (D.R.'s) and alternatively, in terms of fixed-t D.R.'s. By taking differences between such D.R.'s, it is possible to eliminate the subtraction terms, and thereby obtain sum rules equating certain integrals over absorptive parts to zero.

In the present paper, we apply this technique to the $\pi\pi$ system. We obtain sum rules which are similar to, but more powerful than, the sum rules of Wanders,[1] Roskies,[2] and Roy.[3] We use one of these sum rules to extract the $\rho\pi\pi$ Regge residue function from the recent $\pi\pi$ data of Carroll *et al.*,[4] and obtain results with uncertainties of the order of 10%. The residue has a zero near $t = -0.5$ GeV^2, in agreement with duality and with a new prediction which we make from the dual absorption model[5] together with πp and $\bar{p}p$ data.

We show that in the sense of local averages, the $\pi\pi$ charge-exchange amplitude of Carroll *et al.* can be well approximated above 1.0 GeV by Reggeized ρ exchange, with our value for the residue function.

We derive a new representation which expresses $\pi\pi$ amplitudes in terms of a single subtraction parameter and integrals over physical-region absorptive parts. Our representation is valid over the same portion of the physical region as the twice-subtracted representations of Roskies[2] and Roy,[3] and therefore constitutes a powerful tool for studying the $\pi\pi$ interaction.

Finally, we show that a unitarized Veneziano $\pi\pi$ model proposed earlier[6] by the present author is an excellent approximation to nature below about 800 MeV, despite its neglect of Pomeranchon exchange and $K\bar{K}$ production.

II. DISPERSION RELATIONS AND SUM RULES

We denote the $\pi\pi$ elastic amplitude with isospin I in the s channel by $A^I(s,t)$, and the amplitude with isospin I in the t channel by $T^I(s,t)$. According to standard assumptions[7] of analyticity and crossing symmetry, the A^I and T^I are related by

$$A^I(s,t) = T^I(t,s)\,, \tag{1a}$$

and also by

$$A^I(s,t) = \sum_{I'=0}^{2} C_{II'}T^{I'}(s,t)\,, \tag{1b}$$

where $C = C^{-1}$ denotes the s-t crossing matrix

$$C=\begin{pmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{pmatrix}.$$

Bose symmetry implies that

$$A^I(s,t)=(-1)^I A^I(s,u)\,, \tag{1c}$$

$$T^I(s,t)=(-1)^I T^I(u,t)\,, \tag{1d}$$

where s, t, and u may take on any values consistent with

$$s+t+u=4\mu^2\,,$$

where μ denotes the pion mass.

We shall regard the s channel as the direct channel, and we normalize the amplitudes such that

$$A^I=\sum_{l=0}^{\infty}(2l+1)A^{(l)I}(s)P_l(\cos\theta)\,,$$

with

$$A^{(l)I}(s)=[s/(s-4\mu^2)]^{1/2}\exp(i\delta_l^I)\sin\delta_l^I$$

in the elastic region. Equation (1c) implies that $A^{(l)I}$ vanishes for even (odd) I when l is odd (even).

As a final remark on conventions, we shall use units wherein $\hbar=c=1$.

By writing a fixed-s D.R. and exploiting Eqs. (1a) and (1c), we obtain

$$A^I(s,t)=A^I(s,t_0)+\frac{t-t_0}{\pi}\int_{4\mu^2}^{\infty}ds'\,\mathrm{Im}T^I(s',s)\left[\frac{1}{(s'-t_0)(s'-t)}-(-1)^I\frac{1}{(s'+s+t_0-4\mu^2)(s'+s+t-4\mu^2)}\right]. \tag{2a}$$

We assume that $\pi\pi$ double-spectral functions have the boundaries predicted by Mandelstam,[7] in which case Eq. (2a) is valid for arbitrary t and t_0 when s is real $(\pm i\epsilon)$ and $-32\mu^2\leqslant s\leqslant 4\mu^2$. Convergence of the integral in Eq. (2a) is ensured over this range of s by familiar tenets of Regge theory.

By writing a fixed-t D.R. and exploiting Eqs. (1b) and (1d), we obtain

$$A^I(s,t)=A^I(s_0,t)+\frac{s-s_0}{\pi}\int_{4\mu^2}^{\infty}ds'\left[\frac{\mathrm{Im}A^I(s',t)}{(s'-s_0)(s'-s)}-\sum_{I'=0}^{2}C_{II'}(-1)^{I'}\frac{\mathrm{Im}T^{I'}(s',t)}{(s'+t+s_0-4\mu^2)(s'+t+s-4\mu^2)}\right], \tag{2b}$$

which is valid for arbitrary s and s_0 when t is real $(\pm i\epsilon)$ and $-32\mu^2\leqslant t\leqslant 4\mu^2$. Again convergence of the integral is assured by Regge theory.

We could now replace the subtraction term in Eq. (2a) by the right-hand side of Eq. (2b) evaluated at $t=t_0$, and thereby obtain a formula expressing $A^I(s,t)$ as the sum of $A^I(s_0,t_0)$ plus certain integrals.[8] We could also replace the subtraction term on the right-hand side of Eq. (2b) by the right-hand side of Eq. (2a) evaluated at $s=s_0$, and thereby obtain another formula expressing $A^I(s,t)$ as the sum of $A^I(s_0,t_0)$ plus certain other integrals. Upon taking the difference between these two formulas, the subtraction terms would cancel, and we would obtain a sum rule equating certain integrals over absorptive parts to zero.

The procedure outlined in the preceding paragraph has the merit of being quite general. However, we find it simpler for present purposes to exploit a special feature of the $\pi\pi$ system, namely, the fact that Bose symmetry implies the vanishing of $A^1(s,t)$ when $\cos\theta_s=0$. Since $\cos\theta_s=1+2t/(s-4\mu^2)$, we can make the subtraction terms in Eqs. (2a) and (2b) vanish for $I=1$ by setting $t_0=2\mu^2-s/2$ and $s_0=4\mu^2-2t$. With these choices for t_0 and s_0, we obtain

$$A^1(s,t)=\frac{s+2t-4\mu^2}{\pi}\int_{4\mu^2}^{\infty}ds'\frac{\mathrm{Im}T^1(s',s)}{(s'-t)(s'+s+t-4\mu^2)}\,, \tag{3a}$$

$$A^1(s,t)=\frac{s+2t-4\mu^2}{\pi}\int_{4\mu^2}^{\infty}ds'\frac{1}{(s'-t)(s'+s+t-4\mu^2)}\left[\mathrm{Im}T^1(s',t)+\frac{(s-t)(2s'+t-4\mu^2)}{(s'+2t-4\mu^2)(s'-s)}\mathrm{Im}A^1(s',t)\right]. \tag{3b}$$

Equation (3a) is valid for arbitrary t when s is real $(\pm i\epsilon)$ and $-32\mu^2\leqslant s\leqslant 4\mu^2$, while Eq. (3b) is valid for arbitrary s when t is real $(\pm i\epsilon)$ and $-32\mu^2\leqslant t\leqslant 4\mu^2$.

Upon subtracting Eq. (3a) from (3b) and interchanging (for future convenience) the parameters s and t, we obtain

$$\int_{4\mu^2}^{\infty}\frac{ds'}{(s'-s)(s'+s+t-4\mu^2)}\left[\mathrm{Im}T^1(s',s)-\mathrm{Im}T^1(s',t)+\frac{(t-s)(2s'+s-4\mu^2)}{(s'+2s-4\mu^2)(s'-t)}\mathrm{Im}A^1(s',s)\right]=0\,, \tag{4}$$

which holds for real s $(\pm i\epsilon)$ and real t $(\pm i\epsilon)$ when $-32\mu^2\leqslant s\leqslant 4\mu^2$ and, simultaneously, $-32\mu^2\leqslant t\leqslant 4\mu^2$. We remark that the imaginary part of the integral in Eq. (4) vanishes as a direct consequence of Bose symmetry, so it is only the vanishing of the real (principal) part which contains new information.

