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Test of the α-Expansion Method for Extracting the Hadronic Spectrum in QCD: The Harmonic Oscillator*

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The recently proposed α-expansion method for calculation of the hadronic spectrum in QCD is tested by the simple example of the harmonic oscillator. It appears that three terms of the expansion of the generalized S-matrix at large momenta are sufficient to calculate the whole spectrum to very good accuracy.

1. INTRODUCTION

Recently, a method of calculation of the hadronic spectrum in QCD was proposed by one of the authors [1, 2]. The method uses the asymptotic expansion for 2-point functions of gauge invariant composite operators for large Euclidean momenta, and ends up with the expansion for the hadronic masses in terms of an auxiliary parameter α which is set to one at the end of the calculation. The masses have the form

$$M_i = \mu \exp\left(-\int^{g_\mu^2} dg^2/\beta(g^2)\right) C_i(\alpha), \tag{1.1}$$

where μ is a renormalization point, g_μ is the corresponding renormalized coupling, and $\beta(g^2)$ is the Gell-Mann–Low function. The functions $C_i(\alpha)$ are expanded in α and the expansion coefficients are expressed in terms of Feynman integrals of QCD.

The method is quite general and applies to all theories with asymptotic freedom. If the physical spectrum is discrete, then the functions $C_i(\alpha)$ approach finite limits as $\alpha \to 1$. This corresponds to confinement. If there is no confinement, then the

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$\bar{q}q$ and/or gluon production would lead to a continuous spectrum. In this case the functions $C_i(\alpha)$ should be more singular as $\alpha \to 1$; the mass spacings

$$M_i - M_{i+1} \sim C_i - C_{i+1}$$

would tend to zero to reproduce the continuous spectrum. In the above discussion we ignored the physical thresholds, but in the real world the resonances are narrow, so that it makes sense. The method can be generalized to include these effects [2], but we do not need it in the specific case which we are going to discuss below.

The aim of this paper is twofold. First, we are going to clarify the details of the general method of the α-expansion [2]. We also review the infrared regularization through the Padé equations [1, 2] and establish the relationship with the moment conditions used in Ref. [3]. We apply this to the simple example of the harmonic oscillator. As was noted [3], the oscillator can be viewed as an asymptotically free theory with confinement. The β-function in this case is simply $\beta = -g^2$. We use the generalized S-matrix, introduced there, and then proceed in the same way as in Ref. [2].

The second aim of this paper is to check the convergence of the α-expansion. If one needs, say, ten terms to obtain reasonable accuracy then, of course, the method is useless. To our own surprise, the α-expansion coefficients decrease quite rapidly. The α^3-approximation fits the spectrum within a few percent (see Fig. 1). As is discussed at the end of the paper, there are some reasons to expect the same phenomenon in QCD.

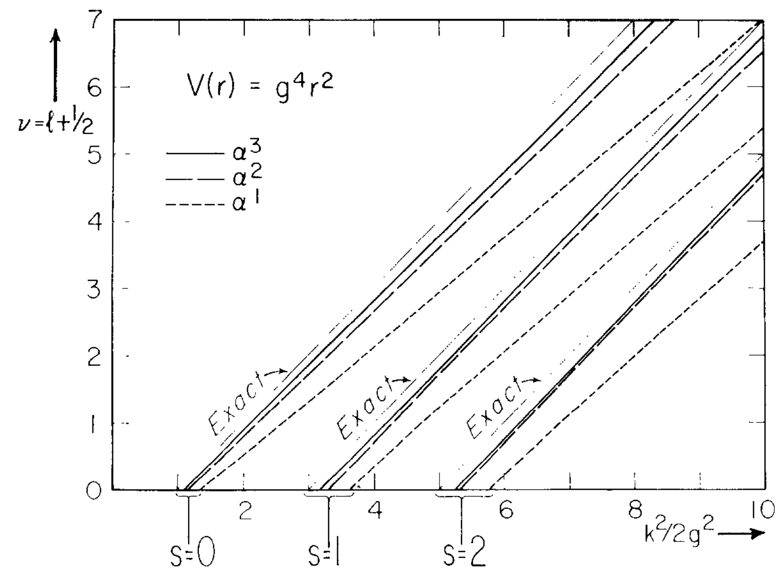

FIG. 1. The leading harmonic oscillator trajectories, labeled by the radial quantum number $s = 0, 1, 2$. Shown are the exact trajectories and the first three approximants, given by α^1, α^2, and α^3; (see (4.15) with $\alpha = 1$).

