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Unitarity constraints on $a + b \rightarrow 1 + 2 + 3$*

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The isobar model for $a + b \rightarrow 1+2+3$ is reexamined in light of the requirements of subenergy unitarity. Discontinuities of the amplitude across the subenergy variables are removed by means of a set of coupled integral equations. We make a comparison of the amplitudes with and without the unitarity corrections and suggest a ratio test to check the validity of the isobar model.

I. INTRODUCTION

In recent years there has been considerable interest in doing partial-wave analysis of the reactions of the type $a+b\rightarrow 1+2+3$. In analyzing such a process, one finds it convenient to assume that the reaction proceeds through an intermediate state dominated by a two-particle resonance or an isobar which ultimately breaks up into its constituents in the final state. Now, it may happen that many such isobars are likely to be present in the intermediate state. In such a case, it has been customary to simply add the various amplitudes corresponding to different isobars to obtain the total amplitude. This is the so-called isobar model which has been widely employed in such reactions as $\pi N \rightarrow \pi\pi N$.[1,2,3] However, this simple scheme is only an approximation and has been criticized lately on grounds that it does not satisfy unitarity.[4]

In the present paper, we outline the isobar model, state the various assumptions that go into it, derive the necessary unitarity constraints on the production amplitudes, and suggest some tests to check their validity. In doing so, we shall confine ourselves to considerations of normal thresholds in subenergy variables only. Our aim is to carry the formal results to a stage where numerical estimates can be easily made. For this reason, we shall present all the necessary details for performing such estimates as we develop the formalism.

In Sec. II we introduce the necessary representations in the Hilbert space of two- and three-particle systems. Then in Sec. III we discuss the isobar model as currently practiced. Next, in Sec. IV we develop the unitarity constraints and write them down in full detail. In Sec. V we deal with the comparison of the isobar amplitude and the unitarized amplitude. Finally, in Sec. VI we offer our concluding remarks.

II. REPRESENTATIONS

We consider particles with spin and use relativistically invariant normalization of states.

A. Two particles

Quite generally, in an arbitrary reference frame, the plane-wave states are normalized as

$$\langle \vec{p}_a'\vec{p}_b';\mu_a'\mu_b'|\vec{p}_a\vec{p}_b;\mu_a\mu_b\rangle = 2E_a 2E_b\delta(\vec{p}_a'-\vec{p}_a)\delta(\vec{p}_b'-\vec{p}_b)\delta_{\mu_a'\mu_a}\delta_{\mu_b'\mu_b}. \tag{2.1}$$

Here μ_i denotes the z component of spin σ_i which we shall suppress. Going over to the angular momentum representation, the states of total momentum $\vec{P}$, energy E, angular momentum J and its z component M have the following normalization:

$$\langle \vec{P}'E'J'M'l'\sigma'|\vec{P}EJMl\sigma\rangle = \frac{4\sqrt{s}}{q}\delta(\vec{P}'-\vec{P})\delta(E'-E)\delta_{J'J}\delta_{M'M}\delta_{l'l}\delta_{\sigma'\sigma}, \tag{2.2}$$

where the center-of-mass (c.m.) momentum and energy are denoted by q and $\sqrt{s}$, respectively. The total spin σ and the relative angular momentum l in the c.m. are coupled in the usual manner,

$$\vec{\sigma}=\vec{\sigma}_a+\vec{\sigma}_b, \qquad \vec{J}=\vec{l}+\vec{\sigma}. \tag{2.3}$$

B. Three particles

The normalization of plane-wave states in an arbitrary reference frame is given by

$$\langle \vec{p}_\alpha'\vec{p}_\beta'\vec{p}_\gamma';\mu_\alpha'\mu_\beta'\mu_\gamma'|\vec{p}_\alpha\vec{p}_\beta\vec{p}_\gamma;\mu_\alpha\mu_\beta\mu_\gamma\rangle = 2E_\alpha\delta(\vec{p}_\alpha'-\vec{p}_\alpha)\cdots 2E_\gamma\delta(\vec{p}_\gamma'-\vec{p}_\gamma)\delta_{\mu_\alpha'\mu_\alpha}\cdots\delta_{\mu_\gamma'\mu_\gamma}. \tag{2.4}$$

In contrast to the case of two particles, a three-particle system has three linearly independent angular momentum representations. We may couple particles β and γ and obtain a state given in (2.2). In particular, we may construct this state in the overall center-of-mass system (o.c.m.) so that $\vec{p}_{\beta\gamma}=-\vec{Q}_\alpha$, where $\vec{Q}_\alpha$ is the momentum of α in the o.c.m. This state, in fact, can be regarded as representing a "particle $\beta\gamma$" which can then be coupled to α, again using the prescription (2.2). Finally, the state thus realized in the o.c.m. can be given a Lorentz boost. We shall indicate the dynamical variables of this state by a superscript α. It is normalized as

$$\langle \vec{P}^{\alpha\prime}E^{\alpha\prime}J^{\alpha\prime}M^{\alpha\prime}L^{\alpha\prime}\Sigma^{\alpha\prime}j^{\alpha\prime}l^{\alpha\prime}\bar{\sigma}^{\alpha\prime}s^{\alpha\prime}\,|\,\vec{P}^{\alpha}E^{\alpha}J^{\alpha}M^{\alpha}L^{\alpha}\Sigma^{\alpha}j^{\alpha}l^{\alpha}\bar{\sigma}^{\alpha}s^{\alpha}\rangle = \frac{4W^\alpha}{Q^\alpha}\frac{4(s^\alpha)^{1/2}}{q^\alpha}\delta(\vec{P}^{\alpha\prime}-\vec{P}^{\alpha})\delta(E^{\alpha\prime}-E^\alpha)\delta_{J^{\alpha\prime}J^\alpha}\cdots\delta_{\bar{\sigma}^{\alpha\prime}\bar{\sigma}^\alpha}\delta(s^{\alpha\prime}-s^\alpha). \tag{2.5}$$

In the above, s^α and q^α for the β,γ pair have the same meaning as defined earlier. W^α is the total energy of the entire system in the o.c.m. The meaning of various angular momenta will be clear from the following coupling scheme which is an extension of (2.3):

$$\begin{aligned}\vec{\bar{\sigma}}^\alpha &= \vec{\sigma}_\beta+\vec{\sigma}_\gamma,\\ \vec{j}^\alpha &= \vec{l}^\alpha+\vec{\bar{\sigma}}^\alpha,\\ \vec{\Sigma}^\alpha &= \vec{j}^\alpha+\vec{\sigma}_\alpha,\\ \vec{J}^\alpha &= \vec{L}^\alpha+\vec{\Sigma}^\alpha.\end{aligned} \tag{2.6}$$

For further details of this canonical representation we refer the reader to Ref. 5, 6, or 7.