Equation (4) is the sum rule which we shall use to extract the $\rho\pi\pi$ Regge residue function from the data. Note that the integrand in Eq. (4) is *independent of S waves,* so that we will be spared from the ambiguities which have plagued experimental studies of the $I=0$ S wave. However, the integrand does involve the P wave, so that Eq. (4) is more powerful than the sum rules of Wanders,[1] Roskies,[2] and Roy.[3]

III. CONTENT OF SUM RULE

We shall make the standard Regge assumption that for large positive s,

$$\mathrm{Im}T^1(s,t)=\gamma_\rho(t)(s/\bar{s})^{\alpha_\rho(t)}, \tag{5}$$

where γ_ρ is related by a well-known factor to the residue of the ρ pole in the J plane, and α_ρ denotes the ρ trajectory. We shall use $\bar{s}=1\ \mathrm{GeV}^2$, which defines the scale of γ_ρ.

Equations (4) and (5) imply that

$$\gamma_\rho(t)=f^{-1}(s,t;t;\Lambda)\left[f(s,t;s;\Lambda)\gamma_\rho(s)+h(s,t)+\mathrm{P}\int_{4\mu^2}^{\Lambda}ds'\frac{\mathrm{Im}T^1(s',s)-\mathrm{Im}T^1(s',t)}{(s'-s)(s'+s+t-4\mu^2)}\right], \tag{6}$$

where

$$f(s,t;x;\Lambda)\equiv\mathrm{P}\int_{\Lambda}^{\infty}ds'\frac{(s'/\bar{s})^{\alpha_\rho(x)}}{(s'-s)(s'+s+t-4\mu^2)}, \tag{7a}$$

$$h(s,t)\equiv(t-s)\mathrm{P}\int_{4\mu^2}^{\infty}ds'\frac{(2s'+s-4\mu^2)\mathrm{Im}A^1(s',s)}{(s'-s)(s'+s+t-4\mu^2)(s'+2s-4\mu^2)(s'-t)}, \tag{7b}$$

and Λ may take on any positive value large enough for Eq. (5) to hold for $T^1(s',s)$ and $T^1(s',t)$ when $s'\geq\Lambda$. Since Eq. (5) is only expected to hold for $\theta\leq 90°$, the minimum suitable value for Λ must satisfy

$$\Lambda\geq(4\mu^2-2s), \tag{8a}$$

$$\Lambda\geq(4\mu^2-2t). \tag{8b}$$

The useful feature of our sum rule is that if γ_ρ is known for any *single* value of its argument, this value of the argument can be substituted for s on the right-hand side of Eq. (6), and then $\gamma_\rho(t)$ can be computed over a wide range of t from a knowledge of $\mathrm{Im}T^1$ between threshold and Λ, together with knowledge of the rapidly convergent integral $h(s,t)$.

IV. ANALYSIS OF DATA AND APPLICATION OF SUM RULE

We turn now to the experimental results of Carroll *et al.*,[4] who have extracted information on $\pi\pi$ scattering from data on the reaction $\pi N\to\pi\pi N$. The results of Carroll *et al.* are presented in the form of S-, P-, and D-wave phase shifts and inelasticities, at 26 points over the energy range $0.60\leq s^{1/2}\leq 1.48$ GeV.

We have assumed that $T^1(s,t)$ is given by the S, P, and D waves of Carroll *et al.* over the energy range of their data, and we have investigated the resulting T^1 for evidence of Regge behavior.

To facilitate our discussion of the data, let us define $s_a(t)$ to be the greater of 1.0 GeV^2 and $(4\mu^2-2t)$, the latter being the value for s above which $\theta\leq 90°$:

$$s_a(t)\equiv\mathrm{Max}[1.0\ \mathrm{GeV}^2,\ (4\mu^2-2t)].$$

We also introduce a symbol for the maximum value of s in the region spanned by the data:

$$s_b\equiv(1.48\ \mathrm{GeV})^2.$$

The significance of $s_a(t)$ is that in the sense of local averages, we shall find the $T^1(s,t)$ of Carroll *et al.* to be dominated by Reggeized ρ exchange over the interval $s_a(t)\leq s\leq s_b$, for a wide range of t. We shall begin by considering the data for $\mathrm{Im}T^1$, and return later to a discussion of $\mathrm{Re}T^1$.

In Fig. 1, we have plotted the data for $\mathrm{Im}T^1$ as a function of s for five different values of t. For $t=0$, the data exhibit a large oscillation as s is varied over the interval $s_a(t)\leq s\leq s_b$. However, as t decreases from zero, the amplitude of the oscillation becomes smaller, and almost vanishes for t near -0.5 GeV^2 *(note the different vertical scales in Fig. 1).* Furthermore, the data exhibit a *very definite zero* near $t=-0.5$ GeV^2 (again note the different vertical scales). We interpret this to mean that the $\rho\pi\pi$ Regge residue function vanishes near $t=-0.5$ GeV^2. In particular, we infer from the data that

$$\gamma_\rho(-0.52\ \mathrm{GeV}^2)=0.00\pm0.10, \tag{9}$$

where the validity of the stated uncertainty may be judged by examining the spread of the data points in Fig. 1 for the case $t=-0.5$ GeV^2, while recalling that $\alpha_\rho(t)$ is approximately zero for this case.

As noted earlier, a simple inspection of the data near $t=0$ does not enable one to infer much about γ_ρ, at least not without large uncertainties. However, the data do appear to make a definite statement about γ_ρ near the point

$$t_z \equiv -0.52\ \mathrm{GeV}^2,$$

namely, Eq. (9). Therefore, we can substitute t_z for s on the right-hand side of Eq. (6), and use Eqs. (6) and (9) to obtain γ_ρ over a wide range of t. This is the procedure which we shall follow to obtain our preferred $\gamma_\rho(t)$ from the data.

In our evaluation of the function f which appears on the right-hand side of Eq. (6), we shall assume that

$$\alpha_\rho(t) = 0.50 + (0.90\ \mathrm{GeV}^{-2})t, \qquad (10)$$

which is in reasonable agreement with all known data [except that $\alpha_\rho(0)$ may be as large as 0.57].

In our evaluation of $h(s,t)$, we shall assume that $\mathrm{Im}\,A^1$ is determined below 1.48 GeV by the P wave, and we use the data of Carroll *et al.* over the interval spanned by their data, i.e., $0.60 \leq s^{1/2} \leq 1.48$ GeV.

In order to estimate $\mathrm{Im}\,A^{(1)1}$ between threshold and 0.60 GeV, we represent $\mathrm{Re}A^{(1)1}$ over this interval by a quadratic form:

$$\mathrm{Re}A^{(1)1}(s) \cong (s-4\mu^2)[a_1/4 + b(s-4\mu^2)].$$

We use the Weinberg scattering length[9] $a_1 = 0.035\mu^{-2}$, and we fit the parameter b to a phase shift of 17° at 0.60 GeV, in accordance with the data of Carroll *et al.* We then use unitarity to obtain $\mathrm{Im}\,A^{(1)1}$ from $\mathrm{Re}A^{(1)1}$. The resulting contributions to the integrals in Eqs. (6) and (7b) are quite small (about 10%) relative to those from the ρ resonance.

