Rise and Fall of Relativistic Trajectories*

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The general problem of rising and falling Regge trajectories is investigated in a fully relativistic model in which an arbitrary two-particle interaction is unitarized by summing generalized ladder diagrams. The trajectories are extracted from the infinite sum by an extension of the technique previously applied to ladder diagrams in an ordinary scalar field theory. Any basic interaction which vanishes faster than any power of the momentum transfer t generates infinitely falling trajectories. A physical explanation of falling trajectories is given in terms of the scattering of infinitely composite particles where the standard single-particle-exchange interaction is modified by exponentially damped form factors. A variety of models for rising trajectories is constructed, all of which violate either the Jaffe bound on form factors or the Froissart bound on forward scattering amplitudes. If the interaction is single-particle exchange with form factors $e^{-\gamma(-t)^\beta}$, the trajectory has $\mathrm{Re}\alpha\to+\infty$ as $s\to\infty$ if $\beta\geq 1$, but it is not of narrow width. Energy-dependent interactions based on Regge-pole exchange, energy-dependent coupling constants, or direct form factors yield rising trajectories, but not narrow widths. Moreover, the energy dependence must be such that the Froissart bound appears to be grossly violated.

I. INTRODUCTION

MUCH of the recent theoretical work on various aspects of Regge theory has incorporated the concept of infinitely rising, and infinitely falling, Regge trajectories.[1] Moreover, the trajectories almost universally are assumed to be linear with small corrections. In particular, the imaginary part is assumed to be small so that narrow-width resonances appear whenever $\mathrm{Re}\alpha$ crosses the appropriate integer or half integer. An examination of the Chew-Frautschi plot for baryons lends credence to the belief that Regge trajectories are infinitely rising and suggests that they are linear functions of s, the square of the energy.[2] There is less evidence for rising boson trajectories. The idea that trajectories are infinitely falling functions as $s\to-\infty$ is based more upon theoretical considerations[3] than experimental evidence, but is supported by the absence of any curvature in Regge-trajectory fits to scattering data at large momentum transfer.[4] Since rising trajectories lie at the heart of so many investigations, the concept should be examined in detail. Mandelstam[3] and Epstein and Kaus[5] have discussed the problem in a dispersion-theoretic framework and conclude that the rising behavior comes from subtraction constants in the dispersion relations for the trajectory function. These subtraction constants are difficult to evaluate. In fact, there is a large class of equivalent theories which assume linear, infinitely rising trajectories, and then impose unitarity and crossing to constrain the parameters of the trajectory function.[6] Although this approach to Regge theory may be the correct one, we do not pursue it here.

Rather than attempt to find self-consistent rising trajectories, we explore the question of whether it is possible to construct dynamical models that generate such trajectories. Our point of view is that if the extension of Regge poles to the relativistic domain is ever to become more than a phenomenological tool, it is necessary to understand the basic dynamical mechanism responsible for the experimentally observed trajectories in the same way that the origin of Regge poles in potential theory is understood. Unfortunately, Regge trajectories in potential theory rapidly turn over.[7] The leading trajectory for a single Yukawa potential starts at $\alpha=-1$ for $s=-\infty$ and moves to the right of $\mathrm{Re}\alpha=-\frac{1}{2}$ at threshold. $\mathrm{Re}\alpha$ continues to increase above threshold and, for strong enough couplings, may cross several positive integers. Ultimately the centrifugal barrier overcomes the attractive force, and $\mathrm{Re}\alpha$ decreases rapidly to $\alpha=-1$. A very similar behavior is found in the standard relativistic models of Regge trajectories and thus the explanation for the rising and falling behavior does not lie in the simple extension to relativistic kinematics.[8,9]

Although there are a number of speculations about rising trajectories, only two basic models have been investigated in detail. In one model, the trajectory is supported by the opening of new channels as the energy is increased.[10] The particles in these new channels have

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increasing spin so that the orbital angular momentum of the internal particles can be low, and there is no problem with a large, repulsive, centrifugal barrier. The sequence of particles with increasing spin is, in turn, described by a rising Regge trajectory. Although this bootstrap scheme insists that multichannel effects are dominant, in the absence of exact solutions, the model becomes another approach constraining the parameters of trajectories that are *a priori* assumed to be rising. The other model keeps the spins of the internal particles small and allows their orbital angular momentum to increase. In order to overcome the angular momentum barrier in such a model, it is necessary that the potential be energy dependent. Trivedi[11] and Tiktopoulos[12] have explored models of this type in potential theory and found that rising trajectories can indeed be generated. There is some question as to whether the potential-theory approximation is valid in the asymptotic energy region of interest.

There has been little work done on the problem of infinitely falling trajectories beyond the conjecture of Mandelstam[13] that it is related to the infinitely composite nature of the scattering particles. Aaron and Teplitz[14] have shown that if one of the particles in a scattering process is treated as a bound state, the position of $\alpha(-\infty)$ is shifted to $\alpha(-\infty)=-3$ from its value $\alpha(-\infty)=-1$ if all the particles are elementary. However, there has been no work on generating $\alpha(-\infty)=-\infty$.

In this paper we explore several different fully relativistic models for generating rising and falling Regge trajectories. We use a recently developed formulation of high-energy perturbation theory[9] or, equivalently, the Bethe-Salpeter equation, as a method of unitarizing a variety of phenomenological interactions which simulate plausible dynamical effects. This method is particularly useful, since it enables us to calculate trajectories above threshold. To understand falling trajectories, we combine the following observations: (i) A Born approximation, or Bethe-Salpeter kernel, which vanishes like $(-t)^{-N}$ as $t\to-\infty$, where t is the momentum transfer, produces trajectories which have $\alpha(-\infty)=-N$.[15] (ii) At least one model of infinitely composite particles in field theory leads to form factors which decrease faster than any power of $-t$.[16] Together, these two points suggest that the infinitely composite nature of the internal particles requires that the single-particle-exchange kernel of the ladder approximation should be modified by the introduction of exponentially damped form factors. Upon unitarization, this modified kernel will produce infinitely falling trajectories. In fact, we investigate a variety of kernels which have the asymptotic form $(-t)^{\delta}e^{-\gamma(-t)^{\beta}}$ and are independent of s. Some are just form-factor modifications of single-particle exchange, and others do not have the particle-exchange pole. In every case, since $\alpha(-\infty)$ is expected to be infinite, weak-coupling techniques cannot be used to investigate the trajectories. Our reformulation of perturbation theory is ideal for investigating the basic question of whether theories which have $\alpha(-\infty)=-\infty$ can have trajectories which reach positive values of α in the region of $s=0$. The answer is that, in all models with exponentially damped kernels, the trajectories do reach $\alpha>0$ for reasonable values of the coupling constant even though $\alpha\to-\infty$ in the weak-coupling limit. From these investigations, we conclude that the dynamics of infinitely falling trajectories is relatively simple compared to that required to generate trajectories with $\mathrm{Re}(+\infty)=+\infty$.

Above threshold we find that almost all of the trajectories generated by energy-independent kernels which have $\alpha(-\infty)=-\infty$ turn over, just as they do for pure single-particle exchange where $\alpha(-\infty)=-1$. This is not surprising since such kernels do not reflect forces which are capable of overcoming the angular momentum barrier. The exceptions to this statement are kernels of the form given above with $\beta\geq 1$. Such kernels produce Regge trajectories which have $\mathrm{Re}\alpha(+\infty)=\infty$, although asymptotically $\mathrm{Im}\alpha>\mathrm{Re}\alpha$ so that they are not of narrow width. Over any finite region of positive s, however, it is possible to obtain trajectories which have $\mathrm{Re}\alpha>\mathrm{Im}\alpha$. The possibility of obtaining trajectories with $\mathrm{Re}\alpha\to\infty$, although the potential does not increase in strength, depends on the fact that the angular momentum barrier is proportional to α^2. If $\alpha\approx Re^{i\theta}$, the real part of the potential is attractive if $\theta>\frac{1}{4}\pi$. We investigate in detail the trajectories arising from the simple kernel $e^{+\gamma t}$ and obtain trajectories which rise smoothly from $-\infty$ to $+\infty$. Among the more interesting results is the fact that the trajectory is asymptotically linear. This is not in any way assumed as input to the calculation. As a form factor, $e^{+\gamma t}$ violates the Jaffe bound.[17] However, Khuri[18] and Jones and Teplitz[19] have shown that infinitely rising trajectories themselves require abandonment of certain commonly held truths, so we feel that violation of the Jaffe bound is not surprising. Form factors such as $e^{-\gamma(-t)^{1/2}}$ which obey the Jaffe bound yield trajectories which turn over above threshold and have $\mathrm{Re}\alpha(+\infty)=-\infty$.

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In addition to s-independent interactions, we investigate a number of other models which might yield rising trajectories. The relativistic analog of the energy-dependent potential of Trivedi[11] and Tiktopoulos[12] is an energy-independent kernel. If we use a single Regge-pole amplitude with an exponentially damped residue function $\beta(t)=e^{\gamma t}$, we find trajectories which look very much like those arising from a pure exponential kernel, even to the ratio of $\mathrm{Re}\alpha$ to $\mathrm{Im}\alpha$. In other words, simple Regge-pole exchange does not generate narrow-width trajectories. In another model that we discuss, the basic interaction has the form $s^{\eta}e^{-\gamma(-t)^{\beta}}$ and corresponds to an energy-dependent coupling constant. The trajectories turn over for $\beta<1$ unless $\eta>1$. However, if $\eta>1$, the sum of generalized ladder diagrams obtained by iteration of this interaction appears to violate the Froissart bound[20] on scattering amplitudes; in the limit $s\rightarrow\infty$ the Nth term in the sum grows like $s^{N(\eta-1)+1}$. While we are willing to contemplate violation of the Jaffe bound for purely hadronic form factors in the presence of infinitely rising trajectories, we are reluctant to tamper with the Froissart bound. An alternate model that introduces a strong energy dependence into the basic interaction includes the effect of direct form factors. As $s\rightarrow\infty$, the Bethe-Salpeter equation samples a kinematic region in which the particles are far off their mass shell in the timelike direction. We choose the direct form factors proportional to $e^{-\gamma(q^2)^{\beta}}$, where the mass shell is defined by $q^2=-\mu^2$. In the limit $q^2\rightarrow-\infty$, $q^2=(k+\frac{1}{2}is^{1/2})^2$, where k is the loop momentum, the direct form factors are equivalent to exponentially growing or damped coupling constants depending on whether $\beta>\frac{1}{2}$ or $\beta<\frac{1}{2}$. If the direct form factors violate the Jaffe bound,[17] the trajectories rise; but the Froissart bound[20] is violated, and we reject such theories. If the Jaffe bound is observed, the trajectories turn over at least as fast as if we did not include direct form factors.