C. Transformation functions

The states introduced so far describe the two- and three-particle systems in an arbitrary frame. Since relativistic normalization is used, the final result will not depend on the choice of the frame which we shall now take as the o.c.m., omitting the label $\vec{P}=0$ from the states.

The states defined by (2.1) and (2.2) are connected by a transformation function which is given by

$$\langle \vec{p}_a\vec{p}_b;\mu_a\mu_b\,|\,WJMl\sigma\rangle = \frac{4W}{q}C(\sigma_a\sigma_b\sigma;\mu_a\mu_b)C(l\sigma J;M-(\mu_a+\mu_b),\mu_a+\mu_b)Y^{(\omega)}_{l,M-(\mu_a+\mu_b)}\delta(\vec{p}_a+\vec{p}_b)\delta(W-(E_a+E_b)), \tag{2.7}$$

where $\omega\equiv(\theta,\phi)$ are the spherical coordinates in the c.m. with arbitrary orientation of the axes. Similarly, the connection between (2.4) and (2.5) is given by

$$\begin{aligned}\langle \vec{p}_\alpha\vec{p}_\beta\vec{p}_\gamma;\mu_\alpha\mu_\beta\mu_\gamma\,|\,W^\alpha J^\alpha M^\alpha L^\alpha\Sigma^\alpha j^\alpha l^\alpha\bar{\sigma}^\alpha s^\alpha\rangle
&= \frac{4W^\alpha}{Q^\alpha}\frac{4(s^\alpha)^{1/2}}{q^\alpha}\sum_{m^\alpha,\nu_\beta,\nu_\gamma} C(\sigma_\beta\sigma_\gamma\bar{\sigma}^\alpha;\nu_\beta\nu_\gamma)C(l^\alpha\bar{\sigma}^\alpha j^\alpha;m^\alpha-(\nu_\beta+\nu_\gamma),\nu_\beta+\nu_\gamma)\\
&\quad\times C(j^\alpha\sigma_\alpha\Sigma^\alpha;m^\alpha,\mu_\alpha)C(L^\alpha\Sigma^\alpha J^\alpha;M^\alpha-(m^\alpha+\mu_\alpha),m^\alpha+\mu_\alpha)\\
&\quad\times Y^{(\omega^\alpha)}_{l^\alpha,m^\alpha-(\nu_\beta+\nu_\gamma)}Y^{(\Omega^\alpha)}_{L^\alpha,M^\alpha-(m^\alpha+\mu_\alpha)}D^{\sigma_\beta}_{\mu_\beta\nu_\beta}(\xi^\alpha_\beta)D^{\sigma_\gamma}_{\mu_\gamma\nu_\gamma}(\xi^\alpha_\gamma)\delta(\vec{p}_\alpha+\vec{p}_\beta+\vec{p}_\gamma)\\
&\quad\times\delta(W^\alpha-(E_\alpha+E_\beta+E_\gamma))\delta(s^\alpha-(p_\beta+p_\gamma)^2),\end{aligned} \tag{2.8}$$

where $\omega^\alpha\equiv(\theta^\alpha,\phi^\alpha)$ and $\Omega^\alpha\equiv(\Theta^\alpha,\Phi^\alpha)$ are the spherical coordinates in the c.m. of $\beta\gamma$ and the o.c.m., respectively (see Fig. 1). The presence of the D functions is due to the fact that the spins undergo Lorentz rotation. Later we shall give explicit formulas for their arguments.

D. Recoupling coefficients

At this point, it is convenient to introduce our choice of the coordinate axes in the o.c.m. For the $|\alpha\rangle$ representation, we take the z axis in the opposite direction to $\vec{Q}_\alpha$, the x axis toward β and orthogonal to z axis, and the y axis out of the paper so that $Oxyz$ forms a right-handed system (see Fig. 2). Similar choices are made for the $|\beta\rangle$ and $|\gamma\rangle$ representations by cyclic permutations of α,β,γ.

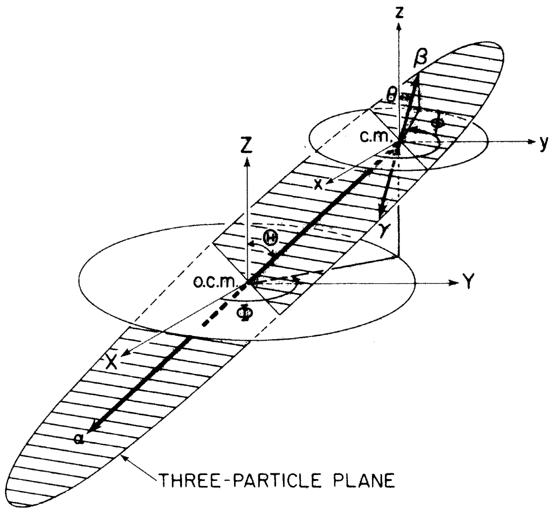

FIG. 1. Three-particle state in the overall center-of-mass frame with arbitrary orientation of the coordinate axes [see Eq. (2.8)].

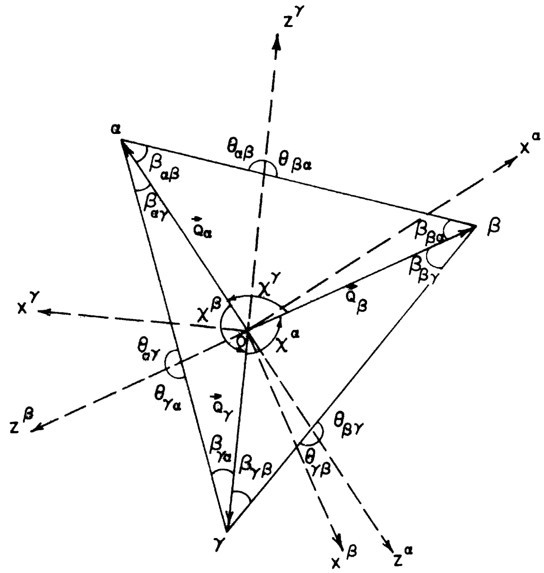

FIG. 2. Three-particle state in the overall center-of-mass frame with the three different sets of coordinate axes as defined in Sec. II D.