The coefficient of $\mathrm{Im}\,A^1$ in Eq. (7b) decreases like s'^{-3} as $s' \to \infty$, so we do not expect the contribution from above 1.48 GeV to be very important. Nevertheless, we shall attempt to make a realistic estimate for this contribution. We begin with a Regge analysis based on the t-channel composition of A^1:

$$A^1 = \tfrac{1}{3}T^0 + \tfrac{1}{2}T^1 - \tfrac{5}{6}T^2.$$

We shall assume that in the Regge region, T^0 is dominated by Pomeranchon and f_0 exchange, and we note that the contribution of f_0 exchange is purely real at the point where $\alpha_f(t)=0$, i.e., near $t=-0.5\ \mathrm{GeV}^2 \cong t_z$. We also assume that T^1 is dominated by ρ exchange, with $\mathrm{Im}T^1(s,t_z)=0$. Finally, we assume that $\mathrm{Im}T^2$ is negligible.

The preceding remarks would lead us to represent $\mathrm{Im}\,A^1(s,t_z)$ by pure Pomeranchon exchange in the Regge region. However, the $g(1680)$ resonance is well established in A^1, so pure Pomeranchon exchange cannot be strictly valid in the g region. Since duality tells us that resonances can be added to Pomeranchon exchange without double-counting, we shall simply add the g contribution to that of Pomeranchon exchange, using the narrow-resonance approximation:

$$\mathrm{Im}\,A^1(s,t_z) = \tfrac{1}{3}\gamma_P(t_z)(s/\bar{s})^{\alpha_P(t_z)} + 7\pi(m_g\Gamma_g/\bar{s})(-0.32)\delta((s-s_g)/\bar{s}) \qquad (11)$$

for $s^{1/2} > 1.48$ GeV, where γ_P is defined in analogy with γ_ρ in Eq. (5), Γ_g denotes the partial width $\Gamma(g\to\pi\pi) \cong 0.06$ GeV,[10] and we have used the fact that $P_3(\cos\theta) = -0.32$ at $s=s_g$, $t=t_z$.

Between $s^{1/2}=1.00$ and 1.48 GeV, the local average of the data for $\mathrm{Im}\,A^1(s,t_z)$ increases from about 0.2 to 0.4, while the local average of the data for $\mathrm{Im}\,A^2(s,t_z)$ increases from about 0.15 to about 0.30. Since neither A^1 nor A^2 contain any resonances in this region, it is reasonable to associate these absorptive parts with Pomeranchon exchange. Assuming that

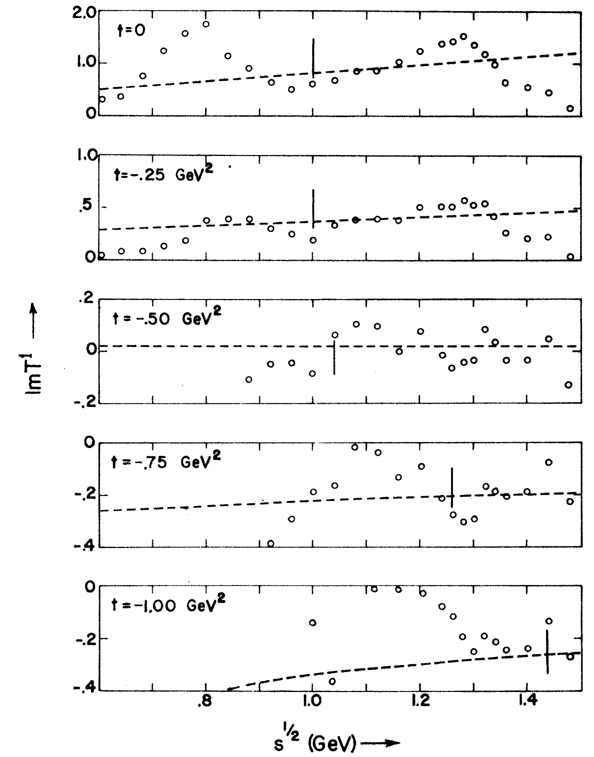

FIG. 1. $\mathrm{Im}\,T^1(s,t)$ of Carroll *et al.*, shown together with prediction of Regge formula (5) using our preferred result for γ_ρ. Statistical uncertainties of data may be inferred from spread of data points. Vertical bars intersecting dashed Regge curves denote $[s_a(t)]^{1/2}$, the energy above which agreement in sense of local averages is claimed to be good. Note the different vertical scales.

$$\alpha_P(t_z) = 0.74\,, \tag{12a}$$

i.e., $\alpha_P' = 0.50\ \mathrm{GeV}^{-2}$, we estimate that

$$\gamma_P(t_z) \cong 0.6\,. \tag{12b}$$

V. RESULTS FOR γ_ρ

We have used the data of Carroll *et al.* to evaluate[11] the right-hand side of Eq. (6), with $s = t_z$, for values of t over the range $-1.00 \le t \le 0.50\ \mathrm{GeV}^2$ [even though Eq. (6) is not strictly valid for $t < -32\mu^2 = -0.61\ \mathrm{GeV}^2$, nor for $t > 4\mu^2 = 0.08\ \mathrm{GeV}^2$; see Ref. 12], and for values of Λ over the range $s_a(t) \le \Lambda \le s_b$.[13] We have plotted the resulting values for γ_ρ as a function of Λ in Fig. 2, for five different values of t.

Note the striking fact that the top three curves in Fig. 2 *remain within* 5% *of their central values* over the entire range of Λ. This strongly suggests that $\mathrm{Im}T^1$ is dominated by ρ exchange for $s \ge 1.0\ \mathrm{GeV}^2$, $-0.25 \le t \le 0.25\ \mathrm{GeV}^2$, with γ_ρ correctly given by Fig. 2.

For $t = t_z$, Eqs. (6) and (9) imply $\gamma_\rho = 0$, regardless of the value chosen for Λ. The fourth curve in Fig. 2, namely, for $t = -0.75\ \mathrm{GeV}^2$, remains within 7% of its central value over the entire range of Λ. For the bottom curve ($t = -1.00\ \mathrm{GeV}^2$), $s_a(t) = (1.44\ \mathrm{GeV})^2$, so the data do not extend to high enough an energy to provide a meaningful test of the stability of γ_ρ against variations of Λ.

Although the magnitude of γ_ρ appears to be increasing for $\Lambda^{1/2} > 1.4$ GeV for the five curves displayed in Fig. 2, we have verified that inclusion of a nonzero F wave in T^1 would tend to stabilize the curves in this region. The $g(1680)$ meson, with a total width of 0.16 GeV and a $\pi\pi$ decay probability of 0.4,[10] implies that the F wave does in fact become appreciable near 1.5 GeV.

In Fig. 3, we display our result for $\gamma_\rho(t)$ for $-1.00 \le t \le 0.25\ \mathrm{GeV}^2$, averaged over Λ for $s_a(t) \le \Lambda \le s_b$.

When judging the precision of any prediction based on a sum rule, it is important to ascertain whether there are any cancellations between major terms in the sum, which might lead to large percentage uncertainties in the result. Fortunately, the present situation could scarcely be more agreeable.

As mentioned earlier, the integrand of our sum rule receives no contribution whatever from S waves. Thus for $\Lambda \le (1.1\ \mathrm{GeV})^2$, the integral in Eq. (6) is determined almost entirely by the ρ resonance, since D-wave absorptive parts are negligible below 1.1 GeV. Furthermore, the integral defining $h(s,t)$ in Eq. (7b) is rapidly convergent, and is thus determined primarily by the ρ resonance. [In fact if $\gamma_P(t_z)$ were simply set equal to zero, instead of being given the value indicated by Eq. (12b), our result for $\gamma_\rho(t)$ would change by less than 0.06 over the interval $-0.65 \le t \le 0.25\ \mathrm{GeV}^2$.[14] The contribution from the $g(1680)$ meson, via $h(s,t)$, is even less important.]