We investigate the effect of a superposition of two-particle thresholds of increasing mass (but zero spin). If such a sequence of thresholds converges in the sense that the coupling to high-mass channels decreases in order that the total contribution to a unitarity integral be finite, the resulting trajectories again turn over. We also investigate a simple model incorporating some aspects of three-particle phase space; again the trajectories turn over. The superposition model and the three-particle phase-space model do have the effect of making $\mathrm{Im}\alpha\rightarrow0$ at threshold with zero slope rather than with the finite or infinite slope that occurs with a single two-particle threshold. In general, these models decrease the imaginary part of the trajectory function relative to the real part. We cannot, at this time, investigate true multiparticle intermediate states or intermediate states involving particles with high spin. Our conclusion is that among the models we have considered, only those with Born approximations which are proportional to $e^{+\gamma t}$ yield rising trajectories, and these trajectories have $\mathrm{Im}\alpha>\mathrm{Re}\alpha$ asymptotically.

In Sec. II, we discuss the technical aspects of the class of models under investigation. In particular, we show that our working equation, previously derived only for single elementary-particle exchange, is valid for a large class of kernels. We then discuss the approximations involved in solving the equation. We analyze, in particular, the effect of using the Blankenbecler-Sugar approximation for the two-particle Green's function and the separable approximation for the kernel. Section III contains the results on falling trajectories as well as a discussion of the trajectories in the region above threshold in models with simple energy-independent exponentially damped kernels. An analytic, as opposed to numerical, treatment is given for the asymptotic behavior of the trajectory generated by the kernel $e^{+\gamma t}$. Section IV is concerned with the other models that we investigated, including energy-dependent kernels and coupled thresholds. The final section contains our conclusions, and two appendices discuss the problems of higher corrections to the separable approximation for arbitrary kernels and the construction of a very approximate three-particle Green's function.

II. FORMALISM OF GENERALIZED KERNELS

Our investigation of rising and falling trajectories proceeds by first assuming a Born approximation to the scattering amplitude, or a kernel to the Bether-Salpeter equation. This amplitude is then used in an integral equation to determine the Regge trajectories. The integral equation we use is a generalization of the equation developed in I to locate the Regge trajectories in a theory based on single-particle exchange. In that case the equation was shown to be mathematically equivalent to the Bethe-Salpeter equation.[21] Since our approach is unconventional, we derive our working equation again for arbitrary interactions. The derivation closely parallels Polkinghorne's complete summation of ladder diagrams in perturbation theory.[8] Let $K(s,\tau;\, q_1^2,q_2^2;\, q_3^2,q_4^2)=K(\tau)$ be an arbitrary off-mass-shell scattering amplitude for two spinless particles with initial four-momenta q_1 and q_2 and final momenta q_3 and q_4. The total center-of-mass energy is $s^{1/2}$ and $\tau=-t$, where t is the momentum transfer. For single-particle exchange without form factors, $K(\tau)=(\lambda^2+\tau)^{-1}$. We derive our equation in a singularity-free region so that all four-vectors are rotated to a Euclidean metric.

The Nth approximation to the scattering amplitude is obtained by iterating $K(\tau)$ N times with two-particle intermediate states described by Feynman propagators. The result is a generalized Nth-order ladder diagram.

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[21] A. R. Swift and R. W. Tucker, Phys. Rev. D **2**, 397 (1970).

The loop integrations are carried out by first parametrizing the direct propagators by the formula

$$1/a=\int_0^\infty e^{-ax}dx\,,$$

and then using the assumption that $K(\tau)$ has a Laplace transform in τ and q_i^2,

$$K(s,\tau;\,q_1^2,q_2^2;\,q_3^2,q_4^2)=\int_0^\infty \rho(x,s,z_1,z_2,z_3,z_4)e^{-\tau x-\Sigma z_i q_i^2}dx dz_i\,. \quad (1)$$

All kernels considered here satisfy (1). The result of the loop integrations is a representation of the Nth-order ladder diagram of the form

$$A_N(s,\tau)=\int_0^\infty \prod_{i=1} dx_i\,\rho(x_i)\frac{e^{-\tau g/\Delta_N-Q_N}}{\Delta_N^2}\,, \quad (2)$$

where we have suppressed the dependence on all Feynman parameters except those conjugate to the momentum transfer. Q_N and Δ_N are functions of the Feynman parameters which characterized the ladder diagram and $g=x_1x_2\cdots x_N$. Next we Mellin-transform the amplitude with respect to τ to obtain

$$A_N(s,\alpha)=\int_0^\infty \tau^{-\alpha-1}A_N(s,\tau)d\tau =\Gamma(-\alpha)\int_0^\infty \prod_{i=0}^N dx_i\,\rho(x_i)x_i^\alpha\frac{e^{-Q_N}}{\Delta_N^{\alpha+2}}\,. \quad (3)$$

The second condition on the kernel K is that

$$\lim_{x\to 0}\rho(x,s;\,z_1,z_2,z_3,z_4)=x^L g_L(s,z_1,z_2)g_R(s,z_3,z_4)\,, \quad (4)$$

where q_1^2, q_2^2 are the squares of the momenta of the particles to the left of K in the ladder diagram, and q_3^2, q_4^2 refer to the particles to the right of K. The limit in (4) is equivalent to the statement that $K\to\tau^{-1-L}g_Lg_R$ as $\tau\to\infty$. This factorization property excludes diagrams such as the cross and other nonplanar kernels. If the right-hand side of (4) is replaced by a sum of factorized terms, the derivation is still valid. Using (3) and (4), we find that $A_N(s,\alpha)$ has an Nth-order pole at $\alpha=-L-1$. For kernels which vanish faster than any power, $L=\infty$. However, such kernels are the limits of functions with finite L.

Standard perturbation-theory techniques are applied to (3) to isolate the pole at $\alpha=-L-1$. The result, when summed over N, becomes

$$A(s,\alpha)=\frac{\Gamma(-\alpha)[G(\alpha,s)]^2}{\alpha+L+1-F(\alpha,s)}\,, \quad (5)$$

which for $L=0$ is just Polkinghorne's expression for a complete sum of ladder diagrams.[8] The poles of $A(s,\alpha)$ are the zeros of the denominator of (5). Following the method in I, we write $F(\alpha,s)$ in the form

$$F(a,s)=\frac{\bar F(\alpha,s)}{1+\bar F(\alpha,s)/(\alpha+1+L)}\,.$$

The poles of $A(s,\alpha)$ are the poles of $\bar F(\alpha,s)$, where

$$\bar F(\alpha,s)=\sum_{N=1}^\infty(-G^2)^N\int_0^\infty\left(\prod_{i=1}^N\frac{\rho(x_i)}{x_i^L}x_i^{\alpha+L}\right)\times g_L{}^{(1)}g_L{}^{(N)}\frac{e^{-Q_N}}{\Delta_N^{\alpha+2}}\,. \quad (6)$$

Next, comparing (2) and (6), we note that the Nth term in this representation for $\bar F(\alpha,s)$ is that of a Feynman ladder diagram with both ends contracted in a space of $2\alpha+4$ dimensions with an α-dependent interaction. The interaction kernel is just

$$K_\alpha(\tau)=\int_0^\infty dz_i dx\,x^\alpha\rho(x,s,z_1,z_2,z_3,z_4)e^{-\tau x-\Sigma z_i q_i^2}\,. \quad (7)$$

$K(\tau)$ and $K_\alpha(\tau)$ have the same dependence on s and q_i^2. To complete the derivation of our final integral equation, we write $\bar F(\alpha,s)$ as a momentum-space integral of an amplitude $V(\alpha,s,p)$ in a $(2\alpha+4)$-dimensional space. The relation between $\bar F$ and V is given in I, except that a factor $g_L((p+iE)^2,\,(p-iE)^2)$ is included to reflect the nature of the contracted line on the end of $\bar F(\alpha,s)$. The infinite sum in (6) is replaced by an integral equation for V,

$$V(\alpha,s,p)=g_R((p+iE)^2,\,(p-iE)^2)+\frac{G^2}{\pi^{\alpha+2}}\int\frac{d^{2\alpha+4}k\,K_\alpha(s,\,(p-k)^2;\,(p+iE)^2,\,(p-iE)^2;\,(k+iE)^2,\,(k-iE)^2)V(\alpha,s,k)}{[(k+iE)^2+\mu^2][(k-iE)^2+\mu^2]}\,, \quad (8)$$

where $s=4E^2$. We have inserted all factors left out heretofore. Equation (8), together with

$$K_\alpha(s,\tau,q_i^2)=\frac{1}{\Gamma(-\alpha)}\int_0^\infty y^{-\alpha-1}K(s,\,\tau+y,\,q_i^2)dy\,, \quad (9)$$

constitute our basic equations. This representation for $K_\alpha(\tau)$ is obtained from (7) by writing

$$x^\alpha=\frac{1}{\Gamma(-\alpha)}\int_0^\infty y^{-\alpha-1}e^{-xy}dy\,,$$

and using (1) to carry out the x integration. Our investigation is based on finding the poles of the solutions of (8) for a variety of interactions. Although the choice of functional form for $K(\tau)$ is arbitrary, the α and τ dependences of $K_\alpha(\tau)$ are strongly correlated. It is not possible to choose $K_\alpha(\tau)$ to be an arbitrary function of α and τ, but rather we must perform the integral transform in (9) as an intermediate step.

The concept of a continuous-dimensional integral equation was discussed in I in detail. Moreover, (8) has been shown to be mathematically equivalent to the Bethe-Salpeter equation, at least for single-particle exchange.[21] Integration in $2\alpha+4$ dimensions is carried out by setting $2\alpha+4$ equal to an arbitrary positive integer and then, upon completion of the integration, allowing it to become continuous and even complex. The rule that $2\alpha+4$ is an integer in all intermediate steps in a calculation can be used to convert (8) into a configuration space differential equation.