The three representations $|\alpha\rangle, |\beta\rangle, |\gamma\rangle$ for the three-particle system are equivalent in the sense that they are connected by unitary transformations. Indeed, it is this transformation function that plays an important role in the partial-wave analysis of a three-body final-state process and also in its unitarity calculations. For helicity representations, this recoupling coefficient has been given by Wick.[8] Calculations for the canonical case proceed along similar lines. Here we only give the final result referring the interested reader to Ref. 7 for details. The recoupling coefficient between the $|\alpha\rangle$ and $|\gamma\rangle$ representations is given by

$$\begin{aligned}
&\langle W^\alpha J^\alpha M^\alpha L^\alpha \Sigma^\alpha j^\alpha l^\alpha \bar\sigma^\alpha s^\alpha | W^\gamma J^\gamma M^\gamma L^\gamma \Sigma^\gamma j^\gamma l^\gamma \bar\sigma^\gamma s^\gamma\rangle \\
&\quad = \delta(W^\alpha - W^\gamma)\delta_{J^\alpha J^\gamma}\delta_{M^\alpha M^\gamma}\left(\frac{\pi}{2J^\alpha+1}\right)[(2L^\alpha+1)(2L^\gamma+1)]^{1/2}\left(\frac{16(s^\alpha s^\gamma)^{1/2}}{q^\alpha q^\gamma Q^\alpha Q^\gamma}\right) \\
&\quad\times \sum_{\substack{\mu_\alpha\mu_\beta\mu_\gamma m^\gamma \\ \mu'_\alpha\mu'_\beta\mu'_\gamma m^\alpha}} [C(\sigma_\beta\sigma_\gamma\bar\sigma^\alpha;\mu'_\beta\mu'_\gamma)C(l^\alpha\bar\sigma^\alpha j^\alpha; m^\alpha, \mu'_\beta+\mu'_\gamma)C(j^\alpha\sigma_\alpha\Sigma^\alpha; m^\alpha+\mu'_\beta+\mu'_\gamma, \mu'_\alpha) \\
&\qquad \times C(L^\alpha\Sigma^\alpha J^\alpha; 0, \Lambda^\alpha)C(\sigma_\alpha\sigma_\beta\bar\sigma^\gamma; \mu_\alpha\mu_\beta)C(l^\gamma\bar\sigma^\gamma j^\gamma; m^\gamma, \mu_\alpha+\mu_\beta) \\
&\qquad \times C(j^\gamma\sigma_\gamma\Sigma^\gamma; m^\gamma+\mu_\alpha+\mu_\beta, \mu_\gamma)C(L^\gamma\Sigma^\gamma J^\gamma; 0, \Lambda^\gamma)d^{J^\alpha}_{\Lambda^\alpha\Lambda^\gamma}(\chi^\beta) \\
&\qquad \times d^{\sigma_\alpha}_{\mu'_\alpha\mu_\alpha}(\chi^\beta+\xi^\gamma_\alpha)d^{\sigma_\beta}_{\mu'_\beta\mu_\beta}(\chi^\beta+\xi^\gamma_\beta-\xi^\alpha_\beta)d^{\sigma_\gamma}_{\mu'_\gamma\mu_\gamma}(\chi^\beta-\xi^\alpha_\gamma)Y^*_{l^\alpha m^\alpha}(\theta_{\beta\gamma})Y_{l^\gamma m^\gamma}(\theta_{\alpha\beta})], \qquad (2.9)
\end{aligned}$$

where

$$\Lambda^\alpha = m^\alpha + \mu'_\alpha + \mu'_\beta + \mu'_\gamma,$$

$$\Lambda^\gamma = m^\gamma + \mu_\alpha + \mu_\beta + \mu_\gamma.$$

The angles θ and χ are shown in Fig. 2. Each angle is to be calculated in the inertial frame located at its vertex. The Lorentz spin rotations are given by

$$\xi^\beta_\alpha = \chi^\gamma - \beta_{\alpha\gamma} - \theta_{\gamma\alpha},$$

$$\xi^\beta_\gamma = -\chi^\alpha + \beta_{\gamma\alpha} + \theta_{\alpha\gamma},$$

with the angles β as indicated in Fig. 2. The spherical harmonics only depend on the polar angles and can be expressed in terms of the associated Legendre polynomials. All angles are in the x-z plane and the entire expression of (2.9) is real. Our convention for the rotation operators is that of Rose.[9] Since we have used cyclic notation throughout, recouplings between other representations can be easily obtained by permutation of the indices in cyclic order.

E. Isospin states

Finally, to complete our discussion of representations, we give the necessary formulas for the isospin states. As usual, the states have unit normalization in terms of Kronecker δ functions. The transformation coefficients, analogous to (2.7) and (2.8), are, in an obvious notation,

$$\langle I_1 I_2; i_1 i_2 | I_1 I_2; Ii\rangle = C(I_1 I_2 I; i_1, i_2), \tag{2.10}$$

$$\langle I_\alpha I_\beta I_\gamma; i_\alpha i_\beta i_\gamma | I_\alpha I_\beta I_\gamma; I^\alpha i^\alpha \bar{I}^\alpha\rangle = C(I_\beta I_\gamma \bar{I}^\alpha; i_\beta i_\gamma) C(\bar{I}^\alpha I_\alpha I^\alpha; i_\beta + i_\gamma, i_\alpha), \tag{2.11}$$

where $\vec{\bar{I}}^\alpha = \vec{I}_\beta + \vec{I}_\gamma$ is the intermediate isospin.

As in the configuration space, there are three equivalent isospin representations whose relationship to the "plane-wave" states in isospin space can be obtained by cyclic permutation in (2.11). The unitary transformation between these representations, similar to (2.9) can be expressed in terms of the Racah coefficients, W.[9]