As Λ increases above $(1.1\ \mathrm{GeV})^2$, D waves begin to contribute to the integral in Eq. (6), but in such a way that $\gamma_\rho(t)$ is highly stable against variations of Λ. This stability constitutes strong evidence that, in the sense of local averages, one has entered the Regge region.

A final source of imprecision lies in the uncertainty of ± 0.10 stated in Eq. (9) for $\gamma_\rho(t_z)$. Inspection of Eq. (6) reveals that the resulting uncertain-

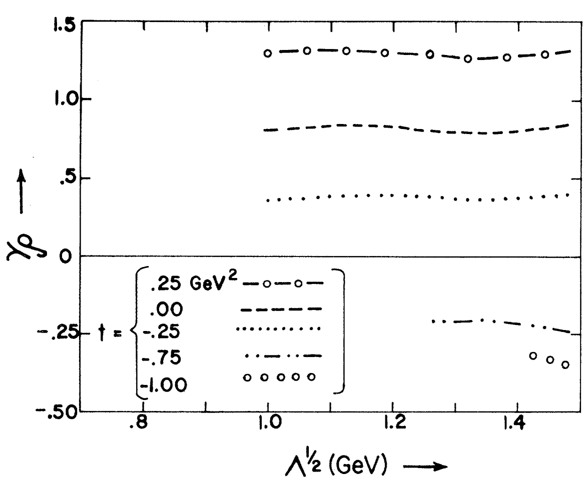

FIG. 2. $\gamma_\rho(t)$ computed from Eqs. (6) and (9), shown as a function of Λ, for $s_a(t) \le \Lambda \le s_b$. Note the different vertical scales for positive and negative γ_ρ.

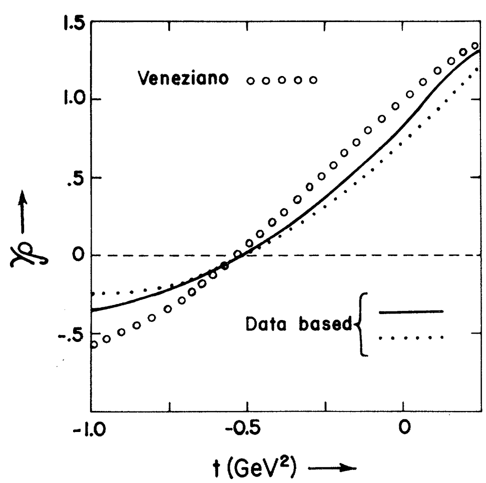

FIG. 3. Solid curve represents preferred γ_ρ based on Eqs. (6) and (9). Dotted curve displays $\bar{\gamma}_\rho$ defined by Eq. (16). The Veneziano γ_ρ is shown for comparison.

ty in $\gamma_\rho(t)$ is determined by the function f, and hence is readily computed. Assigning Λ a typical value of $\Lambda = \frac{1}{2}[s_a(t)+s_b]$, we find that

$$\Delta\gamma_\rho(t) \cong [0.45 - 1.1(t/\bar{s})]\Delta\gamma_\rho(t_z) \quad (13)$$

for $-1.00 \leq t \leq 0.25$ GeV^2,[15] where again we have used $\bar{s} = 1$ GeV^2. Thus the uncertainty of ±0.10 in $\gamma_\rho(t_z)$ generates an uncertainty of only ±0.045 in $\gamma_\rho(0)$.

In view of the stability of our results against variations of Λ, and in view of the preceding discussion of uncertainties, we state our result for $\gamma_\rho(0)$ as

$$\gamma_\rho(0) = 0.82 \pm 0.10\,, \quad (14)$$

assuming that $\alpha_\rho(0)$ is correctly given by Eq. (10), and that the data are free of systematic errors.

To facilitate possible applications of our results, we note that our preferred γ_ρ can be approximated within 4% over the interval $-1.00 \leq t \leq 0.20$ GeV^2 by the simple curve

$$\gamma_\rho(t) \cong 0.82 + 2.04(t/\bar{s}) + 0.88(t/\bar{s})^2\,, \quad (15)$$

again with $\bar{s} = 1$ GeV^2.

Since our results for γ_ρ depend on α_ρ only through the function f, it is easy to obtain from Eq. (15) the values for γ_ρ which would correspond to other assumptions about α_ρ. For example, it is readily verified that if we had assumed $\alpha_\rho(0) = 0.57$, then we would have predicted that $\gamma_\rho(0) = 0.76 \pm 0.10$.

We remark that Eq. (14) is in fairly good agreement with the value $\gamma_\rho(0) \cong 0.68$ which would be required by Morgan and Shaw[16] to describe ImT^1 for their favored solution, if they were to assume ρ dominance above 1.5 GeV with $\alpha_\rho(0) = 0.50$. We also remark that the value deduced by Olsson from πN charge-exchange data together with universality, namely, $\gamma_\rho(0) = 0.17$,[17] is much too small, being only about one-fifth the value indicated by the present analysis.

As we mentioned earlier, Eq. (6) is not strictly valid for $t < -0.61$ GeV^2, nor for $t > 0.08$ GeV^2.[12] The resulting $\gamma_\rho(t)$ develops oscillations for $t > 0.3$ GeV^2, and it has a spurious zero at $t = m_\rho^2$, because the integral defining f diverges when $\alpha_\rho \geq 1$. However, it is reasonable to expect that Eq. (6) be approximately valid for t as negative as -1.00 GeV^2 and as positive as 0.25 GeV^2, and we shall now present evidence that this is indeed the case.

To make a rough comparison of our γ_ρ with the data, we have used the data to compute

$$\bar{\gamma}_\rho(t) \equiv [s_b - s_a(t)]^{-1} \int_{s_a(t)}^{s_b} ds\,(s/\bar{s})^{-\alpha_\rho(t)}\,\mathrm{Im}T^1(s,t)\,. \quad (16)$$

The resulting $\bar{\gamma}_\rho$ is displayed in Fig. 3 along with our preferred γ_ρ based on Eqs. (6), (9), (11), (12a), and (12b). The agreement is good over the entire range $-1.00 \leq t \leq 0.25$ GeV^2, which supports our claim that Eq. (6) is approximately valid over this entire range, and also our claim that Im$T^1(s,t)$ is dominated by ρ exchange when $s \geq s_a(t)$.

As a further comparison of our results with the data, we have inserted our preferred γ_ρ into the right-hand side of Eq. (5), and have compared the resulting ImT^1 with the data. Some of these comparisons are displayed in Fig. 1. In the sense of local averages, the agreement is good for $s \geq s_a(t)$ over the entire range $-1.00 \leq t \leq 0.25$ GeV^2, provided the physical ImT^1 turns up again near $s^{1/2} = 1.5$ GeV for t near zero. Fortunately, the existence and properties of the $g(1680)$ meson ensure that this will happen.

The good agreement for $t < -0.6$ GeV^2 is rather remarkable in light of the fact that $\alpha_\rho(t) < 0$ for $t < -0.6$ GeV^2, and $\alpha_\rho(-1.0\text{ GeV}^2) = -0.4$. Hence there is little reason to expect the contour integral at Re$J = -\frac{1}{2}$ in the Sommerfeld-Watson transform to be negligible for these values of t.

Since the Veneziano model has been a subject of great interest, we have also plotted the Veneziano value[18] for γ_ρ in Fig. 3. The Veneziano model has a somewhat richer resonance spectrum than is indicated by the data [for example, the Veneziano $\rho'(1250)$ does not appear in the data], so the Veneziano γ_ρ is somewhat larger than the γ_ρ computed from the data. However, the qualitative agreement is quite good. In particular, the Veneziano γ_ρ vanishes at precisely the point where $\alpha(t) = 0$,[18] and the data indicate a zero in γ_ρ at nearly the same point.