2. α-EXPANSION IN QCD

Let us describe the general framework of the α-expansion in QCD [2]. The basic idea is to introduce the infrared cutoff, R, in such a way that at finite R the gauge invariant Green's functions possess only poles for positive R^2, while for $-k^2R^2 \gg 1$ they remain unchanged. Then as $k \to \infty$ we return to the original theory and the poles should correspond to the physical spectrum.

A practical way to introduce such a cutoff is to start with the Padé approximant in $k^2 + \Lambda^2$, where Λ^2 is some point in the deep Euclidean domain, and then increase the rank N of the approximant together with Λ. As was shown in Refs. [1] and [2] (also see below), in the limit of N, $\Lambda \to \infty$ only the ratio

$$R = N/\Lambda \tag{2.1}$$

remains in the approximant and the poles M_i^2 depend on R as follows:

$$M_i = (1/R)\, F_i(g_R), \tag{2.2}$$

where g_R^2 is the effective coupling corresponding to the scale R^{-1} in momentum space. It is related to g_μ as usual

$$\ln \mu R = \int_{g_R^2}^{g_\mu^2} dg^2/\beta(g^2). \tag{2.3}$$

It follows from the general theory of the Padé approximant that all the masses are real and all the residues are positive, due to spectral conditions of the original S-function. The functions F_i can be expanded in g_R^2, and expansion coefficients can be expressed in terms of diagrams of the ordinary perturbation theory.

Altogether it means that the perturbation theory is modified in such a way that the masses are finite and can be expanded in the coupling constant at finite R.

Now, if we expand masses in terms of g_μ, rather than g_R, then in each order of g_μ the function F_i would depend on R only logarithmically, so that as $R \to \infty$ the factor R^{-1} in (2.2) takes over and all the masses condense at zero, thus reproducing the continuous spectrum of quarks and gluons. Also, if we expand the approximant itself in g_μ^2 and tend $R \to \infty$ in each order, we obtain the ordinary perturbation theory with the quark–gluon cut in the momentum plane.

But, of course, one should not do it since the effective coupling g_R increases as $R \to \infty$. Actually, the functions F_i depend on the following argument (see (2.3)):

$$x = \mu R \exp\left(-\int^{g_\mu^2} dg^2/\beta(g^2)\right) \sim \mu R \exp(-a/g_\mu^2). \tag{2.4}$$

The perturbation theory corresponds to expansion of F at *small* x, whereas we are interested in the opposite limit, $x \to \infty$. So there is a critical value of coupling constant

$$(g_\mu^2)_{\text{crit}} \sim a/\ln \mu R \tag{2.5}$$

such that $x \sim 1$. The perturbation theory can be applied only at smaller values of g_μ. As $R \to \infty$ the perturbation theory can never be applied. We expect, however, that the functions F_i increase as x in this limit so that the factor R^{-1} in (2.2) is cancelled and the masses tend to the limits (1.1). The detailed discussion of these phenomena is contained in previous papers [1, 2].