$$\begin{aligned}
\langle I_\alpha I_\beta I_\gamma; I^\alpha i^\alpha \bar{I}^\alpha | I_\alpha I_\beta I_\gamma; I^\gamma i^\gamma \bar{I}^\gamma\rangle
&= \delta_{I^\alpha I^\gamma}\delta_{i^\alpha i^\gamma} \sum_{i_\alpha i_\beta i_\gamma} C(I_\beta I_\gamma \bar{I}^\alpha; i_\beta i_\gamma) C(\bar{I}^\alpha I_\alpha I^\alpha; i_\beta + i_\gamma, i_\alpha) C(I_\alpha I_\beta \bar{I}^\gamma; i_\alpha, i_\beta) C(\bar{I}^\gamma I_\gamma I^\gamma; i_\alpha + i_\beta, i_\gamma) \\
&= \delta_{I^\alpha I^\gamma}\delta_{i^\alpha i^\gamma} (-)^{I_\alpha + \bar{I}^\alpha - I^\alpha} \sum_{i_\alpha, i_\beta, i_\gamma} C(I_\beta I_\gamma \bar{I}^\alpha; i_\beta, i_\gamma) C(I_\alpha \bar{I}^\alpha I^\alpha; i_\alpha, i_\beta + i_\gamma) C(I_\alpha I_\beta \bar{I}^\gamma; i_\alpha, i_\beta) C(\bar{I}^\gamma I_\gamma I^\gamma; i_\alpha + i_\beta, i_\gamma) \\
&\equiv \delta_{I^\alpha I^\gamma}\delta_{i^\alpha i^\gamma} (-)^{I_\alpha + \bar{I}^\alpha - I^\alpha} [2\bar{I}^\alpha + 1)(2\bar{I}^\gamma + 1)]^{1/2} W(I_\alpha I_\beta I^\alpha I_\gamma; \bar{I}^\gamma \bar{I}^\alpha).
\end{aligned} \tag{2.12}$$

In what follows, we shall always understand these states to be included in our representations.

III. ISOBAR MODEL

Let T_{23} be the scattering operator for the process $a+b\to\alpha+\beta+\gamma$. In the isobar model, one decomposes this operator into a linear sum of products of two operators:

$$T_{23} = \sum_\beta \frac{M^\beta T^\beta}{\Delta^\beta}, \quad \beta = 1, 2, 3. \tag{3.1}$$

The operator M^β describes the process $\gamma+\alpha\to\gamma+\alpha$ and in the context of the isobar model it is sometimes called the decay operator. The other operator T^β, on the other hand, describes the process $a+b\to\beta+(\gamma\alpha)$ and is often referred to as the production operator. The kinematical factor Δ^β is included for convenience and will be defined shortly.

We can now take the matrix element of (3.1). As we are primarily interested in the partial-wave amplitudes, we use the angular momentum representation. For the final state we may choose any one of the three equivalent representations, say, $|\alpha\rangle$. Then, indicating the initial angular momentum state by $|a\rangle$, we have

$$\begin{aligned}
\langle\alpha | T_{23} | a\rangle &= \sum_\beta \frac{\langle\alpha | M^\beta T^\beta | a\rangle}{\Delta^\beta} \\
&= \sum_\beta \sum_{\beta'} \int \frac{\langle\alpha|\beta'\rangle\rho'\langle\beta' | M^\beta T^\beta | a\rangle}{\Delta^\beta} ds' dW',
\end{aligned} \tag{3.2}$$

where we have inserted the unit operator implied by (2.5), with

$$\rho^\beta = \frac{Q^\beta q^\beta}{16 W^\beta (s^\beta)^{1/2}} \tag{3.3}$$

and the sum β' extending over all the discrete variables in the $|\beta\rangle$ representation. For brevity, we shall omit the superscript β wherever possible. Again, using the unit operator,

$$\langle\beta' | M^\beta T^\beta | a\rangle = \sum_{\beta''} \int \langle\beta' | M^\beta | \beta''\rangle \rho'' \langle\beta'' | T^\beta | a\rangle ds'' dW''. \tag{3.4}$$

Now the meaning of $\langle\beta' | M^\beta | \beta''\rangle$ is that

$$\begin{aligned}
\langle\beta' | M^\beta | \beta''\rangle &= \langle W' J' M' L' \Sigma' j' l' \bar{\sigma}' s'; I' i' \bar{I}' | M^\beta | W'' J'' M'' L'' \Sigma'' j'' l'' \bar{\sigma}'' s''; I'' i'' \bar{I}''\rangle \\
&= \frac{4W'}{Q'} \delta(W' - W'') \delta(s' - s'') \delta_{J' J''} \delta_{M' M''} \delta_{L' L''} \delta_{\Sigma' \Sigma''} \delta_{j' j''} \delta_{I' I''} \delta_{i' i''} \delta_{\bar{I}' \bar{I}''} M^{\beta(s')}_{j' \bar{I}' l' \bar{\sigma}' l'' \bar{\sigma}''},
\end{aligned} \tag{3.5}$$

that is, the matrix element describes the two-body elastic process $\alpha+\gamma\to\alpha+\gamma$. For the T^β term we have

$$\begin{aligned}\langle\beta''|T^\beta|a\rangle &= \langle W''J''M''L''\Sigma''j''l''\bar{\sigma}''s'';I''i''\bar{I}''|T^\beta|WJMl\sigma;Ii\rangle\\ &= \delta(W''-W)\delta_{J''J}\delta_{M''M}\delta_{I''I}\delta_{i''i}T^{\beta(W,s'')}_{IJl\sigma;L''\Sigma''j''l''\bar{\sigma}''\bar{I}''}\,.\end{aligned}\tag{3.6}$$

A similar expression holds for the left-hand side of (3.2). After substituting (3.3) through (3.6) and using (2.9) and (2.12) to replace $\langle\alpha|\beta'\rangle$, one can carry out the sums and integrals in (3.2) utilizing the δ functions to get

$$T_{23}(W,s^\alpha)=\sum_{\beta,\beta',\beta''}\int\left(\frac{4W}{Q'}\right)\left(\frac{\rho'^2}{\Delta^\beta}\right)\langle\alpha|\beta'\rangle M^\beta_{\beta'\beta''}(s')T^\beta_{\beta''}(W,s')ds'\,,\tag{3.7}$$

where, for brevity, the notation is

$$\begin{aligned}&T_{23}(W,s^\alpha)\equiv T_{23}(W,s^\alpha)_{L^\alpha\Sigma^\alpha j^\alpha l^\alpha\bar{\sigma}^\alpha\bar{I}^\alpha;IJl\sigma}\,,\\ &M^\beta_{\beta'\beta''}(s')\equiv M^\beta_{\bar{I}'j'l'\bar{\sigma}'l''\bar{\sigma}''}(s')\,,\\ &T^\beta_{\beta''}(W,s')=T^\beta_{\bar{I}'L'\Sigma'j'l''\bar{\sigma}'';IJl\sigma}(W,s')\,,\\ &\langle\alpha|\beta'\rangle=(2.9)\times(2.12),\quad\text{with }\gamma\to\beta'\text{ and excluding the }\delta\text{ functions }\delta(W^\alpha-W')\delta_{J^\alpha J'}\delta_{M^\alpha M'}\delta_{I^\alpha I'}\delta_{i^\alpha i'}\,,\\ &\sum_{\beta',\beta''}=\sum_{j'l'\bar{\sigma}'\bar{I}'L'\Sigma';l''\bar{\sigma}''}\,.\end{aligned}\tag{3.8}$$