If in the limit $x\to 0$, $\rho(x)$ in (1) is proportional to a sum of M separable terms, the derivation can be carried out by using $M\times M$ matrices for $F(\alpha,s)$ and $\bar{F}(\alpha,s)$. $V(\alpha,s,p)$ becomes an M-component column vector. Each component of V satisfies an uncoupled integral equation with a kernel identical to that in (8), with only the inhomogeneous term depending on M. Thus the Regge trajectories in this case are also given by (8). Furthermore, since (8) does not depend explicitly on L, we can let $L=\infty$ if necessary. Thus, Eq. (8) with $K_\alpha(t)$ defined by (9) is valid for those kernels which (a) have a Laplace transform in the momentum transfer variable and (b) have the asymptotic form $\tau^{-L-1}\sum_i g_L{}^i g_R{}^i$; g_L and g_R depend on the incident and final momentum variables, respectively, and L can be infinite.

Before discussing the solution of (8), we mention our reasons for preferring this equation to either the partial-wave Bethe-Salpeter equation or some other more conventional approach. We find the Mellin-like transform in (9) easier to analyze than a partial-wave projection with Legendre functions. In particular, for many of the kernels that we discuss, $K_\alpha(\tau)$ has a simple dependence on the invariant τ. Compare single-particle exchange where $K_\alpha(\tau)=\Gamma(\alpha+1)/(\lambda^2+\tau)^{\alpha+1}$ with the partial-wave kernel $Q_\alpha(X)/2pq$, where $X=[(P_0-q_0)^2+p^2+q^2+\mu^2]/2pq$. $K_\alpha(\tau)$ is calculated from $K(\tau)$ directly, and it can be continued into the complex α plane without further manipulation. The apparent pole at $\alpha=0$ in the integral is canceled by the zero from $[\Gamma(-\alpha)]^{-1}$. A separable approximation to $K_\alpha((p-k)^2)$ leads to Regge trajectories that compare quite well with trajectories obtained by exact methods. Higher-order corrections to the separable approximation require derivatives of $K_\alpha(\tau)$ with respect to τ. Given a computing routine for $K_\alpha(\tau)$, we find it trivial to evaluate these derivatives with the relation

$$\frac{\partial}{\partial\tau}K_\alpha(\tau)=-K_{\alpha+1}(\tau)\,, \tag{10}$$

which follows from differentiating (9) and then integrating by parts. The final and most important reason for preferring this approach is that it enables us to calculate trajectories above the elastic threshold in a reliable fashion.

As mentioned above, to solve (8) we resort to approximating the kernel by one of finite rank[22]:

$$K_\alpha((p-k)^2)\approx K_\alpha(p^2)K_\alpha(k^2)/K_\alpha(0)\,. \tag{11}$$

When either p^2 or k^2 is small, this approximation becomes exact. It also retains the large p^2 and k^2 properties of $K_\alpha((p-k)^2)$ necessary for proper convergence of the integrals. For this approximation to be valid, the dominant contribution to the integral in (8) must come from the region of small k^2. With exponentially damped kernels, this should be an excellent approximation; it was quite good even for single-particle exchange.[9] If (11) is used in (8), we find that, in the absence of any $q_i{}^2$ dependence in $K(\tau)$, the poles of $V(\alpha,s,p)$ are given by the solutions of

$$1=\frac{2G^2}{\sqrt{\pi}}\frac{1}{\Gamma(\alpha+\frac{3}{2})K_\alpha(0)}\int_{-\infty}^{\infty}dk_0 \times\int_0^\infty\frac{k^{2\alpha+2}dk[K_\alpha(k^2+k_0{}^2)]^2}{[(k_0+iE)^2+k^2+\mu^2][(k_0-iE)^2+k^2+\mu^2]}, \tag{12}$$

where we have taken advantage of the fact that the angular part of the integral in $2\alpha+3$ dimensions can be performed to give a factor[21]

$$\int d\Omega=\frac{2\pi^{\alpha+3/2}}{\Gamma(\alpha+\frac{3}{2})}.$$

Solution of this deceptively simple transcendental equation yields Regge trajectories which can be continued above threshold. As discussed in I, (12) is to be solved by varying α to obtain the desired value of G^2. In Appendix A, we treat the general problem of determining higher-order corrections to the first-rank approximation. If (11) is replaced by the second-rank approximation developed in Appendix A, the trajectories are the solutions of

$$\det(1-D)=0\,. \tag{13}$$

The matrix D is given by

$$D=\begin{pmatrix}[K_{\alpha+2}(0)I_{00}{}^0-K_{\alpha+1}(0)I_{01}{}^0]/H & [K_\alpha(0)I_{01}{}^0-K_{\alpha+1}(0)I_{00}{}^0]/H & 2I_{01}{}^1/K_{\alpha+1}(0)\\ [K_{\alpha+2}(0)I_{01}{}^0-K_{\alpha+1}(0)I_{11}{}^0]/H & [K_\alpha(0)I_{11}{}^0-K_{\alpha+1}(0)I_{01}{}^0]/H & 2I_{11}{}^1/K_{\alpha+1}(0)\\ [K_{\alpha+2}(0)I_{01}{}^1-K_{\alpha+1}(0)I_{11}{}^1]/H & [K_\alpha(0)I_{11}{}^1-K_{\alpha+1}(0)I_{01}{}^1]/H & 2I_{11}{}^2/K_{\alpha+1}(0)\end{pmatrix}, \tag{14}$$

[22] M. M. Lévy, Phys. Rev. 98, 1470 (1955).

where $H=K_\alpha(0)K_{\alpha+2}(0)-[K_{\alpha+1}(0)]^2$ and

$$I_{nm}{}^j=\frac{2G^2}{\sqrt{\pi}}\frac{1}{\Gamma(\alpha+\frac{3}{2})}\int_{-\infty}^{\infty}k_0{}^j dk_0 \times\int_0^\infty \frac{k^{2\alpha+2}dk\, K_{\alpha+n}(k^2+k_0{}^2)K_{\alpha+m}(k^2+k_0{}^2)}{[(k_0+iE)^2+k^2+\mu^2][(k_0-iE)^2+k^2+\nu^2]}.$$

If the scattering particles have equal mass, then the third row and column of D are zero. For a given α there will be three values of G^2 which satisfy (13). Higher-order approximations will have more solutions. These extra solutions are just the secondary trajectories in the theory. A full discussion of them is reserved for a future publication.

In general, we use (12), but we compare the solutions of (12) with those of (13) in one case to check the validity of the approximation. Although (12) is a simple transcendental equation, it must be attacked numerically. As it stands it contains a double integral. A search for the correct value of α, given the desired G^2, particularly above threshold where both Reα and Imα must be varied, requires that the integral must be performed a large number of times in order to map out a complete trajectory. Thus, we make still another approximation in (12). Restricting ourselves to equal masses, we write the integral over k_0 in the form

$$\int_{-\infty}^{\infty}\frac{dk_0[G_\alpha(k,k_0)-G_\alpha(k,0)]}{[k_0{}^2+(E+E_k)^2][(k_0{}^2+(E-E_k)^2]} +G_\alpha(k,0)\frac{\pi}{2E_k}\frac{1}{k^2-q^2}, \quad (15)$$

where $G(k,k_0)$ contains the other factors in the integrand of (12), $q^2=E^2-\mu^2$, and $E_k{}^2=k^2+\mu^2$. We drop the integral in (15). This is, of course, just the Blankenbecler-Sugar[23] approximation. The two-particle threshold effects are contained in the second term. Since our procedure of unitarizing the input kernel imposes elastic unitarity only, it is consistent to neglect the higher threshold effects which are removed by this approximation. As a numerical approximation, the Blankenbecler-Sugar approximation[23] turns out to be quite adequate for our purposes. In Fig. 1 below we compare the solutions of

$$1=\frac{(\sqrt{\pi})G^2}{\Gamma(\alpha+\frac{3}{2})K_\alpha(0)}\int_0^\infty\frac{k^{2\alpha+2}dk[K_\alpha(k^2)]^2}{[k^2+\mu^2]^{1/2}(k^2-q^2)} \quad (16)$$

with those of (12) for an exponentially damped kernel. The agreement between the two solutions is qualitative below threshold but becomes quantitatively excellent above threshold. The integral in (15) proves to be small compared to the second term. We also present in Fig. 1 solutions of (13) in the Blankenbecler-Sugar approximation. Given that all three solutions have $\alpha=-\infty$ in the weak-coupling limit and the $s\to-\infty$ limit, the agreement among the solutions is satisfactory and gives us confidence in our method.

The analytic continuation of any of these equations above threshold is straightforward. Since the two-particle threshold is contained in the second term of (15), the analytic continuation of (16) can be used for (12) also. Wherever it occurs, q^2 has a $+i\epsilon$ attached to it. Thus, above threshold we have the standard result

$$\frac{1}{k^2-q^2-i\epsilon}=\frac{P}{k^2-q^2}+i\pi\delta(k^2-q^2),$$

where P denotes a principal-value integral which is easily carried out numerically. In Secs. III and IV we use (16) to investigate a wide variety of kernels which bear on the problem of rising and falling trajectories.

III. FALLING TRAJECTORIES AND ENERGY-INDEPENDENT INTERACTIONS

An infinitely falling trajectory is one which has $\alpha(s=-\infty)=-\infty$. Any kernel $K(\tau)$ which vanishes faster than any power as $\tau\to\infty$ will produce such a falling trajectory, provided that $K(\tau)$ does not depend strongly on s. The connection between the large-τ behavior of $K(\tau)$ and the asymptotic position of the leading Regge pole can be obtained either from the full scattering amplitude $A(s,\tau)$ or from the integral equation for $V(\alpha,s,p)$. First, in the limit $s\to-\infty$, $A(s,\tau)$, which is equal to an infinite sum of generalized ladder diagrams, becomes equal to the first Born approximation, provided that

$$\lim_{s\to-\infty}K(s,\tau,q_i{}^2)<Ms^{1-\epsilon},$$

where $\epsilon>0$ and M is independent of s. If $K(\tau)$ depends on the square of the direct four-momenta $q_i{}^2$, we assume the dependence is such that $K(\tau)$ vanishes in the limit $q_i{}^2\to+\infty$ or, in other words, when the particles are far off the mass shell ($q_i{}^2=-\mu^2$). The result for the asymptotic form of $A(s,\tau)$ is established, among other ways, by applying the d-line analysis of Halliday[24] to the generalized ladder diagrams. If $K(\tau)$ is independent of s, a ladder diagram with N rungs vanishes like s^{-N+1}; each two-particle intermediate state contributes a factor of s^{-1} to the asymptotic behavior. Thus, if $A(s,\tau)$ is replaced by the first Born approximation, we find, in the limit $s\to-\infty$,

$$A(s,\tau)=g_L(-\mu^2,-\mu^2)g_R(-\mu^2,-\mu^2)\tau^{-L-1}, \quad (17)$$

where $K(\tau)\to(\tau)^{-1-L}g_Lg_R$. On the other hand, an alternate representation of $A(s,\tau)$ is in terms of the

[23] R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).