Now, the α-expansion is continued as follows. We replace the factor R^{-1} in (2.2) by $R^{-\alpha^2}$ and minimize with respect to R:

$$M_i = \min(R^{-\alpha^2} F_i(g_R)). \tag{2.6}$$

The value of R (or g_R) at the minimum is determined by

$$-F_i' \beta({g_R}^2) = F_i \alpha^2. \tag{2.7}$$

At small α one may expand F_i and β in ${g_R}^2$

$$F_i = F_i(0) + F_i'(0)\, {g_R}^2 \leftarrow \cdots, \tag{2.8}$$

$$\beta = -a{g_R}^4 - ba^2{g_R}^6 + \cdots, \tag{2.9}$$

and one arrives at the α-expansion:

$${g_R}^2 = \alpha[F_i(0)/F_i'(0)a]^{1/2} + \cdots, \tag{2.10}$$

$$\begin{aligned}\ln M_i = {} & \alpha^2 \left(\ln \mu + b \ln \alpha - \int^{g_\mu^2} \frac{dg^2}{\beta(g^2)}\right) \\ & + \ln F_i(0) + 2\alpha \left(\frac{F_i'(0)}{aF_i(0)}\right)^{1/2} + \frac{\alpha^2}{2}\left(\frac{F_i''}{aF_i'} - \frac{F_i'}{aF_i} - b \ln \frac{F_i'}{aF_i}\right) \\ & + \cdots. \end{aligned} \tag{2.11}$$

Now, if the functions F_i increase like R as $R \to \infty$ [4], then the position R_α of the minimum tends to infinity as $\alpha \to 1$. Say, for

$$F_i \to A_i R + B_i \tag{2.12}$$

we find

$$R^\alpha = \frac{\alpha^2}{1-\alpha^2} \frac{B_i}{A_i}, \tag{2.13}$$

$$\begin{aligned} M_i & \to \left(\frac{B_i}{1-\alpha^2}\right)^{1-\alpha^2} \left(\frac{A_i}{\alpha^2}\right)^{\alpha^2} \\ & \to A_i[1 - (1-\alpha^2)\ln(1-\alpha^2) + \cdots]. \end{aligned} \tag{2.14}$$

Thus, in the limit $\alpha \to 1$ one returns to the original theory. One may put $\alpha = 1$ in the first line of (2.11) and then one arrives at the form (1.1) of the α-expansion. In Section 4 we apply this method to the harmonic oscillator.

3. Padé Equations and Moment Conditions

Let us consider some 2-point function, e.g.,

$$S(k^2) = \int d^4 \exp(ikx)\langle T\bar{\psi}\psi(x)\; \bar{\psi}\psi(0)\rangle, \tag{3.1}$$

where ψ is a quark field and $\bar{\psi}\psi$ is a gauge invariant composite field. We are going to construct the Padé approximant to $S(k^2)$ in the deep Euclidean domain

$$-k^2 \to +\Lambda^2 \gg (\text{mass})^2. \tag{3.2}$$

The Padé approximant is the ratio of two polynomials

$$[S]_N^N = P_N(k^2)/Q_N(k^2) \tag{3.3}$$

which coincides with S at $k^2 = -\Lambda^2$ within $2N$ derivatives

$$S - [S]_N^N = O(k^2 + \Lambda^2)^{2N+1}. \tag{3.4}$$

The coefficients of the polynomials are to be determined from (3.4) or, equivalently, from

$$(d/dk^2)^l\, [Q_N S - P_N]_{k^2=-\Lambda^2} = 0, \qquad l = 0, 1,..., 2N. \tag{3.5}$$

The last N equations involve only Q_N and can be rewritten in terms of dispersion integrals as follows

$$\int_0^\infty dt\, \frac{Q_N(t)\,\mathrm{Im}\, S(t + i0)\, t^r}{(t/\Lambda^2 + 1)^{2N+1}} = 0, \qquad r = 0, 1,..., N - 1. \tag{3.6}$$

Once Q_N is known, P_N is given by the following expression:

$$P_N(t) = Q_N(t)\, S(t) - \frac{1}{\pi} \int_0^\infty ds\, \frac{Q_N(s)\,\mathrm{Im}\, S(s + i0)}{(s - t)} \left(\frac{t + \Lambda^2}{s + \Lambda^2}\right)^{2N+1}. \tag{3.7}$$

Equations (3.6) imply that (3.7) is indeed an Nth-degree polynomial in t.

Thus the problem is reduced to the solution of the integral equations (3.6).