We now choose

$$\Delta^\beta=\frac{q'}{4(s')^{1/2}}\tag{3.9}$$

so that $(\rho')^2(4W/Q')1/\Delta^\beta=\rho'$, and we have

$$T_{23}(W,s^\alpha)=\sum_{\beta,\beta',\beta''}\int\langle\alpha|\beta'\rangle M^\beta_{\beta'\beta''}(s')T^\beta_{\beta''}(W,s')\rho'ds'\,.\tag{3.10}$$

This is the basic expression for the total partial-wave amplitude in terms of the production and decay amplitudes. The decay amplitude $M^\beta_{\beta'\beta''}$ is usually a known function so that the production parameters $T^\beta_{\beta''}$ can be determined by using (3.10) in the expression for cross section (which we shall not go into). In the rest of the paper we shall be primarily interested in $\pi N\to\pi\pi N$ for which we have, when conservation of parity is taken into account,

$$\bar{\sigma}'=\bar{\sigma}''\,,$$

$$l'=l''\,,$$

so that the β'' label becomes superfluous and will be dropped from now on.

The parameters T^β are functions of continuous variables W and s^β. In order to further simplify the task of fitting the data, it has been customary to approximate T^β by a threshold factor times another parameter which is independent of subenergy:

$$T^\beta_{\beta'}(W,s')\approx f^\beta_{\beta'}(W,s')\tilde{A}^\beta_{\beta'}(W)\,.\tag{3.11}$$

We shall call this "minimal approximation." Equation (3.10) now becomes

$$T_{23}(W,s^\alpha)=\sum_{\beta\beta'}\tilde{A}^\beta_{\beta'}(W)\int\langle\alpha|\beta'\rangle M^\beta_{\beta'}(s')f^\beta_{\beta'}(W,s')\rho'ds'\,.\tag{3.12}$$

With a suitable choice of barriers f, the integral can now be carried out to obtain

$$T_{23}(W,s^\alpha)=\sum_{\beta\beta'}F^{\alpha\beta}_{\beta'}(W,s^\alpha)\tilde{A}^\beta_{\beta'}(W)\,,\tag{3.13}$$

where

$$F^{\alpha\beta}_{\beta'}(W,s^\alpha)=\int f^\beta_{\beta'}(W,s^{\beta'})[\langle\alpha|\beta'\rangle M^\beta_{\beta'}(s^{\beta'})\rho^{\beta'}]ds^{\beta'}\,.\tag{3.14}$$

Index β', signifying the sum over different isobars in the $|\beta\rangle$ representation, will henceforth be absorbed in the index β. Expression (3.13) is a direct outcome of (3.1) and (3.11). It contains the recoupling coefficients explicitly whose presence is due to the fact that we have expressed the entire amplitude T_{23} in one final-state representation $|\alpha\rangle$. Indeed, if we carry out the partial-wave expansion of Eq. (3.1), we get

$$\langle f|T_{23}|i\rangle = \sum_{\alpha,a} \int \langle f|\alpha\rangle \rho^{\alpha} \langle\alpha|T_{23}|a\rangle \rho_a \langle a|i\rangle ds^{\alpha}, \tag{3.15}$$

where the sum and integral are over the relevant variables in the two- and three-particle states, and the transformation functions $\langle f|\alpha\rangle$ and $\langle a|i\rangle$ are as given by (2.8) and the complex conjugate of (2.7), respectively. Then, making use of (3.10) and (3.11) in the above, we get (omitting the isospin part)

$$\langle f|T_{23}|i\rangle = \sum_{\alpha,a} \int \langle f|\alpha\rangle \rho^{\alpha} \sum_{\beta} \int \langle\alpha|\beta\rangle M^{\beta}(s^{\beta}) T^{\beta}(W,s^{\beta}) \rho^{\beta} ds^{\beta} \rho_a \langle a|i\rangle ds^{\alpha} \tag{3.16}$$

$$= \sum_{a} \rho_a \langle a|i\rangle \sum_{\beta} \int \langle f|\beta\rangle M^{\beta}(s^{\beta}) f^{\beta}(W,s^{\beta}) \tilde{A}^{\beta}(W) \rho^{\beta} ds^{\beta} \tag{3.17}$$

$$\begin{aligned} = \sum_{\alpha=1}^{3} \sum_{\substack{L^{\alpha}\Sigma^{\alpha}j^{\alpha}l^{\alpha}\bar{\sigma}^{\alpha} \\ Jl\sigma}} \sum_{m^{\alpha},\nu_{\beta},\nu_{\gamma},M} & C(\sigma_a\sigma_b\sigma;\mu_a\mu_b) C(l\sigma J; M-(\mu_a+\mu_b), \mu_a+\mu_b) C(\sigma_{\beta}\sigma_{\gamma}\bar{\sigma}^{\alpha};\nu_{\beta}\nu_{\gamma}) \\ & \times C(l^{\alpha}\bar{\sigma}^{\alpha}j^{\alpha}; m^{\alpha}-(\nu_{\beta}+\nu_{\gamma}), \nu_{\beta}+\nu_{\gamma}) C(j^{\alpha}\sigma_{\alpha}\Sigma^{\alpha}; m^{\alpha}\mu_{\alpha}) \\ & \times C(L^{\alpha}\Sigma^{\alpha}J; M-(m^{\alpha}+\mu_{\alpha}), m^{\alpha}+\mu_{\alpha}) Y^{(\omega^{\alpha})}_{l^{\alpha},m^{\alpha}-(\nu_{\beta}+\nu_{\gamma})} \\ & \times Y^{(\Omega^{\alpha})}_{L^{\alpha},M-(m^{\alpha}+\mu_{\alpha})} Y^{*(\omega)}_{l,M-(\mu_a+\mu_b)} D^{\sigma_{\beta}}_{\mu_{\beta}\nu_{\beta}}(\xi^{\alpha}_{\beta}) D^{\sigma_{\gamma}}_{\mu_{\gamma}\nu_{\gamma}}(\xi^{\alpha}_{\gamma}) M^{\alpha}(s^{\alpha}) T^{\alpha}(W,s^{\alpha}). \end{aligned} \tag{3.18}$$

Thus Eq. (3.16) with the recoupling coefficient in it is equivalent to Eq. (3.17) which does not contain that term. Because of this reason, the latter is used in the analysis. However, we shall find later that Eq. (3.16) is more suitable for comparison with the unitarized amplitude. Equation (3.18), apart from an overall energy-momentum δ function, is our expression for partial-wave decomposition of the total amplitude. It is entirely in the canonical representation and differs from, for example, the Berkeley-SLAC version[10] in that their spin states are the helicity states. For details see Ref. 7. It should be noted that our procedure for introducing subenergy unitarity [Eqs. (4.7) and (4.8) below] does not, of course, depend upon the specific representation chosen.

