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Low Energy Pion Physics

– Completely revised and updated version –

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Abstract

In these notes we present an introductory review on various topics about low energy pion physics (some kaon physics is discussed as well). Among these, we include the uses of analyticity and unitarity to describe partial wave amplitudes (for which we give accurate and economical parametrizations) and form factors; (forward) dispersion relations; and the use of the Froissart–Gribov representation to evaluate accurately the low energy parameters (scattering lengths and effective ranges) for higher ($l \geq 1$) waves. Finally, we describe some pion physics in QCD and then pass on to study the nonlinear sigma model, and the chiral perturbation theory approach to low energy pion interactions.

Most of the results presented here are well known, but we also give a set of state of the art, precise determinations of some scattering lengths as well as an independent calculation of three of the parameters $\bar{l}_2$, $\bar{l}_4$, $\bar{l}_6$ (to one loop), and one $\bar{f}_2$, to two loops, that appear in chiral perturbation theory. The S waves are discussed and compared with chiral perturbation theory expectations, and the same is done for larger l scattering lengths and effective range parameters. Also new is the evaluation of some electromagnetic corrections.

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1. Introduction

1.1. Foreword

The matter of parametrizations and uses of pion-pion partial waves (p.w.), form factors and correlators, in particular in connection with resonances, received a great deal of attention in the late fifties, sixties and early seventies of last century –until QCD emerged as the theory of strong interactions and such studies were relegated to a secondary plane. In recent times a renewed interest has arisen in this subject and this due to, at least, the following reasons. One is the popularity of chiral perturbation theory calculations (to which the last part of these notes is devoted), in particular of low energy $\pi\pi$ parameters: scattering lengths and ranges, pion charge radius, etc. A second reason is the use of low energy calculations of the pion form factor to get precise estimates of the muon magnetic moment or the value of the QED charge on the Z particle. And last, but certainly not least, we have the appearance of new experimental data on hadronic τ decay, the pion electromagnetic form factor and on K_{e4} decay. The existence of these data allow a much improved determination of low energy pionic observables.

Unfortunately, some of the old lore appears to be lost and indeed many modern calculators seem to be unaware of parts of it. In the present review we do not present much new knowledge, but mostly intend to give an introductory, easily accessible reference to the studies of scattering amplitudes, form factors and correlators involving pions in the low energy region: our aim is, in this respect, mainly pedagogical.

Nevertheless, some very recent results are reported. These include parametrizations of the lowest waves (S, P, D, F) in $\pi\pi$ scattering which are compatible with analyticity and unitarity and, of course, experimental data, depending only on a few (two to four) parameters per wave. This is used to evaluate forward dispersion relations and the Froissart–Gribov representation of the P, D and higher waves. From this there follow very precise determinations of the corresponding scattering lengths and effective range parameters. Using this, as well as the results on the P wave following from the electromagnetic pion form factor and the decay $\tau \to \nu\pi^0\pi^+$, we obtain, in particular, a precise determination (however, only to one loop) of some of the $\bar{l}$ parameters in chiral perturbation theory, as well as an evaluation of some electromagnetic corrections. The scalar radius of the pion is another example.

The plan of this review is as follows. In Chapters 1 and 2 we describe briefly the analyticity, unitarity and high energy properties of various quantities (correlators, form factors, partial waves and scattering amplitudes). The elements of the effective range formalism and the characterization of resonances are given in Chapter 3. These topics are illustrated in a simple model in Chapter 4, while in Chapter 5 we extend the previous analyses (including the requirements of unitarity) to the multichannel case.

The core of the review is contained in the last four chapters. In chapters 6 and 7 we apply the tools described before to the study of partial wave amplitudes and scattering amplitudes for $\pi\pi$ scattering and to fit the pion form factors, electromagnetic and scalar. Here we also implement simple parametrizations of partial wave amplitudes consistent with analyticity and unitarity, and fitting experimental data; this should be useful to people needing manageable representations of $\pi\pi$ phases, as happens e.g. for $J/\psi \to \gamma\pi\pi$ studies. Then we discuss (Chapter 7) how the various theoretical requirements (fixed t dispersion relations and the Froissart–Gribov representation) may be used to check compatibility of the results found with crossing symmetry and analyticity for the

scattering amplitudes. The Froissart–Gribov representation is also used to get precise determinations of low energy parameters for the waves with $l = 1$ and higher. With respect to form factors, the Omnès–Muskhelishvili method is employed to perform an accurate fit to the pion form factor, obtaining in particular precise values of the corresponding low energy parameters, and also to give a reliable determination of the scalar radius of the pion. Something which is missing in this review is the Roy equations analysis; there are in the literature three recent papers (Ananthanarayan et al., 2001, Colangelo, Gasser and Leutwyler, 2001, Descotes et al., 2002) that fill this gap. Some of the results we obtain are summarized in Sect. 7.6 where, in particular, we present a discussion of the results of Descotes et al. (2002) and, especially, Ananthanarayan et al. (2001) and Colangelo, Gasser and Leutwyler (2001); in particular, in view of the criticism of the last by Peláez and Ynduráin (2003a), Ynduráin (2003b).

In Chapter 8 we remember that pions are made of quarks, and that we have a theory for the interactions of these, QCD. We discuss invariance properties of the QCD Lagrangian, in particular chiral invariance that plays a key role for the dynamics of pions. We use this and PCAC to derive relations between the masses of the quarks and the pion and kaon masses, and to study pion decay. Finally, in Chapter 9 we develop the consistent description of pion dynamics based on chiral invariance, known as chiral perturbation theory. In the last sections of this chapter we use the results obtained in Chapters 6 and 7 to test the predictions of chiral perturbation theory, and show how to obtain values for the parameters on which it depends.

Before entering into the main body of these notes, it is convenient to clarify what is to be understood as "low energy." Above energies $s^{1/2}$ of, say, $1.3 \sim 2$ GeV, perturbative QCD (or Regge theory, as the case may be) is applicable; we will be very little concerned with these energies. At very low energies, $s^{1/2} \ll \Lambda_0$, where Λ_0 is a scale parameter that (depending on the process) may vary from $\simeq 600$ MeV to $4\pi f_\pi \sim 1.1$ GeV, chiral perturbation theory is applicable; this we treat in detail in Chapters 8 and 9. Between the two energy scales, analyticity and unitarity allow at least an *understanding* of pionic observables. This understanding certainly holds until inelastic production begins to become important. This means that we are able to cover the energy range of $s^{1/2}$ below 1.42 GeV.

These notes are primarily about pions. However, in some cases kaons and (to a lesser extent) etas are treated as well.

1.2. Normalization; kinematics; isospin

1.2.1. Conventions

Before entering into specific discussions we will say a few words on our normalization conventions.[1] If S is the relativistically invariant scattering matrix we define the scattering amplitude F for particles A, B to give particles C_i by

$$\langle C_1, \ldots, C_n|S|A, B\rangle = \mathrm{i}\delta(P_f - P_i)F(A + B \to C_1 + \cdots + C_n). \tag{1.2.1}$$

We take the states to be normalized in a relativistically invariant manner: if p is the four-momentum, and λ the helicity of a particle, then

$$\langle p, \lambda|p', \lambda'\rangle = 2\delta_{\lambda\lambda'}p_0\delta(\mathbf{p} - \mathbf{p}'). \tag{1.2.2}$$

[1] We assume here a basic knowledge of S matrix theory, in particular of crossing symmetry or partial wave expansions, and of isospin invariance, at the level of the first chapters of the texts of Martin, Morgan and Shaw (1976) or Pilkuhn (1967).

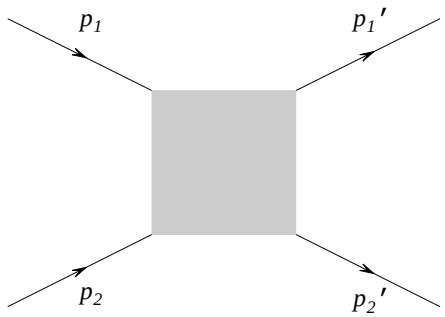

FIGURE 1.1.1. Two particle scattering.

We will seldom consider particles with spin in these notes. Particles with spin pose problems of their own; the generalization of our discussions to spinning particles is not trivial.

It is the function F, defined as in (1.2.1), with the states normalized as in (1.2.2), the one which is free of kinematical singularities and zeros. That is to say, any discontinuity or pole of F is associated with dynamical effects. If we had used a nonrelativistic normalization and defined a corresponding scattering amplitude $T_{\rm NR}$, we could write

$$T_{\rm NR} = \frac{1}{\sqrt{2p_A^0}} \frac{1}{\sqrt{2p_B^0}} \frac{1}{\sqrt{2p_{C_1}^0}} \dots \frac{1}{\sqrt{2p_{C_n}^0}} F. \tag{1.2.3}$$

Then, no matter which field-theoretic interaction we assumed, $T_{\rm NR}$ would show the branch cuts associated with the factors $1/\sqrt{p^0}$ in (1.2.3).