Suppose that

$$\mathrm{Im}\, S \to At^\nu \tag{3.8}$$

as $t \to \infty$. In our case, as $t \to \infty$, S is determined by the free quark loop (due to asymptotic freedom) and

$$\nu = 1. \tag{3.9}$$

However, it will be convenient not to specify ν in what follows. The solutions $Q_N^{(0)}$

of (3.6) for the asymptotic form of S are given by Jacobi polynomials and have the following form [2]:

$$Q_N^{(0)}(t) = \oint_c \frac{dz}{2\pi i} f(z)\left(1 + \frac{t}{\Lambda^2}\right)^z, \tag{3.10}$$

where

$$f(z) = \frac{\Gamma(2N + 1 - z)\,\Gamma(-z)}{\Gamma(N + 1 - \nu - z)\,\Gamma(N + 1 - z)}\,(N^2)^{1-\nu} \tag{3.11}$$

is a meromorphic function with poles at $z = 0,..., N$ and zeros at $z = N + 1 - \nu$, $N + 2 - \nu,..., \infty$. The contour c in (3.10) encloses the poles of f.

Now let us decompose S as

$$\operatorname{Im} S = At^\nu[1 + \sigma(t)], \tag{3.12}$$

where $\sigma(t)$ includes all the corrections to the asymptotic form. As it was shown in [1, 2] the homogeneous equation (3.6) can be transformed to the inhomogeneous form

$$\text{const}\, Q_N^{(0)}(s) = \int_0^\infty \frac{dt}{\Lambda^2}\, Q_N(t)(1 + \sigma(t))\, G(s, t). \tag{3.13}$$

The constant on the left-hand side is arbitrary and can be absorbed in Q. The Green's function $G(s, t)$ is given by the double Mellin–Barns integral

$$G(s, t) = \oint_c \oint_{c'} \frac{dz\,dz'}{(2\pi i)^2} \frac{1}{z' - z} \frac{f(z)}{f(z')} \frac{(1 + s/\Lambda^2)^z}{(1 + t/\Lambda^2)^{z'}}, \tag{3.14}$$

where contour c encloses the poles of $f(z)$ and contour c' encloses the zeros of $f(z')$.

Now one can let N and Λ tend to infinity at fixed ratio R. The Jacobi polynomial reduces to the Bessel function

$$Q_N^{(0)}(s) \to (Rs^{1/2})^{-\nu} J_\nu(2Rs^{1/2}) \tag{3.15}$$

and $G(s, t)$ can be expressed as an integral of two Bessel functions [2]. All we need are the integrals

$$K_\lambda(s) = \int_0^\infty \frac{dt}{\Lambda^2}\,(R^2 t)^\lambda\, G(s, t). \tag{3.16}$$

A straightforward calculation using (3.4) gives as $N = \infty$, R fixed,

$$K_\lambda(s) = \sum_{m=0}^{\infty} (-sR^2)^m \frac{\Gamma(\lambda + \nu)}{m!\,\Gamma(m + \nu)(m - \lambda)\,\Gamma(-\lambda)}. \tag{3.17}$$

Now, if $\sigma(t)$ can be represented by the series [5]

$$\sigma(t) = \sum \sigma_i t^{-\Delta_i} \tag{3.18}$$

with increasing indices Δ_i, then we may expand Q_N as

$$Q_N(t) = \sum_{n=0}^{\infty} q_n(tR^2)^n \tag{3.19}$$

and arrive at the following equation for coefficients q_n:

$$q_m + \sum_{n=0}^{\infty} F_{mn} q_n = q_m{}^0 \tag{3.20}$$

with

$$q_m{}^0 = \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)}, \tag{3.21}$$

$$F_{mn} = \frac{(-1)^m}{m!\,\Gamma(m+\nu)} \sum_i \sigma_i R^{2\Delta_i} \frac{\Gamma(n-\Delta_i+\nu)}{(m-n+\Delta_i)\,\Gamma(\Delta_i-n)}. \tag{3.22}$$

We may now construct the iterative solution for q_m

$$q_m = q_m{}^0 - \sum_n F_{mn} q_n{}^0 + \sum_{p,n} F_{mp} F_{pn} q_n{}^0 - \cdots. \tag{3.23}$$

Since the terms in F_{nm} are proportional to positive powers of R, this perturbation theory makes sense only at small R. For the continuation to large R we need the technique of the α-expansion, described in the previous section.