The shortcoming of the model lies in assuming that the reduced amplitudes $\tilde{A}$ introduced in (3.11) are independent of the subenergy variables. We therefore concentrate on this problem in the next section.

IV. UNITARITY CONSTRAINTS

For the amplitude $a+b \to \alpha+\beta+\gamma$, we shall be primarily interested in the normal threshold singularities in the three-particle subenergy variables s^{α}. For a given subenergy variable, say s^{α}, we have the discontinuity[11] as given in Fig. 3(a), where we have suppressed the signs of the total energy W and the two subenergies s^{β}, s^{γ} which should be fixed at the same values in both amplitudes on the left-hand side, say, $+++$. For the $2\to 3$ bubble on the right-hand side, only $W(+)$ and $s^{\alpha}(-)$ can be specified; since s^{β} and s^{γ} are integration variables they carry a more complicated prescription. For details we refer to Section 4.7 of Ref. 11. Similar expressions can be written down for discontinuities in s^{β} and s^{γ} and the three expressions can be added. The total discontinuity due to subenergy variables is then given by Fig. 3(b). It will also be useful to define the usual two-particle K matrix by Figs. 3(c) and 3(d).

Let us now introduce a reduced amplitude J as in Fig. 3(e) and show that it is free from subenergy discontinuities. Toward this end, we continue the equation in Fig. 3(e) around the subenergy thresholds and let $J \to I$, thus obtaining the result shown in Fig. 3(f), where the minus sign is a consequence of the two-particle phase space. Now, subtracting Fig. 3(f) from Fig. 3(e), we get Fig. 3(g). Furthermore, from Figs. 3(d) and 3(c) we can write down the equations of Figs. 3(h) and 3(i), respectively. Then substitution of these last two results in Fig. 3(g) yields the equation of Fig. 3(j) which, in view of Fig. 3(b), implies that $J=I$, i.e., J has no subenergy discontinuities.

Next, following Smadja,[12] we go a step further and take the $2\to 3$ amplitudes + and J to be of the form given in Figs. 3(k) and 3(l) where division by the two-particle phase-space $\Delta^{\alpha} = q^{\alpha}/4(s^{\alpha})^{1/2}$ ensures the required smoothness of J, so that the equation of Fig. 3(e) can be written as shown in Fig. 3(m), where we used Fig. 3(c) in the last step. Cancellation of the left-hand side with the second term on the right-hand side yields Fig. 3(n) whose one possible solution is indicated in Fig. 3(o).

Decomposition in Fig. 3(k) is similar to the one

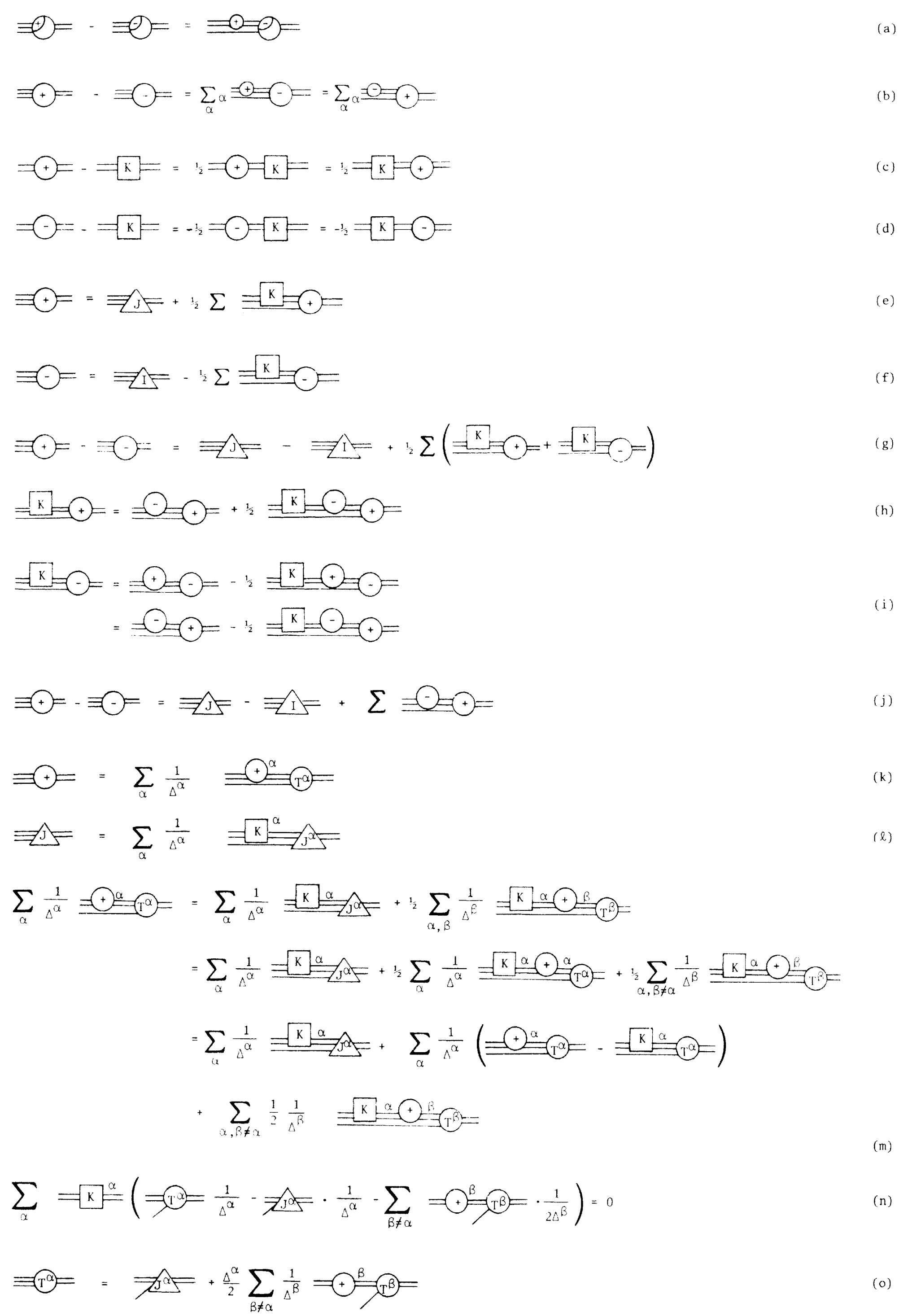

FIG. 3. Development of the unitarity constraints. See Sec. IV for discussion.

used in (3.1). The equation in Fig. 3(o) is a set of coupled integral equations which relates each production amplitude T^α to other amplitudes T^β, $\beta \neq \alpha$. The term J^α is free from subenergy discontinuity and hence represents T_α in the isobar-model approximation. The integral term provides the required correction to the model.