In what regards form factors care has to be exercised to get form factors without kinematic cuts. For the simple case of the e.m. (electromagnetic) form factor of the pion (or any other spinless particle) such form factor is that defined by

$$\langle p_1 | J_\mu^{\rm e.m.}(0) | p_2 \rangle = (2\pi)^{-3} (p_1 + p_2)_\mu F_\pi(t), \quad t = (p_1 - p_2)^2. \tag{1.2.4a}$$

Note that, with this definition, $F_\pi(0) = 1$. Eq. (1.2.4a) is valid for spacelike $t \leq 0$. For timelike $t \geq 4\mu^2$ we write

$$\langle p_1, p_2 | J_\mu^{\rm e.m.}(0) | 0 \rangle = (2\pi)^{-3} (p_1 - p_2)_\mu F_\pi(t), \quad t = (p_1 + p_2)^2. \tag{1.2.4b}$$

Both values of F_π are particular cases of a single function, $F_\pi(t)$, that can be defined for arbitrary, real or complex values of the variable t (see below).

We finish this subsection with a few more definitions. Let us consider scattering of two particles, with masses m_i (Fig. 1.1.1):

$$A_1(p_1) + A_2(p_2) \to A_1'(p_1') + A_2'(p_2').$$

We define the Mandelstam variables

$$s = (p_1 + p_2)^2, \quad u = (p_1 - p_2')^2, \quad t = (p_1 - p_1')^2.$$

They satisfy the equality, when the particles are on their mass shells,

$$s + u + t = \sum_i m_i;$$

for pions,

$$s + u + t = 4\mu.$$

Here, and throughout these notes, $\mu \equiv 138$ MeV is the *average* mass of the pions. When referring specifically to neutral or charged pion masses we will write m_{π^0} or $m_{\pi^\pm}$. The notation M_π for $m_{\pi^\pm}$ will also be used; see below.

In terms of these variables the modulus of the three-momentum, k, and the cosine of the scattering angle (both in the center of mass) are given by

$$k = \frac{\sqrt{s - 4\mu^2}}{2}, \quad \cos\theta = 1 + \frac{2t}{s - 4\mu^2}.$$

With our definitions, the two body differential cross section in the center of mass is given in terms of F as

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\text{c.m.}} = \frac{\pi^2}{4s}\frac{k'}{k}|F(i \to f)|^2, \tag{1.2.5}$$

with k, k' the moduli of the three-momenta of initial, final particles. For particles with arbitrary masses m_i (1.2.5) is still valid but now

$$k = \frac{1}{2s^{1/2}}\sqrt{[s - (m_1 - m_2)^2][s - (m_1 + m_2)^2]}$$

The total cross section, also with the same conventions, is

$$\sigma_{\text{tot}}(s) = \frac{4\pi^2}{\lambda^{1/2}(s, m_1, m_2)}\,\mathrm{Im}\,F(s, 0); \tag{1.2.6}$$

here we define Källén's quadratic form

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc = \left[a - (\sqrt{b} - \sqrt{c})^2\right]\left[a - (\sqrt{b} + \sqrt{c})^2\right].$$

1.2.2. Isospin

As we know, there are three kinds of pions: two charged ones, $\pi^{\pm}$ with a mass $m_{\pi^{\pm}} = 139.57$ MeV, and a neutral pion, with mass $m_{\pi^0} = 134.98$ MeV. If we neglected electromagnetic interactions, and the mass difference between u, d quarks, then the interactions of the three pions would be identical, and they would have a common mass, that we denote by μ and take equal to the average: $\mu = 138$ MeV. Unfortunately, and under the influence of Gasser and Leutwyler, it has become customary to use the mass of the charged pion as the common pion mass. We will (regretfully) follow this convention and will then write $M_\pi = 139.57$ MeV for the common mass of the pions, when identified with the charged pion mass. When distinguishing individual pion masses we will write, as stated above, $m_{\pi^{\pm}}$, m_{π^0}.

The invariance under rotations of the three pions, called *isospin invariance*, is best described by introducing a different basis to describe the pions, $|\pi_i\rangle$, $i = 1, 2, 3$, related to the physical pions by

$$|\pi^0\rangle = |\pi_3\rangle, \quad |\pi^{\pm}\rangle = \frac{\mp 1}{\sqrt{2}}\Big\{|\pi_1\rangle \pm \mathrm{i}|\pi_2\rangle\Big\}.$$

Isospin transformations are then just rotations, $|\pi_j\rangle \to \sum_k R_{jk}|\pi_k\rangle$ with R a rotation matrix. We can then, in the limit of exact isospin invariance, diagonalize the total isospin and its third component and consider e.g. scattering amplitudes corresponding to fixed isospin.

The development of the isospin formalism, including explicit expressions, may be found in the book of Martin, Morgan and Shaw (1976); here we only give, for ease of reference, the isospin crossing matrices:

$$C^{(ts)} = C^{(st)} = \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix}; \tag{1.2.7a}$$

$$C^{(us)}_{II'} = C^{(su)}_{II'} = (-1)^{I+I'} C^{(st)}_{II'}. \tag{1.2.7b}$$

These matrices act according to

$$F^{(I_s=I)} = \sum_{I'} C^{(st)}_{II'} F^{(I_t=I')}, \quad \text{etc.}$$

Note that the order is 0, 1, 2; thus, e.g.,

$$F^{(I_s=2)} = \tfrac{1}{3}F^{(I_t=0)} - \tfrac{1}{2}F^{(I_t=1)} + \tfrac{1}{6}F^{(I_t=2)}.$$

1.3. Field-theoretic, and other models

It is always convenient to illustrate abstract arguments with model calculations in which one can see how the general properties are realized in explicit examples. We will take as a very convenient model one in which pions and rho are realized as elementary fields, ϕ_a, ρ_a (a is an isospin index). The corresponding Lagrangian will be

$$\mathcal{L} = g_{\mu\nu}(\delta_{ab}\partial_\mu - \mathrm{i}g_\rho \epsilon_{abc}\rho^c_\mu)\phi_b(\delta_{ad}\partial_\nu - \mathrm{i}g_\rho\epsilon_{ade}\rho^e_\nu)\phi_d - \mu^2\phi_a\phi_a + \mathcal{L}_\rho. \tag{1.3.1}$$

Here μ is the pion mass and $\mathcal{L}_\rho$ is the pure rho Lagrangian, that need not be specified. The mass of the rho particle, M_ρ, can be assumed to be introduced by a Higgs-type mechanism, with the mass

of the associated Higgs particle so large that it will have no influence on calculations for energies of the order of M_ρ or lower.

The interactions in (1.3.1) induce pion-rho vertices: a $\pi\pi\rho$ vertex, which is associated with the factor $ig_\rho(p_1-p_2)_\mu\epsilon_{abc}$, and a seagull one proportional to $ig_\rho^2 g_{\mu\nu}$. This model is *not* chiral invariant (see later for a chiral invariant version) but it is very simple and thus will be used to illustrate general properties, such as analyticity or unitarity, independent of the underlying dynamics.

Another explicit model of $\pi\pi$ scattering is that of Veneziano (1968); see also Lovelace (1968) and Shapiro (1969). In it the amplitude is written as a combination of beta functions, thus as a sum of poles. These are then unitarized (e.g., à la Lovelace).

We will not spend any time on this model. While it gives reasonably well the masses and widths of the lowest lying resonances, it is a disaster for energies above ~ 1 GeV. It predicts elastic heavy resonances and, while its high energy behaviour contains the ρ and P' Regge poles, it does not admit a Pomeron.

2. Analyticity properties of scattering amplitudes, p.w. amplitudes, form factors and correlators. High energy behaviour and bounds

2.1. Scattering amplitudes and partial waves

Analyticity of partial waves follows from unitarity and causality. In local field theories (such as QCD) both properties are, of course, satisfied, but locality at least would be violated in a theory of strings; although this would occur at energies much higher than the ones in which we are interested here. In the case of the $\pi\pi$ scattering amplitude, $F(s,t)$, one can prove that it is, for t in the Martin–Lehmann ellipse,[2] analytic in the complex s plane with the exception of two cuts: a r.h. (right hand) cut, from $s = 4\mu^2$ to $+\infty$, and a l.h. (left hand) cut from $-\infty$ to $-t$. In addition, if there existed bound states, there would appear poles at the values of s or u given by the square of the mass of the bound state.