As is well known, in QCD the asymptotic expansion at large momenta involves powers of $\ln t$ as well as inverse powers of t. The corresponding integrals can be found by differentiating (3.16) with respect to λ (see Ref. [2] for more details).

In the simple case of the harmonic oscillator we do not meet logarithms, and all the indices Δ_i are integers. In this case F_{mn} simplifies since $\Gamma(\Delta_i - n)$ kills all the terms but those with either

$$n < \Delta_i \tag{3.24}$$

or

$$n = \Delta_i + m. \tag{3.25}$$

Thus, in a given order in R^2 the corrections to $q_m{}^0$ in (3.23) will involve only finite number of terms. For example, if we keep only one term $\sigma_2(R^2)^2$ in (3.22), then

$$q_m = \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left\{1 - \sigma_2 R^4 \left[\frac{\Gamma(\nu-2)(m+\nu)}{(m+2)\,\Gamma(\nu+1)} - \frac{\Gamma(\nu-1)(m+\nu)}{(m+1)\,\Gamma(\nu+2)} + \frac{1}{(m+1)(m+2)(m+\nu+1)(m+\nu+2)}\right]\right\}. \tag{3.26}$$

The corresponding correction to the Q-function is given by the superposition of Bessel functions with shifted index ν. The simplest way to obtain this expression is to start as in Ref. [3] with the Ansatz

$$Q(R^2t) = \Lambda_\nu(R^2t) + \sum_{m=1}^{\infty} \Lambda_{\nu+m}(R^2t)\, C_m\,, \tag{3.27}$$

where

$$\Lambda_\nu(z) = z^{-\nu/2} J_\nu(2z^{1/2}) \tag{3.28}$$

and to find the coefficients C_m directly from (3.6). If one tends $\Lambda \sim N \to \infty$ (3.6), then one arrives at the moment conditions [3, Eq. 2.7]

$$\int_0^\infty dt\, Q(t)\, \mathrm{Im}\, S(t + i0)\, t^n = 0, \qquad n = 0, 1, \ldots, \infty. \tag{3.29}$$

If now one expands $\mathrm{Im}\, S$ in inverse powers of t, and uses (3.26) for Q, then all the integrals can be calculated explicitly

$$\int_0^\infty dt\, t^{\nu+r-\Delta} \Lambda_{\nu+m}(R^2t) = \frac{1}{2} R^{-2(\nu+r-\Delta+1)} \frac{\Gamma(\nu + r - \Delta + 1)}{\Gamma(m - r + \Delta)}\,. \tag{3.30}$$

We arrive thus to the following system of equations for C_m:

$$\sum_{m=0}^{\infty} \sum_{p=0}^{\infty} C_m \sigma_p (R^2)^p \frac{\Gamma(\nu - p + 1 + n)}{\Gamma(m + p - n)} = 0, \qquad n = 0, 1, \ldots, \infty. \tag{3.31}$$

Here by definition

$$C_0 \equiv \sigma_0 \equiv 1.$$

We observe that, for given p and m, all the moment relations starting from $n = m + p$ and higher are automatically satisfied.

Thus, in a given order in R^2 we may leave only a finite number of C_m's. In the first order we leave C_0, C_1 and find

$$C_1 \frac{\Gamma(\nu + 1)}{\Gamma(1)} + \sigma_1 R^2 \frac{\Gamma(\nu)}{\Gamma(1)} = 0, \qquad n = 0. \tag{3.32}$$

The moment relations for $n \geqslant 1$ are automatically satisfied in the first order in R^2. In the higher orders the higher coefficients and the higher-order corrections to the previous coefficients would appear.

The explicit expressions for C_m in the case of harmonic oscillator will be given below.