In the terminology of Sec. III, Fig. 3(o) reads

$$T^\alpha = J^\alpha + \frac{i\Delta^\alpha}{2}\sum_{\beta\neq\alpha}\frac{M^\beta T^\beta}{\Delta^\beta} \tag{4.1}$$

and can be written in the angular momentum representation by a procedure similar to the one used in obtaining (3.10) from (3.1):

$$T^\alpha(W,s^\alpha) = J^\alpha(W,s^\alpha) + \frac{i\Delta^\alpha}{2}\sum_{\beta\neq\alpha}\int\langle\alpha|\beta\rangle M^\beta(s^\beta) \times T^\beta(W,s^\beta)\rho^\beta ds^\beta. \tag{4.2}$$

The above can be written in a more compact form[13]

$$T = J + \mathcal{K}T \tag{4.3}$$

and can be formally solved to yield

$$T - \mathcal{K}T = J$$

or

$$T = (1-\mathcal{K})^{-1}J \equiv HJ. \tag{4.4}$$

We shall refer to H as the mixing matrix.

To deal with the barriers, we set

$$J^\alpha(W,s^\alpha) \approx f^\alpha(W,s^\alpha)\tilde{J}^\alpha(W), \tag{4.5}$$

$$T^\alpha(W,s^\alpha) \approx f^\alpha(W,s^\alpha)\tilde{T}^\alpha(W,s^\alpha). \tag{4.6}$$

Notice that s^α is retained in $\tilde{T}^\alpha$, thus distinguishing it from $\tilde{A}^\alpha$ of Eq. (3.11), but not in $\tilde{J}^\alpha$ which we assume to be constant over the Dalitz plot. This assumption, however, is not crucial to our analysis; that is, we could use a series expansion in s^α for $\tilde{J}^\alpha(W,s^\alpha)$ at the expense, of course, of additional parameters to be determined by the data. Substitution of (4.5) and (4.6) into (4.2) gives

$$\tilde{T}^\alpha(W,s^\alpha) = \tilde{J}^\alpha(W) + \frac{i}{2}\frac{\Delta^\alpha}{f^\alpha(W,s^\alpha)}\sum_{\alpha\neq\beta}\int\langle\alpha|\beta\rangle M^\beta(s^\beta)f^\beta(W,s^\beta)\tilde{T}^\beta(W,s^\beta)\rho^\beta ds^\beta \tag{4.7}$$

or

$$\tilde{T} = \tilde{J} + \tilde{\mathcal{K}}\tilde{T},$$

which again implies a new mixing matrix through

$$\tilde{T} = (1-\tilde{\mathcal{K}})^{-1}\tilde{J} \equiv \tilde{H}\tilde{J}. \tag{4.8}$$

Furthermore, since T^α and J^α are related to $\tilde{T}^\alpha$ and $\tilde{J}^\alpha$, we can derive a relation between H and $\tilde{H}$. Using (4.4) and (4.8) in (4.6), we have

$$\sum_\beta\int H^{\alpha\beta}(W,s^\alpha,s^\beta)J^\beta(W,s^\beta)ds^\beta = f^\alpha(W,s^\alpha)\sum_\beta\int\tilde{H}^{\alpha\beta}(W,s^\alpha,s^\beta)\tilde{J}^\beta(W)ds^\beta.$$

Putting (4.5) in the left-hand side of this equation,

$$\sum_\beta\int H^{\alpha\beta}(W,s^\alpha,s^\beta)f^\beta(W,s^\beta)\tilde{J}^\beta(W)ds^\beta = f^\alpha(W,s^\alpha)\sum_\beta\int\tilde{H}^{\alpha\beta}(W,s^\alpha,s^\beta)\tilde{J}^\beta(W)ds^\beta$$

or

$$\sum_\beta\tilde{J}^\beta(W)\left[\int H^{\alpha\beta}(W,s^\alpha,s^\beta)f^\beta(W,s^\beta)ds^\beta - f^\alpha(W,s^\alpha)\int\tilde{H}^{\alpha\beta}(W,s^\alpha,s^\beta)ds^\beta\right] = 0.$$

Since the $\tilde{J}^\beta$'s are linearly independent parameters, we get

$$\int\tilde{H}^{\alpha\beta}(W,s^\alpha,s^\beta)ds^\beta = \frac{1}{f^\alpha(W,s^\alpha)}\int H^{\alpha\beta}(W,s^\alpha,s^\beta)f^\beta(W,s^\beta)ds^\beta. \tag{4.9}$$

This result can now be incorporated into (4.8), giving

$$\tilde{T}^\alpha(W,s^\alpha) = \sum_\beta\tilde{H}^{\alpha\beta}\tilde{J}^\beta ds^\beta = \sum_\beta\tilde{J}^\beta\int\tilde{H}^{\alpha\beta}ds^\beta = \frac{1}{f^\alpha}\sum_\beta\tilde{J}^\beta\int H^{\alpha\beta}f^\beta ds^\beta, \tag{4.10}$$

where the barriers are explicit. Calculation of H, in contrast to $\tilde{H}$, does not require knowledge of the barrier factors which are somewhat arbitrary. Equation (4.10) is our solution of the unitarity equations (4.7).