The p.w. (partial wave) amplitudes, $f_l(s)$, are related to F though the expansion,

$$F(s,t) = \sum_{l=0}^{\infty} (2l+1) P_l(cos\theta) f_l(s) \tag{2.1.1a}$$

with inverse

$$f_l(s) = \tfrac{1}{2} \int_{-1}^{+1} \mathrm{d}\cos\theta P_l(cos\theta) F(s,t). \tag{2.1.1b}$$

Here θ is the scattering angle in the center of mass, and P_l are the Legendre polynomials. We note that the restriction of, say, s to physical values produces the *physical* $F(s,t)$ and $f_l(s)$ *provided* we take the limit of s real from the upper half plane. That is to say, if s is real and physical (and so is t), the physical values of scattering amplitude and partial waves are obtained for

$$F(s,t) = \lim_{\epsilon \to +0} F(s + \mathrm{i}\epsilon, t); \quad f_l(s) = \lim_{\epsilon \to +0} f_l(s + \mathrm{i}\epsilon).$$

For elastic scattering at *physical* s (which, for $\pi\pi$ scattering means s real and larger than or equal to $4\mu^2$) and below the opening of the first inelastic threshold, that we will denote by s_0, one can express the f_l in terms of phase shifts:

$$f_l(s) = \frac{2s^{1/2}}{\pi k} \sin\delta_l(s) \mathrm{e}^{\mathrm{i}\delta_l(s)} = \frac{2s^{1/2}}{\pi k} \frac{1}{\cot\delta_l(s) - \mathrm{i}}; \quad 4\mu^2 \le s \le s_0. \tag{2.1.2}$$

The previous equations are valid assuming that the scattering particles are *distinguishable*. For $\pi\pi$ scattering, however, the situation is a bit complicated. One may still write (2.1.1) and (2.1.2)

[2] With foci at $t = 0$ and $t = 4\mu^2 - s$ (for pions) and right extremity at $t = 4\mu^2$. This includes the physical region, $4\mu^2 - s \le t \le 0$. For the proof, see Martin (1969) and references there.

for the processes $\pi^0\pi^+ \to \pi^0\pi^+$, but not for $\pi^0\pi^0 \to \pi^0\pi^0$ or $\pi^+\pi^+ \to \pi^+\pi^+$. The general recipe is the following: if $F^{(I_s)}$ is an amplitude with isospin I_s in channel s, one has to replace (2.1.1) by

$$\begin{aligned} F^{(I_s)}(s,t) &= 2 \times \sum_{l=\text{even}} (2l+1) P_l(\cos\theta) f_l^{(I_s)}(s), \quad I_s = \text{even}, \\ F^{(I_s)}(s,t) &= 2 \times \sum_{l=\text{odd}} (2l+1) P_l(\cos\theta) f_l^{(I_s)}(s), \quad I_s = \text{odd}, \\ f_l^{(I)}(s) &= \frac{2s^{1/2}}{\pi k} \sin\delta_l^{(I)}(s) \mathrm{e}^{\mathrm{i}\delta_l^{(I)}(s)}. \end{aligned} \tag{2.1.3}$$

Due to Bose statistics, even waves only exist with isospin $I = 0$, 2 and odd waves must necessarily have isospin $I = 1$. For this reason, we will often omit the isospin index for odd waves, writing e.g. f_1, f_3 instead of $f_1^{(1)}$, $f_3^{(1)}$.

Another very convenient simplification that we will use here is to denote the $\pi\pi$ partial waves by S0, S2, P, D0, D2, F, etc., in self-explanatory notation.

When inelastic channels open (2.1.2) is no more valid, but one can still write

$$f_l(s) = \frac{2s^{1/2}}{\pi k} \left[\frac{\eta_l \, \mathrm{e}^{2\mathrm{i}\delta_l} - 1}{2\mathrm{i}} \right]. \tag{2.1.4a}$$

Here η_l, called the *inelasticity parameter* for wave l, is positive and smaller than or equal to unity. The elastic and inelastic cross sections, for a given wave, are given in terms of δ_l and η_l by

$$\sigma_l^{\text{el.}} = \tfrac{1}{2} \left\{ \frac{1+\eta_l^2}{2} - \eta \cos 2\delta_l \right\}, \quad \sigma_l^{\text{inel.}} = \frac{1-\eta_l^2}{4}; \tag{2.1.4b}$$

$\sigma_l^{\text{el.}}$, $\sigma_l^{\text{inel.}}$ are defined so that, for collision of particles A, B (assumed distinguishable),

$$\sigma_{\text{tot.}} = \frac{4\pi^2}{\lambda^{1/2}(s, m_A, m_B)} \frac{2s^{1/2}}{\pi k} \sum_l (2l+1)[\sigma_l^{\text{el.}} + \sigma_l^{\text{inel.}}].$$

For $\pi\pi$ scattering $s_0 = 16\mu^2$, but the approximation of neglecting inelasticity is valid at the 2% level or better below $s \simeq 1$ GeV, the precise value depending on the particular wave.

The cut structure is more complicated for other processes. For example, for πK scattering the r.h. cut starts at $s = (\mu + m_K)^2$ and the l.h cut also begins at $u = (\mu + m_K)^2$. But, since now $s + u + t = 2\mu^2 + m_K^2$, the l.h. cut in the variable s runs from $-\infty$ to $-t + (m_K - \mu)^2$ for the scattering amplitude, and from $-\infty$ to $(m_K - \mu)^2$ for the p.w. amplitudes.

For $\bar{K}K \to \pi\pi$ or $\pi\pi \to \bar{K}K$ scattering, the u-channel is πK scattering. Therefore, the r.h. cut starts at the (unphysical) value $s = 4\mu^2$, and the l.h. cut at $u = (\mu + m_K)^2$, hence the l.h. cut runs from $-\infty$ to $-t + (m_K - \mu)^2$ for the scattering amplitude, and from $-\infty$ to $(m_K - \mu)^2$ for the p.w. amplitudes. Finally, for $\bar{K}K$ scattering, the l.h. cut runs, as for $\pi\pi$, from $-\infty$ to 0 for p.w. amplitudes; but there is a r.h. cut, both for the amplitude and for p.w.'s associated with the unphysical channel $\bar{K}K \to \pi\pi$, and starting at $4\mu^2$.

Note that from (2.1.1b) it follows that it is $f_l(s)$ that satisfies analyticity properties without kinematical zeros or singularities. These analyticity properties are rather complicated in general; in the simple case of $\pi\pi$ scattering we have that $f_l(s)$ is analytic in the complex s plane except for two cuts, one from $4\mu^2$ to $+\infty$ and another from $-\infty$ to 0 (Fig. 2.1.1), inherited respectively from

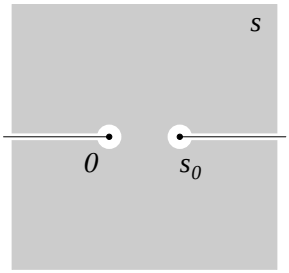

FIGURE 2.1.1.
The domain of analyticity for $\pi\pi$ partial waves (shadowed region) with the cuts of $f_l(s)$ in the complex s plane.

the r.h. and l.h. cuts of $F(s,t)$. If there existed bound states (which is not the case for $\pi\pi$), there would appear poles at the values of s given by the square of the mass of the bound state.

Another property that follows from (2.1.1b) plus the assumption (realized in the real world for $\pi\pi$ scattering) that there is no bound state with zero energy is the behaviour, as $k \to 0$,

$$f_l(s) \underset{k\to 0}{\simeq} \frac{4\mu}{\pi} k^{2l} a_l \tag{2.1.5}$$

where a_l is the so-called l-th wave *scattering length*.

Analyticity is seldom of any use without bounds. Again, on very general grounds one knows that, for t physical, $|F(s,t)|$ is bounded by $C|s|\log^2|s|$ (the Froissart bound). Specific behaviours, particularly those that hold in Regge theory, will be discussed below. For the proof of the Froissart and related bounds we require unitarity, causality and the assumption that Green's functions grow, at most, like polynomials of the momenta. This last assumption holds in renormalizable field theories, to all orders in perturbation theory; but may fail for nonrenormalizable ones. General discussions of analyticity, bounds and expected high energy behaviour of scattering amplitudes may be found in Eden, Landshoff, Olive and Polkinghorne (1966), where the analyticity properties of Feynman graphs are also discussed, Martin (1969), Barger and Cline (1969), Sommer (1970), Ynduráin (1972), etc. For general field theory, cf. Bogolyubov, Logunov and Todorov (1975).

2.2. Form factors

Analyticity of form factors, such as the pion or kaon form factors, can be proved quite generally using only causality and unitarity. In particular the electromagnetic pion form factor $F_\pi(t)$ turns out to be analytic in the complex t plane cut from $t = 4\mu^2$ to $+\infty$ (Fig. 2.2.1). This analyticity, in particular, provides the link between both definitions of F_π, Eqs. (1.2.4). For timelike, physical t, the physical value should in fact be defined as

$$F_\pi(t) = \lim_{\epsilon\to+0} F_\pi(t + \mathrm{i}\epsilon); \quad t \geq 4\mu^2;$$

if we had taken $\lim_{\epsilon\to+0} F_\pi(t - \mathrm{i}\epsilon)$ we would have obtained $F_\pi^*(t)$.

Unlike scattering amplitudes, for which bounds hold in *any* local field theory, one cannot prove bounds for form factors in general. However, bounds can be obtained in QCD, where we can even find the high momentum behaviour with the Brodsky–Farrar counting rules. In particular, for F_π one has the Farrar–Jackson (1979) behaviour

$$F_\pi(t) \underset{t\to\infty}{\simeq} \frac{12 C_F \pi f_\pi^2 \alpha_s(-t)}{-t}, \tag{2.2.1}$$

where f_π is the pion decay constant, $f_\pi \simeq 93$ MeV, α_s the QCD coupling, and $C_F = 4/3$ is a colour factor. For scalar form factors (such as the scalar form factor of the pion), that we denote by F_S, the analyticity is like for F_π, and one can also infer a behaviour of the type

$$F_S(t) \underset{t\to\infty}{\simeq} \frac{\text{Const.}}{-t \log^\nu(-t/\hat{t})}, \tag{2.2.2}$$

with $\hat{t}$ an unknown scale factor and ν also unknown although, probably, one also has $\hat{t} \simeq \Lambda, \quad \nu = 1$. The corrections to (2.2.1), (2.2.2), however, cannot be calculated. These asymptotic behaviours hold, in principle, only on the real axis, but the Phragmén–Lindelöf theorem ensures their validity in all directions of the complex plane.