4. α-Expansion for Harmonic Oscillator

An appropriate analog to the 2-point function in potential theory for confined systems, as discussed in Ref. [3], is given by

$$\hat{S}(\nu, k^2) = D(-\nu, k^2)/D(\nu, k^2), \qquad \nu = l + \tfrac{1}{2} \tag{4.1}$$

where $D(\nu, k^2)$ is the usual Jost function.

To test the method described in the previous section, we take as an example the harmonic oscillator potential

$$V(r) = g^4 r^2, \tag{4.2}$$

and obtain the exact $\hat{S}$, given by

$$\hat{S}(\nu, k^2) = (4g^2)^\nu \frac{\Gamma(\nu/2 + \frac{1}{2} - k^2/4g^2)}{\Gamma(-\nu/2 + \frac{1}{2} - k^2/4g^2)}\,. \tag{4.3}$$

Expanding $\hat{S}$ for large k^2 we obtain the series expression for the "cut":

$$\begin{aligned}\operatorname{Im} \hat{S} = t^\nu \sin \pi\nu \Big[1 &- \left(\frac{y}{R^4 t^2}\right) \frac{\Gamma(\nu + 2)}{\Gamma(\nu - 1)} + \left(\frac{1}{10}\right)\left(\frac{y}{R^4 t^2}\right)^2 \frac{\Gamma(\nu + 2)(5\nu + 7)}{\Gamma(\nu - 3)} \\ &- \left(\frac{1}{210}\right)\left(\frac{y}{R^4 t^2}\right)^z \frac{\Gamma(\nu + 2)(35\nu^2 + 112\nu + 93)}{\Gamma(\nu - 5)} + O(y^4)\Big]\,,\end{aligned} \tag{4.4}$$

where $y = \frac{2}{3}(g^4 R^4)$, $t = k^2$, and R is an arbitrary constant.

Each step here and in what follows involves some tedious but pedestrian algebra and we present the results only.

We now impose confinement by removing the cut in (4.4). This is done by systematically applying the moment conditions (3.29):

$$\int_0^\infty dt\, t^n Q(\nu, t) \operatorname{Im} S(\nu, t) = 0 \tag{4.5}$$

for all integers n, with the Ansatz (3.26). As $R \to \infty$, Q should coincide with D.

These moment conditions are solved by the following Q function:

$$\begin{aligned} Q(\nu, z) = {} & \Lambda_\nu(z) + y[2\Lambda_{\nu+1}(z) + (\nu - 1)\Lambda_{\nu+2}(z)] \\ & + y^2 \left[2\Lambda_{\nu+2}(z) + \left(\frac{2}{5}\right)(5\nu - 7)\left(\Lambda_{\nu+3}(z) + \frac{(\nu - 1)}{4}\Lambda_{\nu+4}(z)\right)\right] \\ & + y^3 \left[\left(\frac{4}{3}\right)\Lambda_{\nu+3}(z) + \left(2\nu - \frac{18}{5}\right)\Lambda_{\nu+4}(z) \right. \\ & \left. + \left(\nu^2 - \frac{16}{5}\nu + \frac{93}{35}\right)\left(\Lambda_{\nu+5}(z) + \frac{(\nu - 1)}{6}\Lambda_{\nu+6}(z)\right)\right] \\ & + O(y^4), \end{aligned} \tag{4.6}$$

where $z = R^2 t$.

The functions $\Lambda_\nu(z)$ are defined for convenience as "cutless" Bessel functions, introduced in (3.28).

The bound states are given by the zeros of $Q(\nu, t)$. The phenomenological approach would be to fix the scale R in $t = z/R^2$ by fitting one "experimental" fact, say, the ground state of the system. This was done in Ref. [3] for $Q(\nu, t)$ to order $y^2 = (\frac{2}{3}g^4R^4)^2$. Here we apply the α-method instead, which fixes the scale R without resort to any fitting.