[24] I. B. Halliday, Nuovo Cimento **30**, 177 (1963); G. Tiktopoulos, Phys. Rev. **131**, 480 (1963).

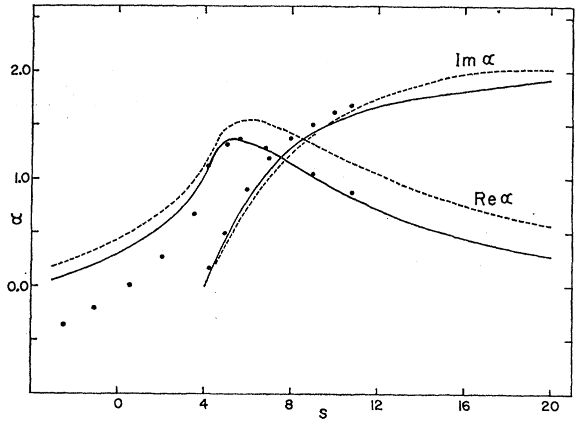

FIG. 1. Real and imaginary parts of the trajectories plotted as a function of s. The full curve corresponds to the solution obtained from the first-rank approximation to the kernel (21), single-particle exchange with a modified-Bessel-function form factor. It has been calculated with the Blankenbecler-Sugar approximation (16) and normalized so that $\alpha(-4)=0.0$. The dashed curve represents the trajectory obtained by using the second-rank approximation to the kernel with the same coupling constant. The isolated points are calculated from the full double integral (12). Threshold is at $s=4$.

leading Regge pole:

$$\lim_{s\to-\infty,\,\tau\to\infty} = \beta(-\infty)\tau^{\alpha(-\infty)}. \tag{18}$$

A comparison of (17) and (18) shows that $\alpha(-\infty) = -L-1$. Hence, for $L\to\infty$, $\alpha(-\infty)=-\infty$. The leading pole has a constant residue if $g_L g_R$ is independent of s.

The alternative argument for the relationship between $\alpha(-\infty)$ and the asymptotic behavior of $K(\tau)$ uses the integral equation (8). If $K(\tau)$ vanishes faster than any power of τ, then the integral representation of $K_\alpha(\tau)$ converges for all $\mathrm{Re}\alpha<0$, and $K_\alpha(\tau)$ has no α singularities in that region. If Fredholm theory is applied to (8), the poles of $V(\alpha,s,p)$ are zeros of the Fredholm denominator. In the limit $s\to-\infty$ for finite α, the Fredholm denominator approaches its first term for the same reason that $A(s,\tau)$ approaches the first Born approximation. The propagators for the two-particle intermediate state appears N times in the Nth term of the Fredholm denominator function. Thus, in the limit $s\to-\infty$, the poles of $V(\alpha,s,p)$ are given by the solutions of

$$1=\frac{(\sqrt{\pi})G^2}{\Gamma(\alpha+\frac{3}{2})}K_\alpha(0)\int_0^\infty \frac{k^{2\alpha+2}dk}{(k^2+\mu^2)^{1/2}(k^2-q^2)}. \tag{18'}$$

Equation (18′) resembles (16), but is not useful for $\alpha>0$ since it diverges at $\alpha=0$. The integral in (18′) converges for negative values of α and $s=4E^2<4\mu^2$. The apparent singularity at $\alpha=-\frac{3}{2}$ is canceled by $[\Gamma(\alpha+\frac{3}{2})]^{-1}$. In the limit $s\to-\infty$, the integral vanishes unless α approaches a singularity, which in this case is at $|\alpha|=\infty$. Thus, (18) is satisfied only if the trajectory is infinitely falling as $s\to-\infty$. Note that $K_\alpha(\tau)=\Gamma(\alpha+1)/(\lambda^2+\tau)^{\alpha+1}$ for single-particle exchange, and (18) is satisfied if $\alpha\to-1$ as $s\to-\infty$.

Having established the mathematical condition for infinitely falling trajectories within the framework of our model, we ask the physical question of why $K(\tau)$ should vanish faster than any power of τ. Clearly the exchange of an elementary particle by elementary particles in a conventional field theory does not lead to this behavior. On the other hand, strongly interacting particles are believed to be composite—infinitely composite in fact. The structure of particles is reflected in their form factors.[16] Hence, one method of taking into account the composite nature of particles is to use a phenomenological field theory in which Feynman diagrams describe scattering processes, but each vertex has a form factor associated with each off-mass-shell line. Presumably these form factors represent a large class of vertex corrections to elementary-particle exchange and, therefore, should be used carefully to avoid serious double counting problems. The simplest model of two-particle scattering in such a phenomenological field theory is patterned on existing field-theory models of Regge poles but incorporates the structure of the scattering particles. It replaces the standard single-particle-exchange interaction by

$$K(\tau)=F(\tau)^2/(\lambda^2+\tau). \tag{19}$$

We do not include form factors on the direct as well as exchange lines, since they would constitute an energy-dependent modification of kernel. We discuss them in Sec. IV.

At least one model[16] of infinitely composite particles leads to form factors which have $F(\tau)\sim e^{-\gamma\tau^\beta}$, $\beta\leq\frac{1}{2}$, as $\tau\to\infty$. The bound on β comes from the theory and is consistent with the bound on form factors established by Jaffe from field theory and by Martin from the requirement that certain dispersion relations exist.[17] All such form factors lead to infinitely falling trajectories, or at least trajectories which have $\alpha(-\infty)=-\infty$. The question remains as to whether trajectories which start at $\alpha=-\infty$ can produce bound states or resonances with $\mathrm{Re}\alpha>0$ for positive values of s. In Fig. 1 we display trajectories generated by the form factor

$$F(\tau)=[(\lambda^2+\tau)^{1/2}\mathcal{K}_1(\gamma(\lambda^2+\tau)^{1/2})]^{1/2}, \tag{20}$$

where $\mathcal{K}_\nu(z)$ is a modified Bessel function of order ν. This form factor has a branch point at $\tau=-\lambda^2$, but is respectable in the sense that it has appeared in the literature.[25] Moreover, it enables us to carry out the integral in (9). When we use (20) as a direct form factor, we choose λ^2 to correspond to a two-particle

[25] M. M. Islam, Nuovo Cimento **48**, 251 (1967).

branch point. The $K_\alpha(\tau)$ corresponding to (20) is[26]

$$K_\alpha(\tau)=(\tfrac{1}{2}\gamma)^\alpha \frac{\mathcal{K}_{\alpha+1}(\gamma(\lambda^2+\tau)^{1/2})}{(\lambda^2+\tau)^{(\alpha+1)/2}}. \tag{21}$$

The direct form factors are suppressed since they are unaffected by the integral transform. The curves in Fig. 1 are solutions of (16). To test the approximations involved in deriving (16), we also present solutions of (13), the second-rank approximation to the kernel, and (12), the full double integral. The trajectories do reach $\alpha=0$, but they also turn over above threshold. It is apparent that a trajectory which has $\alpha(-\infty)=-\infty$ need not have $\mathrm{Re}\alpha(+\infty)=+\infty$.

Having established that exponentially damped, energy-dependent kernels yield infinitely falling trajectories, we next explore whether it is possible to construct basic interactions that generate rising trajectories. We relax the requirement that the interaction have a single-particle pole in τ, since it is not obvious that single-particle exchange is the dominant interaction in the real world. Moreover, if the particle pole is far from the region of integration, it should be unimportant. In effect, what we do is to search for an interaction that leads to rising trajectories, leaving the physical interpretation of the results until we are successful. Thus, we try next a kernel of the form (19) with

$$F(\tau)=\exp[-\tfrac{1}{2}\gamma(\lambda^2+\tau)^\beta](\lambda^2+\tau)^{\delta/2}. \tag{22}$$

If $\delta=\frac{1}{4}$ and $\beta=\frac{1}{2}$, this form is (20), in the limit as τ becomes large. The transform necessary to calculate $K_\alpha(\tau)$ cannot be performed analytically for arbitrary δ and β. Rather we approximate $K_\alpha(\tau)$ by its asymptotic value

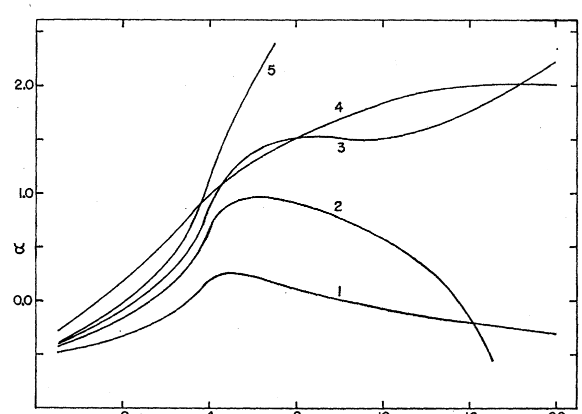

FIG. 2. Real parts of the trajectories obtained from the kernel (23), single-particle exchange with a $z^{\delta/2}e^{-\gamma z^\beta/2}$ form factor, plotted as a function of s. The curves labeled 1, 2, 3, and 5 have $\delta=0$ and $\beta=0.25$, 0.75, 1.0, and 1.3, respectively. Curve 4 has $\delta=1$ and $\beta=0.75$. Each curve is normalized to $\alpha(-4)=-0.5$. Curves with $\beta<1$ turn over. Curve 4 drops rapidly in the region $s>20$. Again threshold is at $s=4$.

[26] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 2, Chap. 7, p. 95.