Since the Veneziano model and the physical P wave have ρ resonance poles with nearly identical residues,[18] the physical curve for γ_ρ should intersect the Veneziano curve near $t = m_\rho^2$. Note that both data-based curves in Fig. 3 are in fact drawing close to the Veneziano curve as t approaches 0.25 GeV^2. Smooth extrapolations of the data-based curves could easily be drawn wherein they intersected the Veneziano curve near $m_\rho^2 = 0.58$ GeV^2, where $\gamma_\rho(\text{Ven.}) = 1.81$. This further supports our claim that our result for γ_ρ is approximately valid up to about 0.25 GeV^2, despite the fact that Eq. (6) is not strictly valid above 0.08 GeV^2.

VI. ANALYSIS OF Re T^1

In addition to analyzing the data for ImT^1, we have also studied ReT^1. We find that in the sense of local averages, ReT^1 is also dominated by ρ exchange for $s \geq s_a(t)$, over a fairly wide range of values for t.

We have computed the average of the data for $\mathrm{Re}T^1(s,t)$ over the interval $s_a(t) \leq s \leq s_b$, and have compared the result with the average value of

$$\mathrm{Re}T^1(s,t) = \left[\frac{1-\cos\pi\alpha_\rho(t)}{\sin\pi\alpha_\rho(t)}\right]\mathrm{Im}T^1(s,t), \tag{17}$$

which is the $\mathrm{Re}T^1$ predicted by ρ dominance. Using Eq. (5) for $\mathrm{Im}T^1$ together with our preferred result for γ_ρ, we find a discrepancy between the two averages for $\mathrm{Re}T^1$ of less than 7% when $-0.20\ \mathrm{GeV}^2 \leq t \leq 0$. As t becomes more negative the percentage discrepancy increases, but the absolute discrepancy is less than 0.10 so long as t lies within the interval $-0.40\ \mathrm{GeV}^2 \leq t \leq 0$.

As t becomes more negative than $-0.40\ \mathrm{GeV}^2$, the resemblance between the data for $\mathrm{Re}T^1$ and the prediction of Eq. (17) rapidly disappears. For example, the average of the data for $\mathrm{Re}T^1$ at $t=t_z$ is -0.18, whereas Eqs. (9) and (17) predict a double zero in $\mathrm{Re}T^1$ at t_z. However, $\alpha_\rho(t)<0.14$ for $t<-0.40\ \mathrm{GeV}^2$, so it would not be surprising if contributions from the contour integral at $\mathrm{Re}J=-\frac{1}{2}$ were appreciable for these values of t.

Because of the strong evidence for ρ dominance of $\mathrm{Im}T^1$, we regard the additional evidence for ρ dominance of $\mathrm{Re}T^1$ near the forward direction as compelling support for our claim of ρ dominance of $T^1(s,t)$ when $s^{1/2} \geq 1.0$ GeV, for the values of t indicated in the preceding discussion.

VII. PREDICTION OF ZERO FROM DUAL ABSORPTION MODEL

Harari has discussed a dual absorption model[5] wherein the contribution of normal trajectory exchange to an elastic, nonflip amplitude is given roughly by $J_0(R\sqrt{-t})$, where J_0 denotes the zeroth-order Bessel function, and R denotes the "interaction radius" for the two colliding particles. The model succeeds in explaining the dip structure in a number of nonexotic elastic cross sections. The dip occurs where J_0 has its first local minimum.

The data for $\pi^\pm p$ and for K^-p scattering indicate dips near $t=-0.8\ \mathrm{GeV}^2$, while the data for $\bar{p}p$ scattering indicate a dip near $t=-0.5\ \mathrm{GeV}^2$.[19] Evidently, the interaction radii $R(\pi^\pm p)$ and $R(K^-p)$ are equal to each other, and smaller than $R(\bar{p}p)$ by a factor of about $(\frac{5}{8})^{1/2}$:

$$R(\pi^\pm p) = R(K^-p), \tag{18a}$$

$$R(\pi^\pm p) = [\tfrac{5}{8} \pm \tfrac{1}{8}]^{1/2} R(\bar{p}p), \tag{13b}$$

where the indicated uncertainty in Eq. (18b) is a rough estimate.

Let us naively assume that $R(AB)=R(A)+R(B)$, where $R(A)$ and $R(B)$ are radii intrinsic to the colliding particles A and B. We shall also assume that $R(\pi^\pm)=R(\pi^0)$, and that $R(\bar{p})=R(p)$. Then Eq. (18b) implies that

$$R(\pi) = (0.58 \pm 0.15) R(p). \tag{19}$$

The first zero of $J_0(x)$ occurs at $x=2.4$, and the first minimum at $x=3.8$. Thus the dip in $\bar{p}p$ scattering at $t=-0.5\ \mathrm{GeV}^2$ indicates that

$$R(\bar{p}p) = (5.4 \pm 0.4)\ \mathrm{GeV}^{-1} \tag{20}$$

(i.e., 1.1 ± 0.1 F), where the indicated uncertainty in $R(\bar{p}p)$ is based on an estimated uncertainty of ±15% in the dip location, which varies somewhat with energy.

From Eqs. (19) and (20), we conclude that

$$R(\pi\pi) = (3.1 \pm 0.9)\ \mathrm{GeV}^{-1}, \tag{21}$$

hence that the $\rho\pi\pi$ Regge residue function should contain a zero at

$$t_{\mathrm{zero}} = (-0.6\,^{+0.2}_{-0.5})\ \mathrm{GeV}^2. \tag{22}$$

This prediction is in good agreement with the zero near $t_z=-0.52\ \mathrm{GeV}^2$ indicated by the $\pi\pi$ data of Carroll *et al.*

From Eq. (18a), we predict that the ρKK Regge residue function vanishes at the same point as the $\rho\pi\pi$ residue function.

We remark that Eqs. (18a) and (19) are both in rough agreement with what one would expect from a simple quark model for the hadrons.

VIII. PREDICTION OF ZERO FROM DUALITY AND THE ABSENCE OF EXOTIC RESONANCES

The observed zero in γ_ρ is in fact a simple consequence of duality together with the absence of exotic resonances.[20] The argument is both important and brief, so we shall repeat it here.

Duality equates resonant absorptive parts with those resulting from exchange of normal Regge trajectories. Since

$$A^2 = \tfrac{1}{3}T^0 - \tfrac{1}{2}T^1 + \tfrac{1}{6}T^2$$

contains no resonances, it follows that

$$\tfrac{1}{3}\gamma_f(t) - \tfrac{1}{2}\gamma_\rho(t) = 0, \tag{23a}$$

$$\alpha_f(t) = \alpha_\rho(t), \tag{23b}$$

where γ_f characterizes the contribution of f_0 exchange to $\mathrm{Im}T^0(s,t)$, in analogy with Eq. (5).

The contribution of f_0 exchange to $T^0(s,t)$ is purely real when $\alpha_f(t)=0$. Therefore, we conclude that γ_f must vanish where $\alpha_f=0$, i.e., somewhere between $t=-0.5$ and $-0.6\ \mathrm{GeV}^2$. From Eq. (23a), it follows that γ_ρ must vanish at the same point. This prediction of duality is in excellent agreement with the zero near $t=-0.52$ GeV^2 indicated by the data of Carroll *et al.*

IX. DIPS IN πN CHARGE-EXCHANGE CROSS SECTIONS

As a final remark on the zero in γ_ρ, we note that the πN charge-exchange helicity-flip amplitude has an approximate zero near $t=-0.6$ GeV^2.[21] Such a zero is to be expected from the zero in the $\rho\pi\pi$ coupling which we have observed in the $\pi\pi$ data of Carroll *et al.* Although one might also expect the πN helicity-nonflip amplitude to have a zero at this value of t, absorptive corrections (Regge cuts) are known to be important in the nonflip amplitude.[21] These corrections may be sufficient to explain why the actual dip in the nonflip cross section is rather shallow and is displaced to about -0.4 GeV^2.[21]

In meson-baryon scattering, there is a general tendency for Regge cuts to be important in nonflip amplitudes. Since $\pi\pi$ scattering is nonflip, it is somewhat surprising that the $\pi\pi$ data analyzed in this paper are consistent with pure ρ exchange. A relative weakness for $\pi\pi$ cuts is partially to be expected from the fact that $\pi\pi$ total cross sections are relatively small (about 15 mb at high energies[22]), since absorptive corrections are typically dependent on total cross sections. However, further investigation of these issues is clearly to be desired.