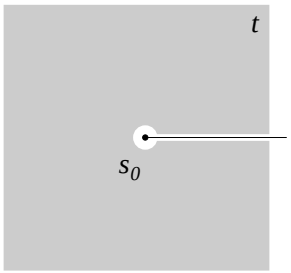

FIGURE 2.2.1.
The domain of analyticity (shadowed region) for $F_\pi(t)$, or $\Pi(t)$, in the complex t plane.

2.3. Correlators

Consider for example the vector current operator $V_\mu(x) = \bar{q}\gamma_\mu q'$ with q, q' quark fields. We associate to it the correlator

$$\begin{aligned} \Pi_{\mu\nu}(p) &= \mathrm{i} \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}p\cdot x} \langle 0|\mathrm{T} V_\mu^\dagger(x) V_\nu(0)|0\rangle \\ &= (-g_{\mu\nu} t + p_\mu p_\nu)\, \Pi_{\mathrm{tr}}(t) + p_\mu p_\nu \Pi_S(t), \quad t = p^2. \end{aligned} \tag{2.3.1}$$

The $\Pi(t)$ can be shown, again using only unitarity and causality, to be analytic in the complex t plane with a cut from t_0 to $+\infty$ where t_0 is the squared invariant mass of the lightest state with the quantum numbers of the current V_μ. If V_μ is the e.m. (electromagnetic) current, that we denote

by J_μ, and we neglect weak and e.m. interactions, then $\Pi_S = 0$ and $\Pi_{\rm tr}$ is analytic except for a cut from $t = 4\mu^2$ to $+\infty$.

There is no bound with validity for arbitrary field theories for the correlators; but, in QCD, we can actually calculate the behaviour for large momentum; it is given by the parton model result for the Π.

2.4. Scattering amplitudes at high energy: Reggeology

Although we are here interested only on low and (at times) intermediate energy, it is clear that calculations of dispersive type, like those we will discuss in coming Chapters, require a model for high energy $\pi\pi$ scattering. Regge pole theory provides such a model and, although outside the scope of this notes, we will describe here those of its features that are of interest to us.

Consider the collision of two hadrons, $A + B \to A + B$. According to Regge theory, the high energy scattering amplitude, at fixed t and large s, is governed by the exchange of complex, composite objects, known as *Regge poles*, related (in some cases) to the resonances that couple to the t channel. The same is true for large t, dominated by the resonances in the s channel (this is the property originally proved, in potential theory, by T. Regge). Thus, for isospin 1 in the t channel, high energy scattering is dominated by the exchange of a "Reggeized" ρ resonance. If no quantum number is exchanged, we say that the corresponding Regge pole is the vacuum, or Pomeranchuk Regge pole; this name is often shortened to *Pomeron*. In a QCD picture, the Pomeron (for example) will be associated with the exchange of a gluon ladder between two partons in particles A, B (Fig. 2.2.2). The corresponding formalism was developed by Gribov, Lipatov and other physicists in the 1970s, and is related to the so-called Altarelli–Parisi, or DGLAP mechanism in deep inelastic scattering (see e.g. Barger and Cline (1969) and Ynduráin (1999) and, more recently, Donnachie et al. (2002), which include references to the original articles).

One of the useful properties of Regge theory is the so-called *factorization*; it can be proved from general properties of Regge theory,[3] or, in QCD, in the DGLAP formalism,[4] as is intuitively obvious from Fig. 2.2.2.

Factorization states that the scattering amplitude $F_{A+B\to A+B}(s,t)$ can be written as a product

$$F_{A+B\to A+B}(s,t) \underset{\substack{s\to\infty\\ t\,\text{fixed}}}{\simeq} C_A(t)C_B(t)(s/\hat{s})^{\alpha_R(t)}. \tag{2.4.1}$$

$\hat{s}$ is a constant, usually taken to be 1 GeV^2 (we will do so here); the functions C_A, C_B depend on the corresponding particles, but the power $(s/\hat{s})^{\alpha_R(t)}$ is universal and depends only on the quantum numbers exchanged in channel t. The exponent $\alpha_R(t)$ is called the Regge trajectory associated to the quantum numbers in channel t and, for small t, may be considered linear:

$$\alpha_R(t) \underset{t\sim 0}{\simeq} \alpha_R(0) + \alpha_R' t. \tag{2.4.2}$$

For the ρ and Pomeron pole, fits to high energy processes give

$$\begin{aligned} &\alpha_\rho(0) = 0.52 \pm 0.02, \quad \alpha_\rho'(0) = 1.01\,\text{GeV}^{-2},\\ &\alpha_P(0) = 1, \quad \alpha_P' = 0.11 \pm 0.03\,\text{GeV}^{-2}, \end{aligned} \tag{2.4.3a}$$

[3] The proof follows from extended unitarity (Gell-Mann, 1962; Gribov and Pomeranchuk, 1962).

[4] Gribov and Lipatov (1972); Dokshitzer (1977); Altarelli and Parisi (1977). See also Kuraev, Lipatov and Fadin (1976) and Balitskii and Lipatov (1978).

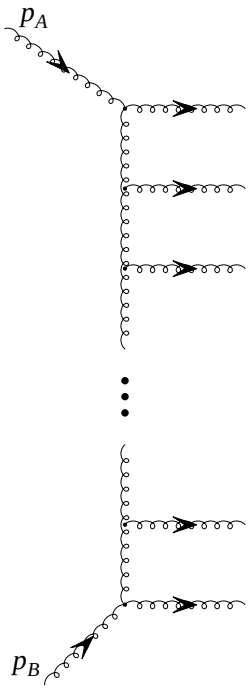

FIG. 2.2.2.
Cut Pomeron ladder exchanged between the partons p_A and p_B in hadrons A, B. The emitted gluons will materialize into a shower of particles.

The Regge parameters taken here are essentially those of Rarita et al. (1968); for $\alpha_\rho(0)$, however, we choose the central value 0.52 which is more consistent with determinations based on deep inelastic scattering. There are indications that the Pomeron is not an ordinary Regge pole but we will not discuss this here.

For the ρ trajectory, we have enough information that we can write a more accurate, quadratic expression that agrees on the mean with (2.4.3a) for spacelike t and satisfies the Regge constraint $\alpha_\rho(M_\rho^2) = 1$:

$$\begin{aligned} \alpha_\rho(t) &= \alpha_\rho(0) + t\alpha'_\rho(0) + \tfrac{1}{2}t^2\alpha''_\rho(0), \\ \alpha'_\rho(0) &= 0.90\,\mathrm{GeV}^{-2},\ \alpha''_\rho(0) = -0.3\,\mathrm{GeV}^{-4}. \end{aligned} \tag{2.4.3b}$$

Let us consider the imaginary part of the spin averaged πN or NN scattering amplitudes (here by NN we also understand $\bar{N}N$), which we recall are normalized so that

$$\sigma_{\mathrm{tot}}(s) = \frac{4\pi^2}{\lambda^{1/2}(s, m_A^2, m_B^2)}\,\mathrm{Im}\,F(s,0).$$

We have, with f_i related to the imaginary part of C_i,

$$\begin{aligned} \mathrm{Im}\,F_{NN}(s,t) &\simeq f_N^2(t)(s/\hat{s})^{\alpha_R(t)}, \\ \mathrm{Im}\,F_{\pi N}(s,t) &\simeq f_\pi(t) f_N(t)(s/\hat{s})^{\alpha_R(t)}, \end{aligned} \tag{2.4.4a}$$

and therefore, using factorization,

$$\operatorname{Im} F_{\pi\pi}(s,t) \simeq f_\pi^2(t)(s/\hat{s})^{\alpha_R(t)}. \tag{2.4.4b}$$

The functions $f_i(t)$ depend exponentially on t for small t and may be written, approximately, as

$$f_i(t) = \sigma_i \mathrm{e}^{bt}, \quad b = (2.4 \pm 0.4)\ \mathrm{GeV}^{-2}. \tag{2.4.5}$$

Consistency requires a more complicated form for the residue functions $f_i(t)$; below we will give expressions that are sufficiently accurate for the small values of t in which we are interested.

From (2.4.4) we can deduce the relations among the cross sections

$$\frac{\sigma_{\pi\pi\to\mathrm{all}}}{\sigma_{\pi N\to\mathrm{all}}} = \frac{\sigma_{\pi N\to\mathrm{all}}}{\sigma_{NN\to\mathrm{all}}}.$$

This relation also holds in the naive quark model[5] in which one considers that scattering of hadrons is given by incoherent addition of scattering of their constituent quarks, so we have

$$\sigma_{\pi\pi\to\mathrm{all}} \ : \ \sigma_{\pi N\to\mathrm{all}} \ : \ \sigma_{NN\to\mathrm{all}} = 2\times 2 \ : \ 2\times 3 \ : \ 3\times 3.$$

From any of these relations one can obtain the parameter σ_π in (2.4.5) in terms of the known πN and NN cross sections.