A particular state will be specified by two quantum numbers, $\nu \equiv l + \frac{1}{2}$ and s, the radial quantum number. Thus a state is given by

$$Q(\nu, t(\nu, s)) = 0, \tag{4.7}$$

where Q is given by (4.6).

First we expand a particular solution of (4.7) in powers of y.

This gives

$$R^2 t(\nu, s) = z_0(\nu, s) + yz_1(\nu, s) + y^2 z_2(\nu, s) + y^3 z_3(\nu, s) + O(y^4) \tag{4.8}$$

where

$$\begin{aligned} z_1 &= 2 + (\nu^2 - 1)/z_0\,, \\ z_2 &= (1/5z_0)[4 - (1/z_0)(7\nu^2 + 17) + (3/2z_0^2)(\nu^2 - 1)(\nu^2 - 9)], \\ z_3 &= (1/7z_0^2)[16/5 + (72/5z_0)(\nu^2 - 9) + (1/15z_0^2)(5357 + 590\nu^2 - 187\nu^4) \\ &\quad + (1/6z_0^3)(\nu^2 - 1)(\nu^2 - 25)(13\nu^2 - 61)]. \end{aligned}$$

The zeroth-order solution $z_0(\nu, s)$ is given by the sth root of $\Lambda_\nu(z)$, or $J_\nu(2(z_0(\nu, s))^{1/2}) = 0$.

For convenience we define

$$\begin{aligned} x \equiv y/z_0(\nu, s) &= \tfrac{2}{3} g^4 R^4 / z_0(\nu, s), \\ \tau(\nu, s) &\equiv t(\nu, s)/2g^2 \end{aligned} \tag{4.9}$$

and rewrite (4.8) as

$$\tau = (6z_0x)^{-1/2}(z_0 + xz_0z_1 + x^2z_0^2z_2 + x^3z_0^3z_3 + \cdots), \tag{4.10}$$

where we have suppressed the quantum numbers ν and s.

To apply the α-method now, we take the logarithm of (4.10) and introduce the auxiliary parameter α, which ultimately must be set equal to one.

$$\begin{aligned} \ln \tau = {} & \tfrac{1}{2} \ln(\tfrac{1}{6}z_0) + xz_1 + x^2(z_2z_0 - \tfrac{1}{2}z_1^2) \\ & + x^3(z_3z_0^2 - z_2z_1z_0 + \tfrac{1}{3}z_1^3) - \tfrac{1}{2}\alpha \ln x \\ & + O(x^4). \end{aligned} \tag{4.11}$$

For a given α, we now minimize the "mass" τ with respect to the scale x and expand the result in powers of α. The result of this operation is

$$x(\nu, s) = \alpha(1/2z_1) - \tfrac{1}{2}\alpha^2(a/z_1^3) + \alpha^3(b/z_1^4) + O(\alpha^4), \tag{4.12}$$

where

$$a = z_2 z_0 - \tfrac{1}{2} z_1^2,$$
$$b = (z_2 z_0 - \tfrac{1}{2} z_1^2)^2/z_1 - \tfrac{3}{8}(z_3 z_0^2 - z_2 z_1 z_0 + \tfrac{1}{3} z_1^3).$$

Since $x = \frac{2}{3} g^4 R^4 / z_0(\nu, s)$, it is interesting to note how the scale R behaves as a function of the quantum numbers (ν, s).

$$R^2(\nu, s) = (\tfrac{3}{2} z_0(\nu, s))^{1/2} \frac{z_0(\nu, s)}{2z_0(\nu, s) + (\nu^2 - 1)} \left(\frac{\alpha}{2g^2}\right) + O(\alpha^2). \tag{4.13}$$

Now, for large ν, the first zero of the Bessel function $2(z_0(\nu, 0))^{1/2} \to \nu$ and therefore when $\nu \gg 1$

$$R^2(\nu, 0) \approx (6^{1/2}/24)(\alpha/2g^2)\nu. \tag{4.14}$$

Thus along the top trajectory, $R(\nu, 0)$ will grow proportional to $\nu^{1/2}$, which compares to the classical turn-around point. This dynamic growth is, of course, absent when R is fixed by one phenomenological fit. This $\nu^{1/2}$ growth of R allows linear trajectories [6]. Here, it appears already in the first order in α, although the slope will not be correct.