X. REPRESENTATION FOR $\pi\pi$ AMPLITUDES

By using Eq. (2a) to evaluate the subtraction term on the right-hand side of Eq. (2b), and by performing the subtractions at the "symmetry point" where $s_0=t_0=\frac{4}{3}\mu^2\equiv c_0$, we obtain for $I=\binom{0}{2}$ the representation

$$A^I(s,t)=\begin{pmatrix}-5\\-2\end{pmatrix}\lambda+\frac{2(t-c_0)^2}{\pi}\int_{4\mu^2}^{\infty}ds'\frac{\operatorname{Im}T^I(s',c_0)}{(s'-c_0)(s'-t)(s'+t-2c_0)}$$
$$+\frac{s-c_0}{\pi}\int_{4\mu^2}^{\infty}ds'\left[\frac{\operatorname{Im}A^I(s',t)}{(s'-c_0)(s'-s)}-\sum_{I'=0}^{2}C_{II'}(-1)^{I'}\frac{\operatorname{Im}T^{I'}(s',t)}{(s'+t-2c_0)(s'+s+t-4\mu^2)}\right], \tag{24}$$

where λ denotes the subtraction parameter first introduced by Chew and Mandelstam.[23] Equation (24) is valid for arbitrary s when t is real[24] ($\pm i\epsilon$) and $-32\mu^2\leqslant t\leqslant 4\mu^2$. Hence Eq. (24) is well suited for studying the physical region.

For $I=1$, Eq. (3b) provides a suitable representation, since it is also valid for arbitrary s when $-32\mu^2\leqslant t\leqslant 4\mu^2$.

The Legendre series for $\operatorname{Im}A^I(s,t)$ converges for all $s\geqslant 4\mu^2$ when $-32\mu^2\leqslant t\leqslant 4\mu^2$.[12] Therefore, the Legendre series can be used to evaluate the integrals in Eqs. (3b) and (24) strictly in terms of physical-region absorptive parts for $-4\mu^2\leqslant s\leqslant 68\mu^2$, $(2\mu^2-s/2)\leqslant t\leqslant 0$ (i.e., $0\leqslant\cos\theta\leqslant 1$). Since Bose symmetry implies that A^I is even (odd) in $\cos\theta$ when I is even (odd), Eqs. (3b) and (24) enable us to express the A^I in terms of λ and physical-region absorptive parts for $-4\mu^2\leqslant s\leqslant 68\mu^2$, $-1\leqslant\cos\theta\leqslant 1$. Partial waves can be projected out by techniques similar to those used in Ref. 25.

It is readily seen that when $\alpha_\rho(t)>[\alpha_P(t)-1]$, the contributions from the asymptotic region to the right-hand sides of Eqs. (3b) and (24) are dominated by the contributions from $\operatorname{Im}T^1$. This condition is satisfied for $t>-29\mu^2$ if $\alpha_P'=0$, and for $t>-66\mu^2$ if $\alpha_P'=0.50$ GeV^{-2}. Therefore, Eqs. (3b) and (24), together with the knowledge of $\gamma_\rho(t)$ presented in this paper, comprise the basis of a powerful technique for studying the $\pi\pi$ interaction.

XI. CROSSING CONSTRAINTS AND AN s-u SYMMETRIC REPRESENTATION

Although crossing symmetry has been used extensively in deriving the representations (3b) and (24), the A^I generated by these representations do not *automatically* have the correct properties under interchange of s with t, s with u, or t with u. Thus crossing symmetry implies nontrivial sum rules for the absorptive parts appearing in Eqs. (3b) and (24). We shall begin by discussing t-u symmetry.

Bose symmetry implies that $A^1(s,t)$ is antisymmetric under interchange of t with u. Since the right-hand side of Eq. (3b) does not automatically have this property, a nontrivial sum rule is implied. However, the resulting sum rule is not a new one. To see this, we note that the right-hand side of Eq. (3a) is manifestly antisymmetric under interchange of t with u. Thus Bose symmetry can be imposed on the A^1 of Eq. (3b) by equating the right-hand side of (3b) with that of (3a). The result is simply Eq. (4), which is the sum rule discussed earlier. Hence nothing new results from the imposition of Bose symmetry on A^1.

Bose symmetry implies that A^0 and A^2 are symmetric under interchange of t with u. Imposing this condition on the right-hand side of Eq. (24), we obtain for $I=0$ and 2 the sum rule

$$\int_{4\mu^2}^{\infty} ds' \left\{ \frac{2(u-t)(s'-c_0)\mathrm{Im}T^I(s',c_0)}{(s'-t)(s'-u)(s'+t-2c_0)(s'+u-2c_0)} + \frac{\mathrm{Im}A^I(s',t)-\mathrm{Im}A^I(s',u)}{(s'-c_0)(s'+t+u-4\mu^2)} - \sum_{I'=0}^{2} C_{II'}(-1)^{I'}\left[\frac{\mathrm{Im}T^{I'}(s',t)}{(s'-u)(s'+t-2c_0)} - \frac{\mathrm{Im}T^{I'}(s',u)}{(s'-t)(s'+u-2c_0)}\right]\right\} = 0\,, \quad (25)$$

which must hold for real t $(\pm i\epsilon)$ and real u $(\pm i\epsilon)$ when $-32\mu^2 \leqslant t \leqslant 4\mu^2$ and, simultaneously, $-32\mu^2 \leqslant u \leqslant 4\mu^2$. Equation (25) is a new sum rule, as may be seen by noting that Eq. (4) contains only $\mathrm{Im}T^1$ and $\mathrm{Im}A^1$, whereas the $\mathrm{Im}T^I(s', c_0)$ in Eq. (25) cannot be expressed in terms of $\mathrm{Im}T^1$ and $\mathrm{Im}A^1$.

It is readily verified that the left-hand side of Eq. (25) receives no contribution from S waves. Furthermore, if u is set equal to c_0, then for both $I=0$ and 2 Eq. (25) reduces to Eq. (4), with $s=c_0$ in the latter. We also note that just as in Eq. (4), the asymptotic contribution to Eq. (25) is dominated by a difference between $\mathrm{Im}T^1$ evaluated at two different values of momentum transfer. Because of these strong similarities between Eqs. (4) and (25), we conjecture that Eq. (25) is reasonably well satisfied when Eq. (4) is satisfied. However, we have no proof that this is so.

Next let us consider s-t crossing. Equations (1a) and (1b) imply that

$$A^I(s,t) = \sum_{I'=0}^{2} C_{II'}A^{I'}(t,s)\,. \quad (26)$$

If we impose Eq. (26) on our representations for the A^I, nontrivial sum rules will follow, since the momentum-transfer variable is t on the left-hand side of Eq. (26), but s on the right-hand side. The sum rules obtained in this way are rather lengthy, so we shall not write them out. We remark that again the integrands are independent of S waves, and are dominated in the asymptotic region by differences between $\mathrm{Im}T^1$ evaluated at different values of momentum transfer. Therefore, we conjecture that these sum rules are also reasonably well satisfied when Eq. (4) is satisfied.