We write explicit formulas for $\pi\pi$ scattering, taken from the expressions of Rarita et al. (1968); to be precise, as given in Peláez and Ynduráin (2003). For $I_t = 0$ exchange,

$$\begin{aligned}
&\operatorname{Im} F^{(I_t=0)}_{\pi\pi\to\pi\pi}(s,t) \underset{\substack{s\to\infty\\ t\ \mathrm{fixed}}}{\simeq} \operatorname{Im} F_P^{(I_t=0)} + \operatorname{Im} F_{P'}^{(I_t=0)};\\
&\operatorname{Im} F_P^{(I_t=0)} = \sigma_\pi(P)\,\frac{(1+\alpha'_P t)\,(2+\alpha'_P t)}{2}\,\mathrm{e}^{bt}(s/\hat{s})^{\alpha_P(0)+\alpha'_P t},\\
&\operatorname{Im} F_{P'}^{(I_t=0)} = \sigma_\pi(P')\mathrm{e}^{bt}(s/\hat{s})^{\alpha_\rho(0)+\alpha'_\rho t}.
\end{aligned} \tag{2.4.6a}$$

We have added the Pomeron and the subleading contribution, the so-called P' pole (associated with the f_2 resonance) that is necessary at the lowest energy range. The slope of the second we have taken as identical to that of the rho. As noticed in Peláez and Ynduráin (2003; see also Appendix C here), this choice gives better consistency for crossing sum rules, besides being what one expects in the QCD version of Regge theory; the experimental information on πN, NN scattering is not enough to fix the slope of the P' with any accuracy.

For $I_t = 1$, we write

$$\begin{aligned}
&\operatorname{Im} F^{(I_t=1)}_{\pi\pi\to\pi\pi}(s,t) \underset{\substack{s\to\infty\\ t\ \mathrm{fixed}}}{\simeq} \operatorname{Im} F^{(\rho)}(s,t)\\
&\qquad = \sigma_\pi(\rho)\frac{1+\alpha_\rho(t)}{1+\alpha_\rho(0)}\ \left[(1+1.48)\mathrm{e}^{bt} - 1.48\right](s/\hat{s})^{\alpha_\rho(0)+\alpha'_\rho t}.
\end{aligned} \tag{2.4.6b}$$

From (2.4.5) and the known cross sections for πN, NN scattering we have[6]

$$\sigma_\pi(P) = 3.0 \pm 0.30; \quad \sigma_\pi(P') = 0.75 \pm 0.08; \quad \sigma(\rho) = 0.84 \pm 0.10. \tag{2.4.6c}$$

[5] Levin and Frankfurter (1965). For a comprehensive review, see Kokkedee (1969). Note, however, that it is not clear why the naive quark model should work, as its mechanism is very different from the orthodox QCD one.

[6] Our Regge parameters $\sigma(i)$ here are slightly smaller than those used in Palou and Ynduráin (1974). This is because we now add a P' contribution to the Pomeron, and a background to the rho piece.

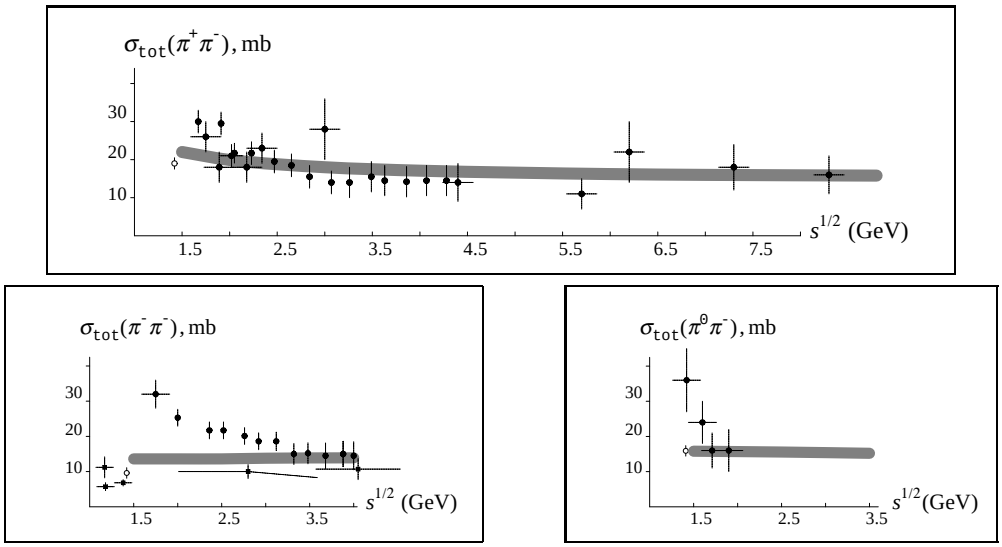

FIGURE 2.2.3. The total cross sections $\sigma(\pi^-\pi^-)$, $\sigma(\pi^+\pi^-)$ and $\sigma(\pi^0\pi^-)$. Black dots and squares: experimental points from Robertson, Walker and Davis (1973), Biswas et al. (1967), Cohen et al. (1973), Hanlan et al. (1976), and Abramowicz et al. (1980). Open dots at 1.42 GeV: the cross sections that follow from the low energy phase shift analyses, see Appendix A. Thick gray lines, from 1.42 GeV: Regge formula, with parameters as in (2.4.6, 7) (the thickness of the line covers the error in the theoretical value of the Regge residue).

For isospin $I_t = 2$ exchange we cannot fix its parameters from factorization, since πN or NN do not contain such amplitude. It must be due to double rho exchange, so we know that its energy dependence, at $t = 0$, should be $s^{2\alpha_\rho(0)-1}$, but we know little more. We use an empirical formula:

$$\mathrm{Im}\, F^{(I_t=2)}(s,t) = (0.6 \pm 0.2)\, \mathrm{e}^{c_2 t} (s/\hat{s})^{2\alpha_\rho(0)+\alpha'_\rho t-1}, \tag{2.4.7a}$$

i.e., a slope like the rho and P'. We have obtained the constant 0.6 ± 0.2 by fitting the difference between the experimental $\pi^0\pi^0$ and $\pi^0\pi^+$ total cross sections at $s^{1/2} = 1.42$ GeV, and the Pomeron plus P' values. We will, somewhat arbitrarily, take the value $c_2 = b$, whose justification is that it produces consistency in crossing sum rules like that in the Appendix C here (see also Peláez and Ynduráin, 2003):

$$c_2 = b = 2.38 \pm 0.02\ \mathrm{GeV}^{-2}. \tag{2.4.7b}$$

Another matter is, when one may apply formulas like (2.4.6). From the DGLAP version of the Pomeron (for example) we expect the following pattern to occur: in the region $|t| \ll s$, $s \gg \Lambda^2$ (with $\Lambda \sim 0.4$ GeV the QCD parameter) the ladder exchange mechanism will start to dominate the collision $A + B$. We then will have the onset of the Regge regime with, at the same time, a large increase of inelasticity and a smoothing of the total cross section according to the behaviour (2.4.6).

For πN, NN scattering this occurs as soon as one is beyond the region of elastic resonances; in fact (as can be seen in the cross section summaries in the Particle Data Tables) as soon as the

kinetic energy in the c.m. is above $1 - 1.2$ GeV. For $\pi\pi$ we thus expect the Regge description to be valid for the corresponding energies, that is to say, for $s^{1/2} \geq 1.4$ GeV, which is the region where we will use it here. Indeed, and as we will see, around this energy, it is possible to calculate the $\pi\pi$ scattering amplitude from experiment and indeed it agrees, within a 15%, with (2.4.6); see for example Peláez and Ynduráin (2003).

It is worth noting that these properties can also be verified directly for $\pi\pi$ scattering, as has been done by Robertson, Walker and Davis (1973), Biswas et al. (1967), Cohen et al. (1973), Hanlan et al. (1976), and Abramowicz et al. (1980). These authors do not attempt at phase shift reconstruction of the amplitude, but measure directly elastic and total $\pi\pi$ cross sections at energies between 1.2 and 6 GeV. They find a pattern identical to that for πN, NN or KN scattering. In particular, a total cross section that flattens out to a value of 15 to 20 mb, precisely as predicted by factorization (and in agreement with our numbers here). Moreover, the elastic cross section becomes less than a third of the total one above 1.7 GeV; see for example fig. 5 in Robertson, Walker and Davis (1973). In fact, and as shown in Fig. 2.2.3, the experimental $\pi\pi$ cross sections agree very well with the prediction obtained from factorization (our equations (2.4.6) here).