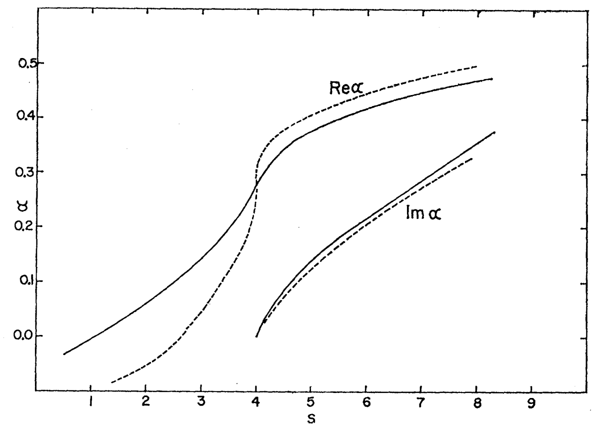

FIG. 3. Full curve is the Blankenbecler-Sugar solution for the trajectory as a function of s with the kernel $K_\alpha(\tau)=\gamma^\alpha e^{-\gamma\tau}$. The dashed curve is obtained from the double integral (12) with the same coupling constant and same value of $\gamma=0.1$. The vertical scale has been expanded relative to the previous figures to display better the difference between the curves. Threshold is at $s=4$.

$$K_\alpha(\tau)=\frac{e^{-\gamma z^\beta}(\beta\gamma z^\beta)^\alpha}{z^{\alpha+1-\delta}} \times\left(1+\frac{\alpha[\alpha(1-\beta)+1-\delta]}{\beta\gamma z^\beta}+\cdots\right), \tag{23}$$

which is valid when $z=\lambda^2+\tau\gg1$ and α is small. If (23) leads to a rising trajectory, the small α approximation breaks down and it becomes necessary to use the exact expression. If the trajectory turns over or remains constant with small α, the approximation should be valid.

In Fig. 2 are shown trajectories for several values of β and δ. The trajectories turn over unless $\beta\geq1$. Varying δ does not effect this conclusion. Hence, it is possible to generate rising trajectories with an energy-independent kernel. However, interpreted as a form factor, the kernels violate the Jaffe bound.[17]

To check our approximations and conclusions, we calculate a trajectory with the simple interaction

$$K(\tau)=e^{-\gamma\tau}, \tag{24}$$

which has the exact transform

$$K_\alpha(\tau)=\gamma^\alpha e^{-\gamma\tau}. \tag{25}$$

In Fig. 3 we show trajectories obtained from (25) in both the Blankenbecler-Sugar approximation[23] (16) and the full double integral (12). Indeed, the trajectory rises and the agreement between the two calculations is good. Moreover, by varying γ and G^2, we can adjust the slope and $s=0$ intercept of the trajectory over a wide range. If $K(\tau)$ in (24) is modified by the factor $(\lambda^2+\tau)^{-1}$ so that $e^{-\gamma\tau/2}$ becomes a form factor, $K_\alpha(\tau)$ is proportional to an incomplete gamma function,[27]

$$K_\alpha(\tau)=\Gamma(1+\alpha,\gamma(\lambda^2+\tau))/(\lambda^2+\tau)^{\alpha+1}. \tag{26}$$

[27] Ref. 26, Vol. 2, Chap. 9, p. 137.

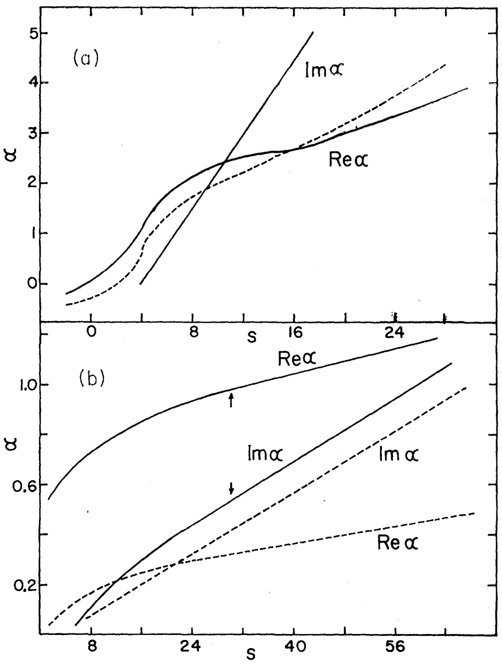

FIG. 4. (a) Full curves are the real and imaginary parts of the trajectory with the incomplete gamma-function kernel (26). The exchanged mass $\lambda=2$, $\gamma=1$, and the coupling is such that $\alpha(-4)=-0.5$. The dashed curve is $\mathrm{Re}\alpha$ resulting from the exponential kernel (25) with $\gamma=1$ and a coupling chosen to make the trajectories coincide at $s=-4$. The curves for $\mathrm{Im}\alpha$ are indistinguishable. The particle-exchange pole has little effect in the asymptotic region. (b) Real and imaginary parts of the trajectories calculated from the full exponential kernel (25) plotted as a function of s. The full curve is normalized to $\alpha(0)=0.5$ and has $\gamma=0.04$ to make $\mathrm{Re}\alpha(30)=1.0$. In units where $m_\pi{}^2=1$, $m_\rho{}^2=30$. This fictitious ρ trajectory yields a width

$$\Gamma=2(\mathrm{Im}\alpha)(d\ \mathrm{Re}\alpha/ds)^{-1}m_\rho{}^{-1}=2.6\ \mathrm{BeV}.$$

The arrow marks the position of the ρ. The dashed curve is a plot of $\frac{1}{5}\alpha$ for the same kernel, but with $\alpha(0)=0.0$ and $\gamma=0.2$. For large s, both the real and imaginary parts of the full and dashed curves are parallel, verifying the asymptotic solution for the trajectory which predicts a slope proportional to γ.

One of the trajectories in Fig. 4(a) is calculated with this kernel. The asymptotic behavior of the trajectories is unmodified. If the exchanged mass is small, a dip is introduced into the real part of the trajectory before the asymptotic rise begins.

The functional form of $K_\alpha(\tau)$ in (25) is sufficiently simple to enable us to analyze the asymptotic behavior of the trajectory analytically. As α and s become large along a trajectory, both (12) and (16) become equal to

$$1=\frac{G^2\gamma^\alpha}{2\Gamma(\alpha+\frac{3}{2})}\frac{\pi^{3/2}(q^2)^{\alpha+1/2}}{(\mu^2+q^2)^{1/2}}e^{-2\gamma q^2}(i+\tan\pi\alpha). \qquad (27)$$

We discovered this asymptotic limit numerically, but presumably it can be established analytically. When α and s both become large, (27) takes on the form

$$1=\frac{i\pi G^2}{\sqrt{2}\alpha}\left(\frac{s}{\alpha}\right)^\alpha e^{\alpha[1+\ln(\gamma/4)]-\gamma s/2}. \qquad (28)$$

If we let $\alpha=Re^{i\theta}$ and $R=as^m$, we find immediately that $m=1$. If $m\neq1$ the factor $(s/\alpha)^\alpha$ prevents the right-hand side of (28) from being constant. In addition, we require that the exponential factors in (28) have constant magnitude and phase. As a consequence both a and θ are determined by the equations

$$\cos\theta[1+\ln(\gamma/4a)]+\theta\sin\theta=\gamma/2a\,, \qquad (29a)$$

$$1+\ln(\gamma/4a)=\theta\cot\theta\,, \qquad (29b)$$

where (29a) comes from the constant magnitude requirement and (29b) comes from the constant phase requirement. When (29) is solved, we find that $\theta=1.13$ and $a=0.40\gamma$, so that the Regge trajectory has the asymptotic form

$$\alpha=0.40\gamma s(0.43+i0.90)\,, \qquad (30)$$

where γ enters through (25). This asymptotic result is verified by actual computation of the trajectory [see Fig. 4(b)]. Since $\theta>\frac{1}{4}\pi$, (30) is not a narrow-width trajectory. However, it is possible to make the ratio $\mathrm{Im}\alpha/\mathrm{Re}\alpha$ take on almost any value in the region just above threshold by varying G^2 and γ. One of the trajectories in Fig. 4(b) has $\alpha(0)=\frac{1}{2}$ and $\mathrm{Re}\alpha(30)=1$ in an attempt to match the ρ trajectory in a theory where $s=4$ is the two-pion threshold. All such trajectories take on the asymptotic form (30). Moreover, if the asymptotic dominance of the on-mass-shell term in (12) depends on the exponential nature of $K_\alpha(\tau)$ and is not affected either by direct form factors or by multiplicative factors of τ, then (29) holds for all kernels proportional to $e^{-\gamma\tau}$, since form factors on the mass shell are constants and (29) was independent of any power behavior in s. Thus, (30) appears to be a universal asymptotic trajectory for a large class of interactions. The linear s dependence of the trajectory is a result of the calculation and is not assumed as input.

We have calculated trajectories with a superposition of kernels containing different τ dependence. It is obvious that if pure single-particle exchange is added to any of the exponentially damped kernels discussed here, the trajectory will start at $\alpha=-1$, rather than $\alpha=-\infty$. We also find that if we add (21) and (25), the behavior of the trajectory is characteristic of (21) even in the region above threshold. We conclude, therefore, that trajectories which rise smoothly from $-\infty$ to $+\infty$ as s increases are possible for energy-independent interactions if *all* terms are proportional to $e^{-\gamma\tau^\beta}$ with $\beta\geq1$. Moreover, if $\beta=1$, the trajectory is asymptotically linear with a slope and direction which are independent of the detailed τ dependence of the interaction.

IV. ENERGY-DEPENDENT INTERACTIONS AND SUPERPOSITION MODELS

In Sec. III we considered interaction kernels which were energy independent, except in so far as direct form factors introduce an energy dependence. Although it is possible to obtain rising trajectories in such theories, the trajectories are not of narrow width in the sense that $\mathrm{Im}\alpha$ and $\mathrm{Im}\alpha/(d\,\mathrm{Re}\alpha/ds)$ are small. Previous work on rising trajectories in potential theory has involved energy-dependent potentials.[11,12] The strength of the coupling increases with energy in order to overcome the effect of the angular momentum barrier. Within the framework of our model, we can treat the relativistic equivalent of energy-dependent potentials. However, we have to be careful in interpreting our results. In I we found that our basic method of calculating Regge trajectories was qualitatively correct for strong coupling, but the quantitative error increased with the coupling strength.