Finally, the result (4.12) for $x(\nu, s)$ is substituted in the expression for the state (4.11). We obtain

$$\begin{aligned} \ln t(\nu, s)/2g^2 = {} & \tfrac{1}{2} \ln(\tfrac{1}{6} z_0) + (\tfrac{1}{2}\alpha)(1 - \ln \alpha + \ln 2z_1) \\ & + (\tfrac{1}{2}\alpha)^2 (z_2 z_0 / z_1^2 - \tfrac{1}{2}) \\ & - (\tfrac{1}{2}\alpha)^3 (2 z_2^2 z_0^2 / z_1^4 - z_3 z_0^2 / z_1^3 - z_2 z_0 / z_1^2 + \tfrac{1}{6}) \\ & + O(\alpha^4), \end{aligned} \tag{4.15}$$

to be evaluated at $\alpha = 1$, where the z_i are the same as in (4.8).

It is the expression (4.15), with $\alpha = 1$, which is to be compared with the familiar result for the harmonic oscillator:

$$t(\nu, s)/2g^2 = 1 + \nu + 2s, \qquad \nu = l + \tfrac{1}{2}, \quad s = 0, 1, 2, \ldots . \tag{4.16}$$

5. Summary and Conclusions

As one can see from Fig. 1, the α-expression works exceedingly well for the harmonic oscillator [7]. In general, it should be an asymptotic expansion (since the Stirling formula was used to construct the perturbation theory), but the first three coefficients are decreasing by a factor of about 4. The actual expansion parameter is not α, but $\frac{1}{4}\alpha$. The origin of this factor can be easily traced. It appears because the perturba-theory expands in $(g/k)^4$. If the expansion parameter would be $(g/k)^N$, then the α-expansion would go in α/N.

In other words, the α-expansion works better the faster the corrections to the asymptotic freedom behave with momentum. In QCD there are two sources of corrections—the quantum fluctuations, which give rise to the logarithmic corrections, and the instanton and meron field configurations [8], which give rise to very rapidly varying corrections,

$$\sim(\mu_0^2/k^2)^{5\div 9}.$$

The estimates [8] show that μ_0 is large enough to neglect the quantum fluctuations (the effective coupling $g^2/8\pi^2$ is less or about $\frac{1}{8}$ at this scale). In this situation, the α-expansion might work even better than for the harmonic oscillator. The problem here is to find a systematic method of calculation of meron and instanton corrections. Note, that in Ref. [2] only quantum fluctuations were taken into account. In the limit of infinite number of colors, it was legitimate since the instantons and merons display themselves later than quantum fluctuations. In the physical world, with three colors it seems to be the opposite.

The above considerations suggest that the most important corrections in QCD are powerlike as in the oscillator case and that the α-expansion in QCD should work well.

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2. A. A. Migdal, *Ann. Phys.* **110** (1978), 46.
3. S.-Y. Chu, B. R. Desai, and P. Kaus, *Phys. Rev. D* **16** (1977), 2631.
4. The mass F/R always decreases due to general properties of Padé approximants. It may, however, approach a constant from above.
5. That is what will happen with the harmonic oscillator.
6. The relevance of the appearance of a scale proportional to $\nu^{1/2}$ for linear trajectories has been discussed recently. See S.-Y. Chu and P. Kaus [*Phys. Rev. D* **14** (1976), 1681; **16** (1977), 503 for discussion of this point in the context of relativistic potential scattering and soliton–Regge trajectories, respectively.
7. The exact trajectories are linear with slope $(2g^2)(d\nu/dt) = 1$. The approximate trajectories α^1, α^2, and α^3 are asymptotically linear with slopes 0.858, 0.972, and 1.001, respectively, and independent of the radial quantum number, s.
8. C. Callan, R. Dashen, and D. Gross, IAS Preprint, COO–2220–115 (1977).