As is clear from this minireview, the reliability of the Regge calculation of high energy pion-pion scattering cannot go beyond this accuracy of $\sim 15\%$, even for small t. The deviations off simple Regge behaviour are expected to be much larger for large t, as indeed the counting rules of QCD imply a totally different behaviour for fixed t/s. This is one of the problems involved in using e.g. the Roy equations that require integration up to $-t \sim 0.7$ GeV^2, $s \sim 2$ GeV^2, where the Regge picture fails completely (we expect instead the Brodsky–Farrar behaviour, $\sigma_{\text{fixed } \cos\theta} \sim s^{-7}$). However, for forward dispersion relations or the Froissart–Gribov representation we will work only for $t = 0$ or $t = 4\mu^2$ for which the largest variation, that of e^{bt}, is still small, $b(t = 4\mu^2) \simeq 0.19$, and we expect no large error due to departure off linearity for the exponent in $f_i(t)$ or for the Regge trajectories, $\alpha_R(t)$.

3. The effective range formalism for p.w. amplitudes. Resonances

3.1. Effective range formalism

We will consider here only the pion-pion case. The discontinuity of $f_l(s)$ across the *elastic* cut is very easily evaluated. Because all functions (scattering amplitudes, form factors and correlators) are real analytic[7] we can calculate their discontinuity as

$$\begin{aligned} \mathrm{disc} f(s) &= 2\mathrm{i}\,\mathrm{Im}\, f(s) = \lim_{\epsilon\to+0}\{f(s+\mathrm{i}\epsilon) - f(s-\mathrm{i}\epsilon)\} \\ &= \lim_{\epsilon\to+0}\{f(s+\mathrm{i}\epsilon) - f^*(s+\mathrm{i}\epsilon)\}\,. \end{aligned} \tag{3.1.1}$$

For p.w. amplitudes, and for physical s below the inelastic threshold s_0, we have

$$\mathrm{Im}\, f_l(s) = \frac{\pi k}{2s^{1/2}}|f_l(s)|^2, \quad 4\mu^2 \le s \le s_0. \tag{3.1.2}$$

This suggests how we can form from f_l a function in which this elastic cut is absent. This is the function $\Phi_l(s)$ defined for arbitrary complex s by

$$\Phi_l(s) = \frac{\mathrm{i}k^{2l+1}}{2\sqrt{s}} + \frac{k^{2l}}{\pi f_l(s)}. \tag{3.1.3a}$$

We assume that $f_l(s)$ does not vanish for $0 \le s < 4\mu^2$, or for $4\mu^2 < s \le s_0$. If f_l vanished below threshold, or on the elastic cut, the function Φ_l would have poles at such zeros; the analysis can be generalized quite easily to cope with this, and, for $\pi\pi$ scattering, we will show explicitly how in the cases of the S waves and the D2 wave.

We can rewrite (3.1.3a) as

$$\Phi_l(s) = -2^{-2-2l}(s-4\mu^2)^l\left(\frac{4\mu^2}{s}-1\right)^{1/2} + 2^{-2l}\frac{(s-4\mu^2)^l}{\pi f_l(s)}. \tag{3.1.3b}$$

In this second form it is obvious that the first term in the r.h. side is analytic for all s, except for a (kinematic) cut running from $-\infty$ to $s=0$ and a cut for $s \ge 4\mu^2$. The second term is also analytic over the segment $0 \le s < 4\mu^2$, and it presents a dynamical cut from $-\infty$ to 0 due to the l.h. cut of f_l (Fig. 3.1.1).

We next check that $\Phi_l(s)$ is analytic over the elastic cut. We have, for $4\mu^2 < s \le s_0$,

$$\mathrm{Im}\,\Phi_l(s) = \frac{k^{2l+1}}{2\sqrt{s}} + k^{2l}\frac{-\,\mathrm{Im}\, f_l(s)}{\pi f_l^*(s) f_l(s)}.$$

[7] A complex function $f(z)$ is *real analytic* if it satisfies $f^*(z^*) = f(z)$. The theorem of Painlevé ensures that, if a function analytic for $\mathrm{Im}\, z \neq 0$ is real analytic, and is real on a segment $[a,b]$ of the real axis, it is also analytic on the segment. For more information on matters of complex analysis, we recommend the texts of Ahlfors (1953) and Titchmarsh (1939).

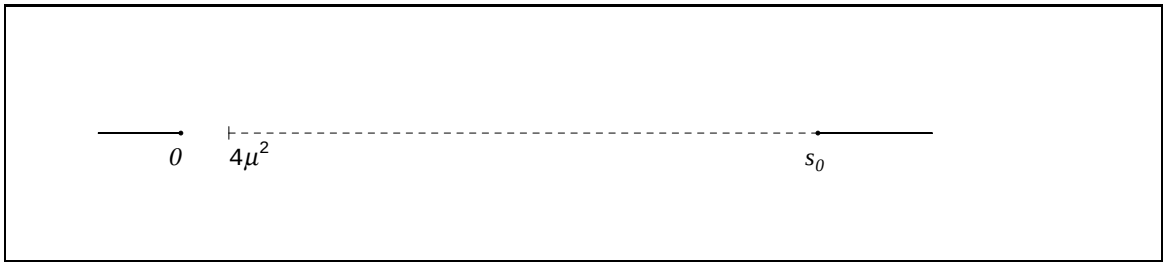

FIGURE 3.1.1. The cuts in the complex s plane for $\Phi_l(s)$. The dotted line shows the absent elastic cut. We have taken $s_0 = 1$ GeV^2, and the drawing is to scale.

Using then (3.1.2), the r.h. side is seen to vanish. The only point which appears dangerous is the threshold, $s = 4\mu^2$, because here f_l vanishes for $l \geq 1$; but this zero is exactly compensated by the zero of the factor k^{2l}; cf. (2.1.5). Therefore it follows that the function $\Phi_l(s)$ is analytic along the elastic cut. Its only singularities are thus (apart from poles due to zeros of f_l), a r.h. cut from $s = s_0$ to $+\infty$; and a l.h. cut, formed by two superimposed cuts, namely, the kinematic cut of

$$2^{-2-2l}(s-4\mu^2)^l\sqrt{\frac{4\mu^2}{s}-1},$$

and the dynamical cut of

$$\frac{k^{2l}}{\pi f_l(s)}$$

due to the l.h. cut of $f_l(s)$.

Eq. (3.1.3b) defines $\Phi_l(s)$ for all complex s; in the particular case where s is on the elastic cut, we can use Eq. (2.1.2) to get

$$\Phi_l(s) = \frac{k^{2l+1}}{2\sqrt{s}}\cot\delta_l(s), \quad 4\mu^2 \leq s \leq s_0. \tag{3.1.4}$$

In general, i.e., for any value (complex or real) of s, we can solve (3.1.2) and write

$$f_l(s) = \frac{2s^{1/2}}{\pi k}\frac{1}{2s^{1/2}k^{-2l-1}\Phi_l(s) - \mathrm{i}} = \frac{k^{2l}}{\pi}\frac{1}{\Phi_l(s) - \mathrm{i}k^{2l+1}/2s^{1/2}}. \tag{3.1.5}$$

$\Phi_l(s)$ is real on the segment $0 \leq s \leq s_0$, but it will be *complex* above the inelastic threshold, s_0, and also for $s \leq 0$.

The fact that $\Phi_l(s)$ is analytic across the elastic region is valid not only for $\pi\pi$, but also for other p.w. amplitudes; for example, for pion-nucleon, nucleon-nucleon or even nucleon-nucleus. This implies that, at low energies ($k \to 0$), one can expand

$$\Phi_l(s) = \frac{1}{4\mu a_l} + R_0k^2 + R_1k^4 + \cdots. \tag{3.1.6}$$

This is the so-called effective range formalism, widely used in low energy nucleon and nuclear physics. The quantity a_l is the scattering length (cf. Eq. (2.1.4)) and the R_i are related to the

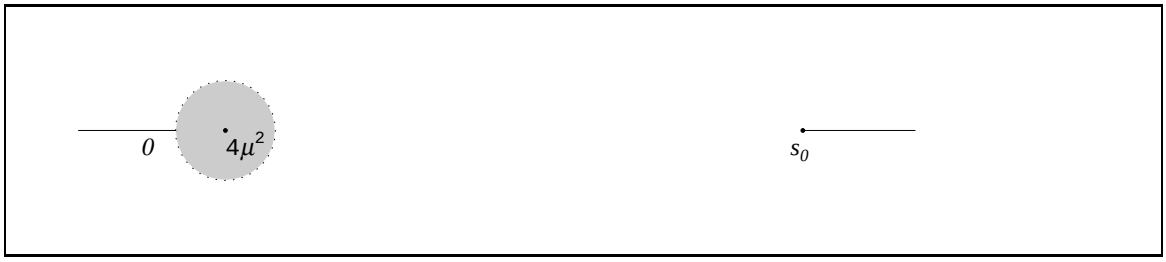

FIGURE 3.1.2. The circle of convergence for the effective range expansion for $\Phi_l(s)$; $s_0 = 1$ GeV^2.

range of the potential (if the scattering is caused by a short-range potential). For $\pi\pi$ scattering, the expansion is convergent in the disk $|s - 4\mu^2| < 1$, shown shaded in Fig. 3.1.2.