An example of a physically motivated, energy-dependent interaction is Regge-pole exchange,

$$K(\tau)=\beta(\tau)(s/s_0)^{\tilde{\alpha}(\tau)}. \qquad (31)$$

In the absence of a signature factor, (31) represents an exchange-degenerate pair of Regge poles.[28] Since we use (31) in the region $t<0$ or $\tau=-t>0$, we suppress the poles that occur when $\tilde{\alpha}(\tau)$ crosses a positive integer. If we make the conventional choice $\tilde{\alpha}(\tau)=\alpha_0-\alpha'\tau$, $\beta(\tau)=G^2e^{-\gamma'\tau}$, and $s_0=(\alpha')^{-1}$,[29] then (31) takes on the form of (24), and $K_\alpha(\tau)$ is given by (25), with an energy-dependent $\gamma=\gamma'+\alpha'\ln(s/s_0)$. In Fig. 5 are shown the rising trajectories generated by this simple model of Regge-pole exchange. Clearly (31) is valid only for s substantially above threshold and cannot be used to calculate trajectories which rise smoothly from $s=-\infty$. In fact, this is a general difficulty with energy-dependent potentials. They describe the dominant part of the dynamics at high energy, but some other mechanism is more important at lower energies.

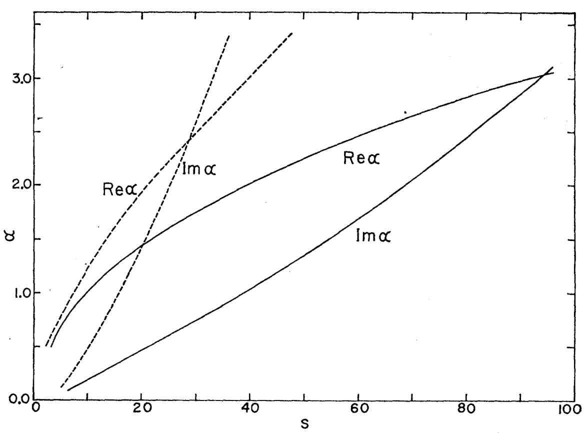

FIG. 5. Trajectories calculated from Regge-pole exchange (31). Both sets of curves are rising, but are not of narrow width. The solid curve has $\gamma'=0.1$, $\alpha_0=0.5$, and $\alpha'=0.02$ in order to match the exchanged trajectory to the observed ρ trajectory. The dashed curve is calculated from an exchanged trajectory with the same intercept but $\alpha'=0.1$ and the residue parameter $\gamma'=0.2$.

[28] V. Barger, Rev. Mod. Phys. **40**, 129 (1968).

[29] R. J. N. Phillips and W. Rarita, Phys. Rev. Letters **15**, 807 (1965).

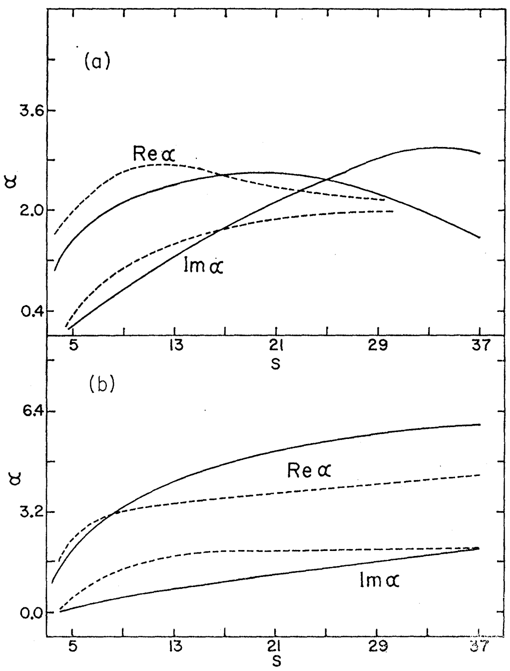

FIG. 6. Trajectories calculated from energy-dependent coupling constants shown as a function of s. In (a) the coupling constants are proportional to s, the maximum power that is allowed by the Froissart bound. Both the solid curve calculated from the β-dependent kernel (23) ($\beta=0.75$) and the dashed curve calculated from the Bessel-function kernel (21) turn over. In (b) the coupling constant is proportional to s^3; otherwise all parameters are the same. The trajectories are now rising.

Given the close relation between (31) and (25), it is not surprising that the trajectories are rising. More interesting is the observation that the analysis on the asymptotic form of the trajectory function can be repeated for (31). The on-mass-shell, δ-function part of the integral dominates the integral in (12) as $s\to\infty$. The result is that $\alpha\sim Re^{i\theta}$, where $R=as\ln s$, and θ and a are determined by (29). In other words, the asymptotic trajectory is not of narrow width for Regge-pole exchange, and it is not a linear function of s. This result is independent of the choice of α_0.

Proceeding in the spirit of Sec. III, we investigate the effect of allowing the coupling constant in the energy-independent models to grow as a power of the energy. The physical basis of such an assumption is obscure compared to that of choosing Regge-pole exchange, but it might be a way of incorporating spin effects. Thus, we multiply (22) by s^η and calculate trajectories in the large-s limit. The results are shown in Fig. 6. $K_\alpha(\tau)$ is given by (19) with either (20) or (22) for $F(\tau)$. For η large enough, the trajectories do appear to rise. However, these large values of $\eta>1$ constitute a violation of the Froissart bound.[20] If the coupling constant is proportional to s^η, the Nth generalized ladder diagram approximation to the full scattering amplitude is proportional to $s^{N\eta+1-N}$ up to powers of

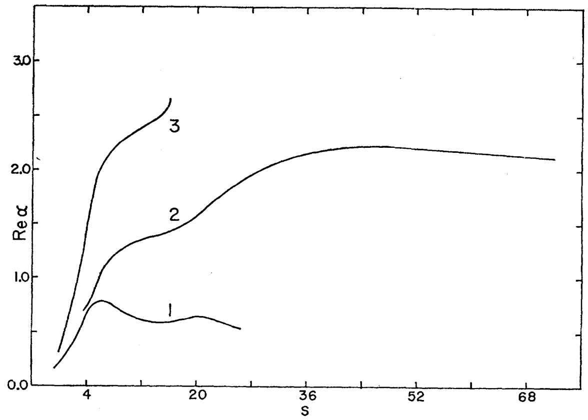

FIG. 7. Effect of including direct form factors illustrated in three cases where the kernel is single-particle exchange with form factors. The form-factor singularities appear at $s=16$, where the elastic threshold is at $s=4$. Curves 1 and 3 combine the kernel (23) with the form factor (22). Both have $\lambda=2$ and $\delta=0$, while 1 has $\beta=0.25$ and 3 has $\beta=0.75$. Curve 2 combines the modified-Bessel-function form factor (20) with the kernel (21) ($\lambda=1$). $\gamma=1.0$ for all curves. If the form factors obey the Jaffe bound (1 and 2), the real part of the trajectory turns over as a function of s. The real part of 3 is rising; and although it is now shown, the imaginary part does not rise as fast as the real part. If the difficulties with the Froissart bound were overcome, this trajectory would be a candidate for a narrow-width rising trajectory.

$\ln s$. If $\eta>1$, $A(s,\tau)$ is unbounded term by term as $s\to\infty$, and the Froissart bound appears to be grossly violated.[30] Hence, our result says that rising trajectories are not possible with energy-dependent coupling constants unless the full scattering amplitude generated by this same dynamical interaction violates the Froissart bound.

Although we have used form factors for particle exchange, we have yet to discuss direct form factors. If we write $K_\alpha(\tau)=K_\alpha'(\tau)F(q_1^2)F(q_2^2)F(q_3^2)F(q_4^2)$ in order to display the explicit dependence on the direct form factors, the Blankenbecler-Sugar approximation[23] (16) becomes

$$1=\frac{(\sqrt{\pi})G^2}{\Gamma(\alpha+\frac{3}{2})K_\alpha'(0)}\times\int_0^\infty\frac{k^{2\alpha+2}dk[K_\alpha'(k^2)\,|F(k^2-E^2)\,|^2]^2}{(k^2+\mu^2)^{1/2}(k^2-q^2)},\quad(16')$$

where $s=4E^2$ and the absolute square of the form factor, $|F(k^2-E^2)|^2$, indicates how we continue past the singularities of the form factor. We always choose the direct form factors $F(q_i^2)$ to have the same functional form as the exchange form factor $F(\tau)$. A direct form factor introduces a strong energy dependence, but a dependence which is smooth in the sense that it is well defined for all s, unlike the previous cases. In Fig. 7, we see the effect of introducing direct form factors. In the previous section, we found that form factors proportional to $e^{-\gamma\tau^\beta/2}$, with $\beta<1$, lead to trajectories that turn over. If $\beta<\frac{1}{2}$, the direct form factors do not alter this conclusion. On the other hand, if $\beta>\frac{1}{2}$, the introduction of direct form factors leads to rising trajectories; however, the Jaffe bound is violated.[17] These results are not surprising when we consider that when E^2 is such that the integral in (16′) overlaps a portion of the form-factor cut, $z^\beta=(-z)^\beta(\cos\pi\beta+i\sin\pi\beta)$, $z=k^2-E^2+\Lambda^2$. We adjust Λ so that the branch point $z=0$, $E^2=k^2+\Lambda^2$, occurs at an inelastic threshold. If $\beta<\frac{1}{2}$, then $\cos\pi\beta>0$, and the energy dependence from $e^{-\gamma z^\beta}$ is exponentially damped[31]; we do not expect the direct form factor to produce rising trajectories. If $\beta>\frac{1}{2}$, we have the equivalent of an exponentially growing coupling constant, and the trajectory rises. It even appears to be of narrow width. However, as pointed out above, the Froissart bound[20] will be violated in the limit $s\to+\infty$. If $\beta=\frac{1}{2}$, the trajectory turns over, although not as rapidly as if there were no direct form factor present. In every case, the direct form factor increases the slope of the trajectory below threshold as expected from the increased energy dependence. All the trajectories have $\alpha(-\infty)=-\infty$ in the presence of direct form factors if $K_\alpha(\tau)$ is exponentially damped.