Finally, we consider s-u symmetry. We begin by noting that if s-t and t-u symmetry were satisfied exactly, then s-u symmetry would automatically be satisfied, since the $A^I(s,t)$ are either even or odd under interchange of t with u.

In practice, s-t and t-u symmetry will sometimes be satisfied only approximately. In this case, it is advantageous to have a representation in which s-u symmetry is manifest. Such a representation can be obtained by first setting $s_0=c_0$ in Eq. (2b), subsequently setting $s_0=4\mu^2-t-c_0$, and then using half the sum of the resulting expressions.

For the sake of brevity in writing the result, we shall use a matrix notation, with A^I forming a 3-vector denoted by $\vec{A}$. The s-t crossing matrix C was given earlier. We shall now denote it by C_{st}, and we define C_{tu} and C_{su} by

$$(C_{tu})_{II'} \equiv (-1)^I\delta_{II'}\,,$$
$$C_{su} \equiv C_{st}C_{tu}C_{st}\,,$$

in terms of which crossing symmetry may be stated as

$$\vec{A}(s,t) = C_{st}\vec{A}(t,s) = C_{tu}\vec{A}(s,u) = C_{su}\vec{A}(u,t)\,.$$

We can then write our s-u symmetric representation as

$$\vec{A}(s,t) = \tfrac{1}{2}(1+C_{su})\left[\vec{A}(c_0,c_0) + \frac{t-c_0}{\pi}\int_{4\mu^2}^{\infty} ds'\left(\frac{1}{s'-t} - \frac{C_{tu}}{s'+t-2c_0}\right)\frac{C_{st}}{s'-c_0}\,\mathrm{Im}\vec{A}(s',c_0)\right] + \frac{1}{2\pi}\int_{4\mu^2}^{\infty} ds'\left[\frac{1}{s'-c_0}\left(\frac{s-c_0}{s'-s} + \frac{u-c_0}{s'-u}C_{su}\right) - \frac{1}{s'+t-2c_0}\left(\frac{s-c_0}{s'-u}C_{su} + \frac{u-c_0}{s'-s}\right)\right]\mathrm{Im}\vec{A}(s',t)\,, \quad (27)$$

where

$$\vec{A}(c_0,c_0) = \begin{pmatrix} -5\lambda \\ 0 \\ -2\lambda \end{pmatrix}.$$

Since $(C_{su})^2=1$, the $\vec{A}$ generated by Eq. (27) manifestly satisfy $\vec{A}(s,t)=C_{su}\vec{A}(u,t)$. Bose $(t-u)$ symmetry implies a (vector) sum rule for the absorptive parts appearing in Eq. (27). If Bose symmetry were satisfied exactly, then s-t symmetry would automatically be satisfied.

Do the sum rules considered in this paper constitute sufficient conditions for crossing symmetry to be satisfied? Unfortunately, the answer is negative. The reason is that all the sum rules implied by crossing symmetry hold only for restricted ranges of s and/or t and/or u. If the amplitudes generated by our representations were analytic in these variables, then the sum rules would imply complete crossing symmetry by analytic continuation. However, the integrands of our representations involve $\mathrm{Im}A^I(s',t)$. Therefore, one must use absorptive parts with the proper analyticity in t in order to obtain amplitudes which are fully crossing-symmetric. In practice, this is difficult to do. (Similar remarks apply to the representations of Roskies and Roy.)

Notwithstanding the difficulty of constructing amplitudes which are fully crossing-symmetric, we believe that satisfaction of the sum rule (4) is sufficient to ensure results consistent with crossing in the low-energy region.

XII. UNITARIZED VENEZIANO $\pi\pi$ AMPLITUDES

Recently a model[6] (henceforth UV) has been proposed for the A^I in which the following conditions are satisfied *exactly*: analyticity, crossing symmetry, positivity of partial-wave absorptive parts with definite isospin, satisfaction of fixed-t D.R.s with one subtraction, and satisfaction of fixed-s D.R.s with two subtractions. It follows from the preceding conditions that the inequalities of Martin[26] and others are satisfied, including the highly stringent inequalities of Yen and Roskies.[27]

When s is given the symmetry-point value $s=\frac{4}{3}\mu^2=c_0$, the model also satisfies fixed-s D.R.s with only one subtraction, and hence satisfies the $I=1$ Regge sum rule[28] for the symmetry-point derivative parameter λ_1 of Chew and Mandelstam.[29]

In addition to the preceding *exact* properties, the S waves and P wave are unitary within 2% from threshold up to energies exceeding 1 GeV. Solutions in agreement with data above 600 MeV were constructed for values of the isoscalar scattering length ranging from 0.0 to $1.1\mu^{-1}$, thereby disproving various claims in the literature about alleged uniqueness for the S-wave scattering lengths.

The absorptive parts of all partial waves with $l \geq 2$ are given in UV by the δ-function absorptive parts corresponding to resonances in the Veneziano model.[30] Thus through duality, asymptotic absorptive parts are given by the Regge exchanges of the Veneziano model.

From the preceding remarks, it is clear that UV satisfies *exactly* all the hypotheses upon which the representations (3b), (24), and (27) are based. Thus all solutions of UV are automatically solutions to Eqs. (3b), (24), and (27).

Although the rigorous representations derived herein are satisfied exactly by the solutions of UV, we have seen in the present paper that the Veneziano value for γ_ρ is about 25% larger than the physical value (Fig. 3). To see how this affects the solutions of UV, we begin by noting that the A^I are rigorously determined within a neighborhood of the symmetry point by λ and the derivative parameter λ_1.[29] In UV, λ is treated as a phenomenological subtraction parameter, while λ_1 is determined by an integral over $\mathrm{Im}T^1(s, c_0)$.

One readily finds that the net contribution to λ_1 from above 1 GeV is $0.04\mu^{-2}$, which is only about one-third the total value of λ_1. (For the Weinberg[9] value $\lambda=-0.01$, $\lambda_1=0.11\mu^{-2}$ in UV.) Thus an error of 25% in γ_ρ implies an error of only about 8% in λ_1. When we recall that the amplitudes are fixed at the symmetry point by the subtraction parameter λ, we see that an error of 8% in λ_1 cannot lead to significant errors within the low-energy region.

Furthermore, we note that the ρ resonance width used in UV was 120 MeV, whereas the value now favored [and used in $\gamma_\rho(\mathrm{Ven.})$[18] in Fig. 3] is 135 MeV.[10] Since the Veneziano contributions are normalized to Γ_ρ in UV, the Veneziano γ_ρ is really only about 11% too large in UV, thereby generating an error of only $0.004\mu^{-2}$ in λ_1. Another consequence of using 120 MeV for Γ_ρ in UV is that the ρ resonance contribution to λ_1 was *underestimated* by $0.004\mu^{-2}$. Hence by a fortuitous circumstance, the UV values for λ_1 are actually in perfect accord with a ρ width of 135 MeV and with the γ_ρ reported in this paper.

A potentially significant limitation of UV lies in its neglect of Pomeranchon exchange. The corrections which one should make are given by the Pomeranchon contributions to the right-hand sides of Eqs. (3b) and (24). Therefore, let us estimate these contributions.

An asymptotic total cross section of[22] 15 mb implies that $\mathrm{Im}T^0(s,0)=1.2(s/\overline{s})$ for large s, where $\overline{s}=1\ \mathrm{GeV}^2$. However, the data of Carroll *et al.* for $\mathrm{Im}A^2$ indicate that Pomeranchon exchange does not achieve full strength below 1.5 GeV. Furthermore, nonresonant absorptive parts are primarily S wave below 1.5 GeV, and S-wave absorptive parts are already present in UV. Therefore, Pomeranchon corrections to the solutions of UV should be estimated by computing contributions from Pomeranchon exchange above 1.5 GeV to the right-hand sides of Eqs. (3b) and (24).