Besides (3.1.6) we will also use the parameters b_l defined by

$$\frac{\pi}{4\mu k^{2l}} \operatorname{Re} f_l(s) \underset{k\to 0}{\simeq} a_l + b_l k^2 + \cdots \tag{3.1.7}$$

3.2. Resonances in (nonrelativistic) potential scattering

Consider scattering by a spherical potential, $V(r)$, that we assume to be of short range. We will simplify the discussion by working in the nonrelativistic approximation. The nonrelativistic energy E is $E = s^{1/2} - m_1 - m_2$ with m_i the masses of the particles, and we shall let m be the reduced mass. To lighten notation, we take mass units so that $2m = 1$.

The l-wave Schrödinger equation is

$$\frac{\mathrm{d}^2\psi_l(r)}{\mathrm{d}r^2} + \left[k^2 - V(r) - \frac{l(l+1)}{r^2}\right]\psi_l(r) = 0. \tag{3.2.1}$$

One may express its solutions as

$$\psi_l(r) \underset{r\to\infty}{\simeq} \frac{1}{2\mathrm{i}}\left\{\mathrm{e}^{\mathrm{i}kr - \mathrm{i}l\pi/2 + \mathrm{i}\delta_l(E)} - \mathrm{e}^{-\mathrm{i}kr + \mathrm{i}l\pi/2 - \mathrm{i}\delta_l(E)}\right\}. \tag{3.2.2a}$$

In principle, (3.2.2a) is valid only for physical $k \geq 0$. However, because (3.2.1) depends explicitly on k, we can take the solution to be valid for arbitrary, even complex k.

From (3.2.2a) we can find the p.w. amplitudes. First, we rewrite it as

$$\psi_l(r) \underset{r\to\infty}{\simeq} j^-(k,l)\mathrm{e}^{\mathrm{i}kr} + j^+(k,l)\mathrm{e}^{-\mathrm{i}kr}; \tag{3.2.2b}$$

the $j^{\pm}$, known as the Jost functions, are identified, at large r, comparing with (3.2.2a). In terms of these we can write the S-matrix element,

$$s_l(E) \equiv \mathrm{e}^{2\mathrm{i}\delta_l},$$

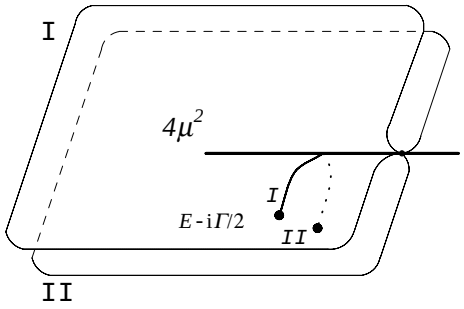

FIGURE 3.2.1. The Riemann sheet for p.w. amplitudes.

as

$$s_l(E) = (-1)^{l+1}\frac{j^-(k,l)}{j^+(k,l)}. \tag{3.2.3}$$

Now, the exchange of $k \to -k$ does not alter the Schrödinger equation, but it exchanges the exponentials in (3.2.2a). Therefore, one must have

$$j^-(-k,l) = j^+(k,l). \tag{3.2.4}$$

If we start from the k plane, then the energy plane will be a two-sheeted Riemann surface (Fig. 3.2.1). We designate *physical* sheet (sheet I) to that coming from $\mathrm{Im}\, k > 0$, and *unphysical* sheet (sheet II) to that obtained from $\mathrm{Im}\, k < 0$. When considering $s_l(E)$ as a function of E it then follows that we have two determinations. Since obviously $k^{\mathrm{II}} = -k^{\mathrm{I}}$, we find

$$s_l^{\mathrm{II}}(E) = \left[s_l^{\mathrm{I}}(E)\right]^{-1}. \tag{3.2.5}$$

The physical value is

$$s_l(E) = \lim_{\epsilon\to+0} s_l^{\mathrm{I}}(E+\mathrm{i}\epsilon) = \lim_{\epsilon\to+0} s_l^{\mathrm{II}}(E-\mathrm{i}\epsilon).$$

We shall now look for singularities of $s_l(E)$. For physical $E > 0$ we cannot have poles because $|s_l(E)| = 1$. For $E = -E_B < 0$, a pole of $s_l(E)$ means a zero of j^+. If the pole occurs in the first sheet, this means that the corresponding value of the momentum will be $k_B = \mathrm{i}|k| = \mathrm{i}\sqrt{E_B}$ and hence (3.2.2b) becomes

$$\psi_l(r) \underset{r\to\infty}{\simeq} j^-(k_B,l)\mathrm{e}^{-|k|r},$$

i.e., the wave function of a bound state with binding energy E_B. We thus conclude that poles of the S-matrix in the physical sheet for energies below threshold correspond to bound states.[8]

[8] We will not be interested in poles in the unphysical sheet with negative energies, known as *antibound* states. More details on the subject of this section may be found in the treatises of Omnès and Froissart (1963) and Goldberger and Watson (1964).

We next investigate the meaning of poles in the lower half-plane in the unphysical sheet, that is to say, poles located at $E^{\rm II} = E_R = E_0 - {\rm i}\Gamma/2$ with E_0, $\Gamma > 0$. If there is a pole of $s_l^{\rm II}(E)$ for $E = E_R - {\rm i}\Gamma/2$, then (3.2.5) implies that the physical S-matrix element has a zero in the same location:

$$s_l^{\rm I}(E_R - {\rm i}\Gamma/2) = 0$$

(Fig. 3.2.1). The corresponding wave function is not as easily obtained as for the bound state case; a detailed discussion may be found in Godberger and Watson (1966) or Galindo and Pascual (1978), but an essentially correct result may be obtained by replacing, in the standard time dependent wave function for stationary states

$$\Psi = {\rm e}^{-{\rm i}Et}\psi(\mathbf{r}),$$

E by the complex value $E_R - {\rm i}\Gamma/2$. So we get

$$\Psi = {\rm e}^{-\Gamma t/2}{\rm e}^{-{\rm i}E_R t}\psi(\mathbf{r}) :$$

the probability to find the state decreases with time as $|\Psi|^2 = {\rm e}^{-\Gamma t}$, which can be interpreted as a metastable state that decays with a lifetime $\tau = 1/\Gamma$; that is to say, a *resonance*. Γ is called the *width* of the resonance, and is equal to the indetermination in energy of the metastable state.

Let us consider now the corresponding physical phase shift. The pole and zero of $s_l^{\rm II}$, $s_l^{\rm I}$ imply corresponding zeros of the Jost functions. We will assume that Γ is very small; then, in the neighbourhood of E_R we can write

$$s_l^{\rm I}(E) \underset{E\sim E_R}{\simeq} \frac{E - E_R - {\rm i}\Gamma/2}{E - E_R + {\rm i}\Gamma/2}. \tag{3.2.6a}$$

For the phase shift this implies

$$\cot\delta_l(E) \underset{E\sim E_R}{\simeq} \frac{E_R - E}{\Gamma/2}. \tag{3.2.6b}$$

This means that at E_R the phase shift goes, *growing*, through $\pi/2$ and that it varies rapidly.

We can write the corresponding formulas for the p.w. amplitudes, now for the relativistic case. We profit from the analyticity of the effective range function over the elastic cut to conclude from (3.2.6b) and the proportionality between $\cot\delta_l$ and Φ_l that, for $s = M_R^2$ (where M_R is the invariant mass corresponding to the energy E_R), $\Phi_l(s)$ must have a zero:

$$\Phi_l(s) \underset{s\simeq M_R^2}{\simeq} \frac{M_R^2 - s}{\gamma}. \tag{3.2.7a}$$

So we may write the p.w. amplitude in its neighbourhood as

$$f_l(s) \underset{s\simeq M_R^2}{\simeq} \frac{1}{\pi}\,\frac{k^{2l}\gamma}{M_R^2 - s - {\rm i}k^{2l+1}\gamma/2s^{1/2}}. \tag{3.2.7b}$$

The residue of Φ_l, γ, can be related to the width of the resonance:

$$\Gamma = [k(M_R^2)]^{2l+1}\gamma/2M_R^2. \tag{3.2.7c}$$

Eq. (3.2.7) is the (relativistic) *Breit–Wigner* formula for the p.w. scattering amplitude near a resonance. Note however that Eqs. (3.2.7) are only valid in the vicinity of the resonance; away from it, the ratio $\Phi_l(s)/(M_R^2 - s)$ will not be a constant, so in general we will have to admit a dependence of γ (and Γ) on s.

Let us consider another characterization of a resonance. Returning to nonrelativistic scattering, one can prove that the time delay that the interaction causes in the scattering of two particles in angular momentum l and with energy E is

$$\Delta t = 2\frac{\mathrm{d}\delta_l(E)}{\mathrm{d}E}.$$

We can say that the particles resonate when this time delay is maximum. In the vicinity of a zero of the effective range, we can use (3.2.7) to show that $\Delta t(s)$ is maximum for $s = M_R^2$ and then the time delay equals $1/\Gamma$.