Another possible approach to generating rising trajectories is through the use of coupled channels. The effect of a two-particle threshold is such as to keep the trajectory rising until s reaches threshold. Thus a superposition of suitably chosen thresholds with appropriate coupling constants might force the trajectories to continue rising. Within the framework of the model used here, the particles in the intermediate states must have zero spin. A model of this type was briefly investigated in I for elementary-particle exchange with two or three thresholds, and the results were not encouraging. Multiple thresholds enter the model as a sum over propagators of the intermediate states in the integral equation in (8). If this effect is to lead to infinitely rising trajectories, there must be an infinite number of terms in the sum. Since the spectrum of intermediate states is arbitrary, we make the smoothest possible assumption and replace the infinite sum by an integral over a continuous distribution of two-particle, equal-mass states. In the Blankenbecler-Sugar approximation[23] (16), this involves the replacement of the propagator factor

$$P(k^2)=\frac{G^2}{(k^2+\mu^2)^{1/2}(k^2-\frac{1}{4}s+\mu^2)}$$

[30] These statements on the violation of the Froissart bound are based on a term-by-term analysis of the series representation of the scattering amplitude $A(s,\tau)$ in the limit $s\to\infty$. A careful analysis of ϕ^3 field-theory ladder diagrams shows that the Nth term in the series has the form $(-s)^{-N}f_N(\tau)$, $f_N(\tau)>0$. Since the terms alternate in sign, it is possible that the series sums to a bounded function. However, the dependence of $f_N(\tau)$ on N and τ is very complicated, and we consider it very unlikely that the sum of the series is bounded. It is a point that merits a more detailed investigation.

[31] The argument here is a simplified version of that given by Martin in Ref. 17.

by

$$P(k^2)=\int_{\mu_0^2}^{\infty}\frac{G^2(\mu^2)d\mu^2}{(k^2+\mu^2)^{1/2}(k^2-\frac{1}{4}s+\mu^2)},\qquad(32)$$

where $G^2(\mu^2)$ is the coupling to the intermediate state of particles with mass μ. The lowest threshold is at $s=4\mu_0^2$. If the integral in (32) is to converge, there is a limit to the rate at which the coupling $G^2(\mu^2)$ can increase with mass. We make the simplest possible choice, $G^2(\mu^2)=G^2/\mu_0^2$, so that we can perform the integral to obtain

$$P(k^2)=\frac{2G^2}{\mu_0^2\sqrt{s}}\left[\ln\left(\left|\frac{2(k^2+\mu_0^2)^{1/2}+\sqrt{s}}{2(k^2+\mu_0^2)^{1/2}-\sqrt{s}}\right|\right)+i\pi\theta[s-4(k^2+\mu_0^2)]\right].\qquad(33)$$

Then using first (21) and then (25) for $K_\alpha(\tau)$, we obtain the trajectories in Fig. 8. The trajectories do not rise, even when $K_\alpha(\tau)$ is given by (25). One noticeable effect, however, is that the rapid threshold rise of $\mathrm{Im}\alpha$ as a function of s seen in Figs. 1–3 is replaced by a more gradual rise. This is a result of the fact that the imaginary part of (33) above threshold is no longer a δ function. The discontinuity in s above threshold involves an integral over a finite region. The effect on the slope of $\mathrm{Im}\alpha$ at $s=0$ is analogous to what would be expected at a three-particle threshold. The absence of the δ-function discontinuity also explains why $K(\tau)=e^{-\gamma\tau}$ does not produce a rising trajectory. The right-hand side of (27) becomes an integral with a corresponding weaker dependence on s. Thus, subject to the constraint that the superposition converges, an infinite set of two-particle spinless thresholds does not lead to rising trajectories for any interaction.

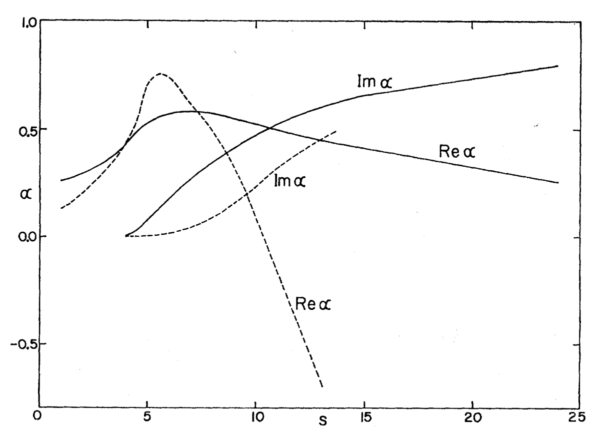

FIG. 8. Trajectories from the model of a continuous superposition of equal-mass intermediate states plotted as a function of s. The full curve uses (21) for the kernel, while the dashed curve is calculated with (25). Although the trajectories turn over the imaginary part of the trajectory function has zero slope at $s=4$, the lowest threshold, in contrast to the trajectories in Figs. 1 and 3.

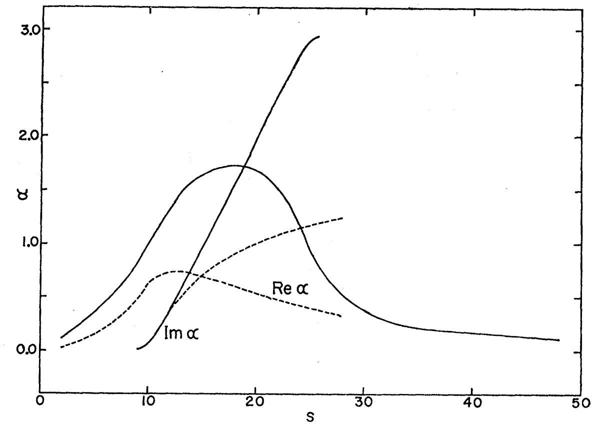

FIG. 9. Trajectories calculated from the model of three-particle intermediate states shown as a function of s. Threshold is at $s=9$. The full curve is generated by the pure exponential kernel (25) with $\gamma=1.0$ and $\alpha(0)=0.0$. The dashed curve has the Bessel-function kernel (21) with $\gamma=1.0$, $\lambda=1$, and $\alpha(1.5)=0.0$. The trajectories turn over, but the imaginary parts start with zero slope at threshold.

Finally we investigate a model of a three-particle threshold treated as a two-particle threshold where one of the particles has variable mass. The effective propagator in this case is given in Appendix B, and the resulting trajectories appear in Fig. 9. Again they turn over, but they show the same decrease in the slope of $\mathrm{Im}\alpha$ that was seen in the continuous-threshold model. The ultimate model of a rising trajectory will undoubtedly require a much more careful treatment of three-particle states than the one developed here.

V. CONCLUSION

We have investigated the general problem of finding a fully relativistic dynamical model which leads to Regge trajectories which are infinitely falling, $\alpha=-\infty$ as $s\to-\infty$, and infinitely rising, $\mathrm{Re}\alpha=+\infty$ as $s\to+\infty$. We would like such trajectories, if they rise, to be of narrow width. Our approach has been to use a phenomenological field theory in which a basic interaction is unitarized by summing generalized ladder diagrams. We find that it is possible to construct models for infinitely falling trajectories that do not violate any of our strongly held beliefs about the nature of the world. In particular, there appears to be a very natural connection between the concept of infinitely composite particles and falling trajectories. The composite nature of particles suggests that standard single-particle exchange should be modified by form factors which are exponentially damped in the spacelike region.[16] In fact, any interaction which is an exponentially damped function of the momentum transfer leads to falling trajectories. Of course, the relevance of this analysis to the real world depends on a knowledge of the nature of other singularities in the complex angular momentum plane which could interact with the falling trajectories.

The problem of cuts in such theories obviously deserves consideration.

The strongest statement we can make about rising trajectories is that they probably do not exist; but if they do, something has to give. In other words, our results are consistent with the conclusion reached by Khuri[18] and modified by Jones and Teplitz.[19] If the interaction is independent of the energy, the trajectory will not rise unless the kernel, viewed as a form factor, violates the Jaffe bound[17] which states that the maximum rate at which a form factor can vanish is $e^{-\gamma\tau^\beta}$ with $\beta\leq\frac{1}{2}$. We find rising trajectories only if $\beta\geq 1$. The Jaffe bound is based on either field-theoretic arguments or the existence of dispersion relations with a finite number of subtractions.[17] It would not surprise us if this bound is violated in the presence of rising trajectories. Moreover, there is little direct experimental evidence on form factors for purely hadronic processes. However, it is worth mentioning that if we view our generalized ladder diagrams in the crossed channel physical region, $t\to\infty$, $s<0$, violation of the Jaffe bound constitutes violation of the Froissart bound.[20] Fortunately our integral equation does not probe this region, and it is conceivable that there are other contributions to a complete theory that could cancel the large-t divergence but which do not contribute to the binding of Regge poles.

Such a cancellation seems inconceivable in those theories which produce rising trajectories by introducing direct energy dependence. If the coupling constant is energy dependent, not only are there problems as $s\to-\infty$, but we find that the energy dependence necessary for rising trajectories is such that the sum of generalized ladder diagrams violates the Froissart bound.[20] This conclusion is consistent, qualitatively at least, with that reached by Trivedi[11] and by Tiktopolous[12] in potential theory. The energy dependence introduced by direct form factors leads to an even more gross violation of the Froissart bound. Since this bound on the scattering amplitude follows from more general considerations than the Jaffe bound[17] on form factors, and since there is relevant experimental information on hadronic cross sections, we do not consider seriously theories that lead to scattering amplitudes which increase faster than s^η with $\eta>1$. There is, of course, the possibility of cancellation against other contributions. However, our integral equation probes this kinematic region, and we prefer not to invoke unknown mechanisms for the necessary drastic cancellations.

All our statements in this conclusion are subject, of course, to the caveats made in the introduction about the type of theories we are willing and able to discuss.

Finally we note that the rising trajectories we do find, although not of narrow width, are asymptotically linear. A basic interaction of the form $K(\tau)=e^{-\gamma\tau}$ is hard to interpret physically, except perhaps as the zero spin equivalent to the exchange of a fixed Pomeranchuk trajectory in a theory with spin. (Spin in field-theory models has the effect of introducing powers of s and shifting singularities to the right in the complex angular momentum plane.[32]) On the other hand, the Regge-pole amplitude $\exp[-\gamma\tau-(\alpha'\tau-\alpha_0)\ln s/s_0]$ commonly appears in the literature[29] and presumably represents the sum of ladders in the t channel. In the $\tau\to-\infty$ limit this amplitude diverges. There must be terms which cancel the singularity. In any case this theory has at least a veneer of respectability. Since these theories are the only relativistic models of rising trajectories, it is worthwhile investigating them in more detail. In particular, is the currently popular concept of a sequence of parallel secondary trajectories supported by this model? Work is in progress on this question.