V. ISOBAR MODEL AND UNITARITY

The production amplitude $T^\alpha(W,s^\alpha)$ of Eq. (4.2) satisfies subenergy unitarity, and since it reduces to the isobar amplitude in the absence of unitary corrections it would be most tempting to replace $\tilde{A}^\beta$ in (3.17) by $\tilde{T}^\beta$ of (4.10) to unitarize the isobar amplitude:

$$\langle f|T_{23}|i\rangle=\sum_a \rho_a\langle a|i\rangle\sum_\beta\int\langle f|\beta\rangle M^\beta(s^\beta)f^\beta(W,s^\beta)\frac{1}{f^\beta(W,s^\beta)}\sum_\alpha \tilde{J}^\alpha(W)\int H^{\beta\alpha}(W,s^\beta,s^\alpha)f^\alpha(W,s^\alpha)ds^\alpha\rho^\beta ds^\beta$$

$$=\sum_a \rho_a\langle a|i\rangle\sum_{\alpha\beta}\tilde{J}^\beta(W)\int\int\langle f|\alpha\rangle M^\alpha(S^\alpha)f^\beta(W,s^\beta)H^{\alpha\beta}(W,s^\alpha,s^\beta)\rho^\alpha ds^\alpha ds^\beta. \tag{5.1}$$

This sort of unitarization of the isobar model, however, would lead to spurious effects since analyticity has not been included. As Aitchison has pointed out,[14] neglect of analyticity tends to produce some unwanted rapid variation of T^α with respect to s^α. This is especially the case when M^β of Eq. (4.2) represents a rather narrow resonance.[15] Aaron and Amado have recently proposed a formalism that includes analyticity as well as unitarity.[16]

The present formalism, however, can be used to test if unitarity corrections will be important. To obtain this test, it is more convenient to work with Eq. (3.16) and include unitarity through Eq. (4.10) than to compare the above equation with the isobar model. Thus,

$$\langle f|T_{23}|i\rangle=\sum_a \rho_a\langle a|i\rangle\sum_\alpha\int\langle f|\alpha\rangle\rho^\alpha ds^\alpha\sum_\beta\int\langle\alpha|\beta\rangle M^\beta\rho^\beta f^\beta\frac{1}{f^\beta}\sum_\gamma\tilde{J}^\gamma\int H^{\beta\gamma}f^\gamma ds^\gamma ds^\beta$$

$$=\sum_a \rho_a\langle a|i\rangle\sum_\alpha\int\langle f|\alpha\rangle\rho^\alpha ds^\alpha\sum_\gamma\int\langle\alpha|\gamma\rangle M^\gamma\rho^\gamma\sum_\beta\tilde{J}^\beta\int H^{\gamma\beta}f^\beta ds^\beta ds^\gamma$$

$$\equiv\sum_a \rho_a\langle a|i\rangle\sum_\alpha\int\langle f|\alpha\rangle\rho^\alpha ds^\alpha\sum_\beta\tilde{J}^\beta G^{\alpha\beta}(W,s^\alpha), \tag{5.2}$$

with

$$G^{\alpha\beta}(W,s^\alpha)=\int f^\beta\Big[\sum_\gamma\langle\alpha|\gamma\rangle M^\gamma\rho^\gamma H^{\gamma\beta}ds^\gamma\Big]ds^\beta. \tag{5.3}$$

To recast the isobar amplitude, we use (3.13) in (3.15):

$$\langle f|T_{23}|i\rangle=\sum_a \rho_a\langle a|i\rangle\sum_\alpha\int\langle f|\alpha\rangle\rho^\alpha ds^\alpha\sum_\beta\tilde{A}^\beta F^{\alpha\beta}(W,s^\alpha). \tag{5.4}$$

Expressions (5.2) and (5.4) are now similar; their only difference comes from the F and G functions. Indeed, if the mixing matrix H is weak, we can write it as

$$H^{\gamma\beta}(W,s^\gamma,s^\beta)\approx\delta_{\gamma\beta}\delta(s^\gamma-s^\beta) \tag{5.5}$$

and easily verify that $G\approx F$. Thus, the effective strength of mixing may be defined by a ratio of the two functions. Following our practice of separating out the barrier factors, we can set

$$R^{\alpha\beta}(W,s^\alpha,s^\beta)=\frac{\sum_\gamma\int\langle\alpha|\gamma\rangle M^\gamma H^{\gamma\beta}\rho^\gamma ds^\gamma}{\langle\alpha|\beta\rangle M^\beta\rho^\beta}, \tag{5.6}$$

which is the ratio of the bracket terms in (5.3) and (3.14). The full ratio R is, on the other hand,

$$R^{\alpha\beta}(W,s^\alpha)=\frac{G^{\alpha\beta}(W,s^\alpha)}{F^{\alpha\beta}(W,s^\alpha)} \tag{5.7}$$

and includes the barrier terms in it. If these ratios are much different from unity or their subenergy dependence is appreciable or rapid, unitarity corrections will be important.

VI. CONCLUSION

We have presented the formalism of the isobar model and the subenergy unitarity constraints in a systematic manner with sufficient details. For the most part, the results derived here are quite general and can be applied to many reactions of interest of the type $a+b\to 1+2+3$. There are several versions of the three-body partial-wave analysis as described in Ref. 2; the one used here corresponds to the Berkeley-SLAC version in all but one respect: we use canonical, instead of helicity, representation.

We have dealt with only the subenergy discontinuities here; other discontinuities, in the total energy, arising from the two- and three-particle intermediate states have been removed by many authors.[17,18] Our main results are the set of coupled integral equations for production amplitudes, (4.7), their formal solution (4.8), and the ratio (5.6) or (5.7). The method presented here essentially involves the calculation of the mixing matrix H.

We have paid special attention to the handling of the barrier factors. Pulling them out from the production amplitudes will involve them in the mixing matrix. Our preliminary results indicate that the mixing matrix can be quite sensitive to small changes in the barrier factors. For this reason we have tried to separate them out as far as possible.

The test suggested in Sec. V should help determine if the unitarity effects will be important for the isobar model. If the H matrix is roughly diagonal, unitarity corrections are not necessary. If it is not diagonal, then one must see how their mixing actually modifies the isobar amplitude. This is the motivation for the ratio test. An important feature of this test is that it can be carried out before any fit is performed, i.e., it does not depend on any fitting parameters at all. Thus it provides an answer to the question often asked: How much is the overlap between two given isobars? Unitarity as presented here tends to overestimate the overlaps; only analyticity can remove this difficulty. But if the overlaps are small, one need not embark upon a full program of unitarity and analyticity.

Finally, we have left out the important discussion on identical particles in the final state. The kernels $\mathcal{K}$ have a certain symmetry property with respect to the interchange of identical particles. This, along with the use of properly symmetrized amplitudes, enables us to reduce the number of independent integral equations. This and other related topics are discussed in Ref. 7.

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