Denoting the aforementioned Pomeranchon contributions by A_P^I, we find for $I=0$ and 2 that

$$A_P^I(s,0) \cong 0.11[3\delta_{I0}(c_0)^2 + s(s-c_0)]/\overline{s}^{\,2}, \qquad (28a)$$

where δ_{I0} denotes the Kronecker δ, and the approximation is valid for $s \ll (1.5\ \mathrm{GeV})^2$. For $I=1$, we find

$$A_P^1(s,0) \cong 0.11 s(s-4\mu^2)/\overline{s}^{\,2}. \qquad (28b)$$

Thus between threshold and 1 GeV, the Pomeranchon contribution to all three of the A^I is given approximately by

$$A_P^I(s,0) \cong 0.11(s/\overline{s})^2. \qquad (28c)$$

For $E_{\mathrm{c.m.}}=600$, 800, and 1000 MeV, Eq. (28c) indicates that $A_P^I(s,0)$ equals 0.01, 0.04, and 0.11, respectively. Thus the contributions of Pomeranchon exchange are negligible below 600 MeV, and quite small below 800 MeV.

Finally, we must consider the effect of strong $K\bar{K}$ production in the $I=0$ channel above 1 GeV, since inelasticity was neglected in the actual calculations performed in Ref. 6.

As might be expected, numerical estimates indicate that inelasticity above 1 GeV has very little effect on A^0 below 600 MeV. Furthermore, A^0 is constrained in UV to agree with data in the ρ region. Therefore, we conclude that the A^I of UV are in *excellent* accord with nature below 800 MeV, notwithstanding the neglect of Pomeranchon exchange and $K\bar{K}$ production. The only uncertainty remaining in the low-energy region is nature's choice for λ.

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[11]Since principal parts are involved, we represent each phase shift and each inelasticity by a continuous 17-parameter function over the region spanned by the data, and determine the 17 parameters by making a least-squares fit to the 26 data points. Below 0.60 GeV, we represent the P wave as described earlier for our evaluation of $h(s,t)$, and we neglect the D and S waves [we remind the reader that S waves contribute nothing to the integrals in Eqs. (6) and (7b)]. We then use the resulting S, P, and D waves in a truncated Legendre series for $\mathrm{Im}T^1$.

[12]The domain of validity is determined by the boundaries of the double-spectral functions (Ref. 7). Loosely speaking, $\mathrm{Im}A^I(s,t)$ becomes complex when s and t lie within the domain of a double-spectral function, and formulas based on the assumption that $\mathrm{Im}A^I$ is the discontinuity across a cut are no longer valid. [Contrary to the claim of Roy, Eq. (14) of Ref. 3 is not valid for all $t<4\mu^2$; it is invalid for $t<-32\mu^2$. However, Roy concentrates on using Eq. (14) within the domain of convergence of a certain Legendre series, which in fact coincides with the domain of validity of Eq. (14).] One finds that for $-32\mu^2 \le t \le 4\mu^2$, $\mathrm{Im}A^I(s,t)$ is real for all $s \ge 4\mu^2$, and the Legendre series for $\mathrm{Im}A^I$ converges. As t decreases below $-32\mu^2$, $\mathrm{Im}A^I$ becomes complex in neighborhoods of $s=12\mu^2$ and $24\mu^2$. As t continues to decrease, the sizes of these neighborhoods grow. As t increases above $4\mu^2$, $\mathrm{Im}A^I$ becomes complex for large positive s. If the double-spectral functions are small near the relevant boundaries, then Eqs. (4) and (6) will be *approximately* valid over a wider range than $-32\mu^2 \le (s \text{ and } t) \le 4\mu^2$, and the Legendre series for $\mathrm{Im}A^I$ will provide an *asymptotic* series over some interval outside the domain of convergence.

[13]Condition (8a) with $s=t_z$ corresponds to $\Lambda \ge (1.06\text{ GeV})^2$, but we consider slightly smaller values for Λ, namely, down to 1.0 GeV^2. Condition (8b) is automatically satisfied when $s_a(t) \le \Lambda$.

[14]For a typical value $\Lambda = \frac{1}{2}[s_a(t)+s_b]$, the contribution of $\gamma_P(t_z)$ to $\gamma_\rho(t)$ can be described to excellent approximation by

$$\gamma_\rho(t) \sim [0.09 - 0.35(t/\bar{s})^2]\,\gamma_P(t_z)$$

for $-0.65 \le t \le 0.25$ GeV^2 (with $\bar{s}=1$ GeV^2). When t becomes more negative than -0.65 GeV^2, the contribution of $\gamma_P(t_z)$ to $\gamma_\rho(t)$ grows more rapidly than indicated above. For our assumed value of $\gamma_P(t_z)=0.6$, the contribution to γ_ρ reaches -0.10 at $t=-0.80$ GeV^2, and -0.25 at $t=-1.00$ GeV^2.

[15]The coefficient of $\gamma_\rho(t_z)$ in Eq. (13) is correct within ± 0.03 for $-0.75 \le t \le 0.25$ GeV^2. For $t=-1.00$ GeV^2, the indicated coefficient is 20% smaller than the correct value.

[16]D. Morgan and Graham Shaw, Nucl. Phys. B10, 261 (1967).

[17]M. G. Olsson, Phys. Rev. 162, 1338 (1967). Olsson's result is $\sum C_{1I}\,\sigma^I_{\text{total}} = 0.6\nu^{-1/2}\mu^{-2}$, where $\nu \equiv \frac{1}{4}(s/\mu^2-4)$. With Olsson's normalization of amplitudes, the optical theorem (cf. Ref. 16) implies $\mathrm{Im}A^I_F(\nu) = (\nu/8\pi)\mu^2\sigma^I_{\text{total}}(\nu)$ for large ν, where A^I_F denotes the forward amplitude. It follows that $\mathrm{Im}T^1_F = 0.024\nu^{1/2}$. However, Olsson consistently multiplies σ by $(\nu/4\pi)$ instead of $(\nu/8\pi)$ in his implicit uses of the optical theorem, so one might infer from his statement about cross sections that he would say $\mathrm{Im}T^1_F = 0.048\nu^{1/2}$. This corresponds to $\gamma_\rho(0)=0.17$, which is smaller than our result by a factor of nearly five. [Olsson's result was misquoted by E. P. Tryon, Phys. Lett. 38B, 527 (1972), but no conclusion of that work is affected.]

[18]Cf. C. Lovelace, Phys. Lett. 28B, 264 (1968). We follow Lovelace in keeping only the leading term of the Veneziano series and in setting $\alpha(t) = 0.483 + 0.017 \times (t/\mu^2)$. We choose the over-all coefficient such that the δ-function absorptive part of the ρ resonance corresponds to $\Gamma_\rho = 0.135$ GeV.

[19]Cf. H. Harari, Ref. 5, and references cited therein. Harari ignores the differences in dip locations, presumably because he believes the model to have only qualitative, or at most semiquantitative, validity. We

wish to emphasize that differences *do* exist, and furthermore that minor differences in dip locations can imply *major* differences in zero locations, because the argument of the Bessel function is proportional to the *square root* of t.

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[24]The variable t must be given a small imaginary part in Eq. (24) because of the twice-occurring denominator factor $(s' + t - 2c_0)$, which would otherwise vanish within the range of integration when $t < -\frac{4}{3}\mu^2$. However, these vanishing denominators do not generate a cut in t; the net discontinuity vanishes because of Eqs. (1a), (1b), and (1d).

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[30]Cf. C. Lovelace, Ref. 18.