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APPENDIX A: CORRECTIONS TO SEPARABLE APPROXIMATIONS

The basic rationale behind the particular separable approximation used here for the kernel $K((p-k)^2)$ is that the approximation should become exact in the limit in which either p or k approach zero, yet retain the large-p or -k properties of the exact kernel necessary to make the integral converge.[22] The problem is to calculate higher-order corrections to the first-rank approximation in order to estimate the validity of the approximation. For simplicity, we work with a kernel in one dimension, and then generalize the results to the case where p and k are vectors in N dimensions. The correction to the first-rank approximation is defined by

$$\Delta(p,k)=K(p,k)-K(p,0)K(0,k)/K(0,0)\,, \quad \text{(A1)}$$

where $K(p,k)$ is an arbitrary, symmetric kernel. We denote the first approximation to $\Delta(p,k)$ by $\Delta^1(p,k)$. Clearly $\Delta^1(p,k)$ is a separable, symmetric function of p and k which vanishes as p and k separately go to zero. Hence, we write

$$\begin{aligned}\Delta^1(p,k)&=kpg(p)g(k)\\&=K(p,k)-K(p,0)K(0,k)/K(0,0)\,, \quad \text{(A2)}\end{aligned}$$

where $g(0)\neq 0$. If we expand both sides of (A2) in powers of k and equate terms of order k, we find

$$g(p)=\frac{1}{pg(0)}\left[\frac{\partial}{\partial k}K(p,k)-\frac{K(p,0)}{K(0,0)}\frac{\partial}{\partial k}K(0,k)\right]_{k=0}, \quad \text{(A3)}$$

[32] See Ref. 15, Chap. 3, p. 170.

and (A3) can, in turn, be used to evaluate $g(0)$:

$$g(0)^2=\left[\frac{\partial}{\partial p}\frac{\partial}{\partial k}K(p,k)-\frac{1}{K(0,0)}\frac{\partial}{\partial p}K(p,0)\frac{\partial}{\partial k}K(0,k)\right]_{k=0,p=0}. \quad \text{(A4)}$$

Together, (A3) and (A4) determine $\Delta^1(p,k)$. The procedure is easily generalized to evaluate

$$\Delta^n(p,k)=p^nk^ng_n(p)g_n(k).$$

The extension of this procedure to N-dimensional integral equations is straightforward. In particular, if the kernel is a function of the scalar quantity $(p-k)^2$, then (A2) is replaced by

$$\begin{aligned}\Delta^1(p,k)&=k\cdot pg(k^2)g(p^2)\\&=K((p-k)^2)-K(p^2)K(k^2)/K(0),\end{aligned} \quad \text{(A5)}$$

since $p\cdot k$ is the only scalar function of order p and k. Expanding both sides of (A5) to order k and equating coefficients, we obtain

$$g(p^2)g(0)=-2K'(p^2), \quad \text{(A6)}$$

and

$$\Delta^1(k,p)=-2k\cdot pK'(p^2)K'(k^2)/K'(0), \quad \text{(A7)}$$

where $K'(x)=dK(x)/dx$. We also need $\Delta^2(p,k)$ which must have the form

$$\Delta^2(k,p)=p^2k^2g(p^2)g(k^2)+(p\cdot k)^2f(p^2)f(k^2). \quad \text{(A8)}$$

Repeating our procedure, we find

$$\Delta^2(k,p)=\frac{[K'(p^2)K(0)-K(p^2)K'(0)][K'(k^2)K(0)-K(k^2)K'(0)]}{K(0)[K(0)K''(0)-K'(0)^2]}+2(p\cdot k)^2\frac{K''(p^2)K''(k^2)}{K''(0)}. \quad \text{(A9)}$$

The angular integrations involved in an N-dimensional integration suppress terms proportional to $(p\cdot k)^n$, so that $\Delta^1(p,k)$ and the part of $\Delta^2(p,k)$ proportional to p^2k^2, denoted by $\tilde{\Delta}^2(p,k)$, are of the same order when used to calculate Regge trajectories. Thus, we refer to $\Delta^1(p,k)+\tilde{\Delta}^2(p,k)$ as the first-order correction to the simple separable approximation. This prescription is consistent with that used in I to obtain the higher-order corrections for single-particle exchange and is equivalent to the statement that the order is identified with the highest order of the derivative of $K(p^2)$ that appears in a given term. To generate the second-order approximation to (12), we substitute

$$\begin{aligned}K_\alpha((p-k)^2)=&\frac{K_\alpha(p^2)K_\alpha(k^2)}{K_\alpha(0)}+2k\cdot p\frac{K_{\alpha+1}(p^2)K_{\alpha+1}(k^2)}{K_{\alpha+1}(0)}\\&+\frac{[K_{\alpha+1}(p^2)K_\alpha(0)-K_\alpha(p^2)K_{\alpha+1}(0)][K_{\alpha+1}(k^2)K_\alpha(0)-K_\alpha(k^2)K_{\alpha+1}(0)]}{K_\alpha(0)[K_\alpha(0)K_{\alpha+2}(0)-K_{\alpha+1}(0)^2]}\end{aligned} \quad \text{(A10)}$$

into the integral equation (8). After a moderate amount of algebra we find the poles of $V(\alpha,s,p)$ are given by (13) and (14). We have used (10) to relate $K_\alpha'(p^2)$ to $K_{\alpha+1}(p^2)$. Moreover, we have made use of the fact that all integrals are even functions of the $2\alpha+3$ coordinates of the vector k, so that only the k_0p_0 term in $k\cdot p$ survives. If the scattering particles have equal mass, the integrals are even in k_0 as well; and the matrix D reduces to 2×2. Finally we mention that the second-approximation terms which are proportional to p^2k^2 vanish for the pure exponential kernel (26).

APPENDIX B: THREE-PARTICLE INSERTION

A model of a three-particle intermediate state can be constructed if we consider that two of the three particles are correlated to form an effective variable mass system with zero spin. We apply the Blankenbecler-Sugar approximation[23] to a system with particles of mass μ and w, where w is continuous, and find

$$\begin{aligned}B(k^2,w)&=\int_{-\infty}^{\infty}\frac{dk_0}{[(k_0+iE)^2+k^2+\mu^2][(k_0-iE)^2+k^2+w^2]}\\&=\frac{\pi(E_\mu+E_w)}{E_\mu E_w[(E_\mu+E_w)^2-4E^2]},\end{aligned} \quad \text{(B1)}$$

where $E_\mu{}^2=k^2+\mu^2$ and $E_w{}^2=k^2+w^2$. If the system with effective mass w is composed of two particles, each of mass μ, then the three-particle approximation we consider is defined by the superposition

$$P_3(k^2)=\tfrac{1}{2}G^2\pi\int_{2\mu}^{\infty}\rho(w)(w^2-4\mu^2)^{1/2}B(k^2,w)dw, \quad \text{(B2)}$$

where the weight function $\rho(w)$ modulates the two-particle phase-space factor $\frac{1}{2}\pi(w^2-4\mu^2)^{1/2}$ and $P_3(k^2)$ is the propagator factor analogous to $P(k^2)$ defined in (32). If

$$\rho(w)=\mu^2/w(w^2-4\mu^2)^{1/2},$$

the integral becomes

$$P_3(k^2)=\tfrac{1}{4}G^2\pi^2\frac{\mu^2}{E_\mu}\int_{E_\mu+E_{2\mu}}^{\infty}\frac{EdE}{(E^2-s)[(E-E_\mu)^2-k^2]},\quad\text{(B3)}$$

where $s=4E^2$. This choice of $\rho(w)$ enables us to evaluate the integral and obtain

$$P_3(k^2)=\frac{G^2\pi^2\mu^2}{8E_\mu}\bigg[\frac{E_\mu-k}{(E_\mu-k)^2-s}\frac{1}{k}\ln(E_{2\mu}+k)-\frac{E_\mu+k}{(E_\mu+k)^2-s}\frac{1}{k}\ln(E_{2\mu}-k)-\frac{1}{(E_\mu-\sqrt{s})^2-k^2}$$
$$\times\ln(|E_\mu+E_{2\mu}-\sqrt{s}|)-\frac{1}{(E_\mu+\sqrt{s})^2-k^2}\ln(E_\mu+E_{2\mu}+\sqrt{s})+\frac{4i\pi\theta[s-(E_\mu+E_{2\mu})^2]}{(E_\mu-\sqrt{s})^2-k^2}\bigg].\quad\text{(B4)}$$

To obtain Regge trajectories in this approximation, we replace the factor $\pi G^2/[2(k^2+\mu^2)^{1/2}(k^2-q^2)]$ by $P_3(k^2)$ in (16).

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Diagonalization of the Natural Parity in the Multiperipheral Equations*

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The crossed partial-wave analysis of the multiperipheral equations is extended to include the natural-parity quantum number. At nonvanishing momentum transfer, a constraint imposed by reflection symmetry on the two-Reggeon–particle vertex function is shown to diagonalize a discrete index κ, which therefore assumes the meaning of a parity index. At vanishing momentum transfer, the result is a selection rule for the vertex function between input Regge poles and an output Regge pole of Toller quantum number $M=0$. It turns out that, to leading order in the asymptotic expansion in the subenergies, the product of the three natural parities at the vertex has to be positive. The $t=0$ limit and some possible implications of this selection rule are also discussed.

I. INTRODUCTION

THE crossed partial-wave analysis[1–4] of the multiperipheral equations[5–8] has provided a mathematical framework for the Regge bootstrap.[9–13] Though the resulting equations are rather complicated in the multi-Regge-pole case, and their physical basis is being questioned,[13] they still provide a bootstrap model for arbitrary spin configurations and a reasonable motivation for factorized production amplitudes. If the Regge bootstrap is to be extended to lower-ranking trajectories, the use of multi-Regge-pole models will be useful to describe such nonvacuum quantum numbers as unnatural parity or Toller's quantum number $M\neq 0$.

The purpose of this paper is to fill a gap left by the previous group-theoretical analysis, by presenting a diagonalization of the crossed-channel natural parity

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