We have therefore three definitions of an *elastic* resonance: a pole of the scattering amplitude in the unphysical Riemann sheet; a zero of the effective range function; or a maximum of the quantity

$$\mathrm{d}\delta_l(s)/\mathrm{d}s.$$

These three definitions agree, to order γ^2, when γ is small, and neglecting variations of γ; but a precise description of broad resonances requires discussion of these variations. In these notes, however, we will only give the value of $s^{1/2}$ at which the phase crosses $\pi/2$. Since we will also give explicit parametrizations, to find e.g. the location of the poles should not be a difficult matter for the interested reader.

Unstable elementary particles may also be considered a special case of resonances; thus, for example, one may treat the Z particle as a fermion-antifermion resonance. We discuss this for a simple model in Sect. 4.2.

4. The P p.w. amplitude for $\pi\pi$ scattering in the elementary rho model

4.1. The ρ propagator and the $\pi^+\pi^0$ scattering amplitude

Before continuing with general properties of pion interactions, it is convenient to illustrate what we have already seen with a simple, explicit model. In the present chapter we do precisely this; specifically, we consider the elementary rho model and take π^0, π^+ interactions to be given by the Lagrangian given in (1.3.1). We start by calculating the ρ^+ propagator in dimensional regularization, to lowest order and neglecting the rho self-interactions. We therefore consider only the diagrams in Fig. (4.1.1). The corresponding vacuum polarization function is then

$$\begin{aligned}\Pi^{(\rho)}_{\mu\nu}(q) = {}& \mathrm{i}^2 g_\rho^2 \int \mathrm{d}^D\hat{p}\,(2p+q)_\mu (2p+q)_\nu \frac{\mathrm{i}}{p^2-\mu^2}\frac{\mathrm{i}}{(p+q)^2-\mu^2} \\ & + 2\mathrm{i} g_\rho^2 g_{\mu\nu}\int \mathrm{d}^D\hat{p}\,\frac{\mathrm{i}}{p^2-\mu^2};\end{aligned}$$

we have defined

$$\mathrm{d}^D\hat{p} \equiv \frac{\mathrm{d}^D p}{(2\pi)^D}\nu_0^{4-D},$$

and ν_0 is an arbitrary mass parameter. After standard manipulations, and with $D = 4-\epsilon$, we find

$$\begin{aligned}\Pi^{(\rho)}_{\mu\nu}(q) = {}& (-q^2 g_{\mu\nu} + q_\mu q_\nu)\frac{\mathrm{i} g_\rho^2}{16\pi^2}\left\{\tfrac{1}{3}\left(\frac{2}{\epsilon} - \gamma_{\mathrm{E}} + \log 4\pi - \log\nu_0^2\right)\right. \\ & \left. - \int_0^1 \mathrm{d}x (1-2x)^2 \log(\mu^2 - x(1-x)q^2)\right\}.\end{aligned} \tag{4.1.1a}$$

Here $\gamma_{\mathrm{E}} \simeq 0.5772$ is Euler's constant.

We then calculate the dressed rho propagator. For this, we first rewrite (4.1.1a) as

$$\Pi^{(\rho)}_{\mu\nu}(q) = (-q^2 g_{\mu\nu} + q_\mu q_\nu)\mathrm{i}\Pi_D(s), \quad s = q^2. \tag{4.1.1b}$$

The dressed propagator is then,

$$D^{(\rho;0)}_{\mu\nu} = \frac{-\mathrm{i} g_{\mu\nu}}{s-M_0^2} + \frac{-\mathrm{i} g_{\mu\nu}}{s-M_0^2}(-\mathrm{i}s\Pi_D)\frac{-\mathrm{i}}{s-M_0^2} + \cdots + \text{gauge terms..}$$

The gauge terms are terms proportional to $q_\mu q_\nu$. Summing this we find

$$D^{(\rho;0)}_{\mu\nu} = \frac{-\mathrm{i} g_{\mu\nu}}{s - M_0^2 + s\Pi_D} + \text{gauge terms.}$$

This is still unrenormalized, and M_0 is the unrenormalized rho mass. We renormalize in the $\overline{\mathrm{MS}}$ scheme, with scale parameter the (renormalized) rho mass, $\nu_0^2 = M^2$. Thus,

$$\Pi_{\mathrm{ren.}}(s) = -\frac{g_\rho^2}{16\pi^2}\int_0^1 \mathrm{d}x (1-2x)^2 \log\frac{\mu^2 - x(1-x)s}{\bar{M}^2} \tag{4.1.2a}$$

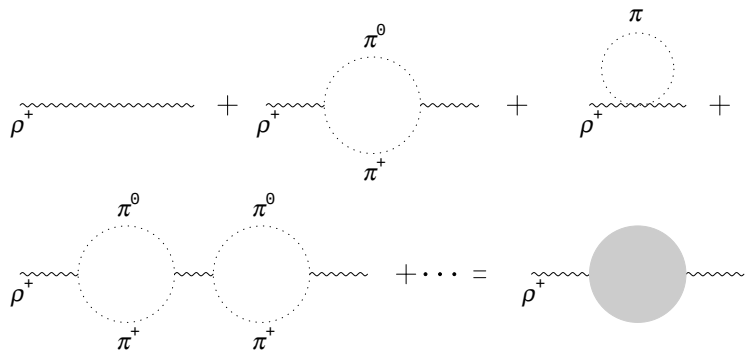

FIGURE 4.1.1. The sum of one loop corrections to the rho propagator.

and the renormalized, dressed rho propagator is

$$D^{(\rho)}_{\mu\nu} = \frac{-\mathrm{i}g_{\mu\nu}}{s - \bar{M}^2 + s\Pi_{\mathrm{ren.}}(s)} + \text{gauge terms}; \quad \bar{M} = \bar{M}(M^2). \tag{4.1.2b}$$

For s real and larger than $4\mu^2$ we can split $\Pi_{\mathrm{ren.}}$ into a real and an imaginary part as follows:

$$\Pi_{\mathrm{ren.}}(s) = -\frac{g_\rho^2}{16\pi^2}\int_0^1 \mathrm{d}x(1-2x)^2 \log\left|\frac{\mu^2 - x(1-x)s}{\bar{M}^2}\right| + \mathrm{i}\frac{g_\rho^2}{16\pi^2}\frac{8k^3}{3s^{3/2}}, \quad s \geq 4\mu^2. \tag{4.1.3}$$

We next evaluate the scattering amplitude, with the fully dressed propagator. We have to calculate the amplitudes $F^{(s)}$ and $F^{(u)}$ associated with diagrams (s), (u) in Fig. 4.1.2, so that the scattering amplitude is $F = F^{(s)} + F^{(u)}$. For the first we find,

$$F^{(s)} = -16\frac{g_\rho^2}{16\pi^2}\,\frac{k^2}{s - \bar{M}^2 + s\Pi_{\mathrm{ren.}}}\cos\theta, \tag{4.1.4}$$

where θ is the scattering angle in the c.m.

Projecting $F^{(s)}$ onto the P wave we get

$$f_1^{(s)}(s) = \tfrac{16}{3}\frac{g_\rho^2}{16\pi^2}\,\frac{k^2}{\bar{M}^2 - s - s\Pi_{\mathrm{ren.}}(s)}. \tag{4.1.5}$$

We have to add the contribution of diagram (u); note that, in this model, there is no contribution from the t channel, at leading order, because you cannot make a ρ with two π^0s. We have,

$$F^{(u)} = 4\frac{g_\rho^2}{16\pi^2}\left(\frac{3s - 4\mu^2}{2} - 2k^2\cos\theta\right)\frac{1}{u - \bar{M}^2 + u\Pi_{\mathrm{ren.}}(u)} \tag{4.1.6}$$

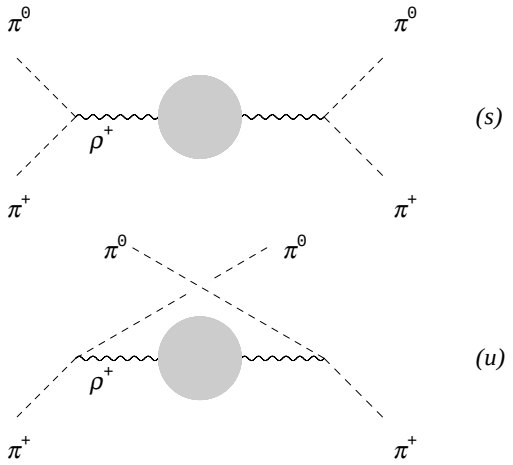

FIGURE 4.1.2. Diagrams for $\pi^0\pi^+$ scattering mediated by the ρ.

and we recall that $u = 4\mu^2 - s - t = -2k^2(3 + \cos\theta)$. Projecting into the P wave we find

$$f_1^{(u)}(s) = \tfrac{1}{2}\frac{4g_\rho^2}{16\pi^2} \times \int_{-1}^{+1} \mathrm{d}\cos\theta\, \cos\theta \left(\frac{3s - 4\mu^2}{2} - 2k^2\cos\theta\right) \frac{1}{u - \bar{M}^2 + u\Pi_{\text{ren.}}(u)}. \tag{4.1.7a}$$

The complete partial wave amplitude is

$$f_1(s) = f_1^{(s)}(s) + f_1^{(u)}(s). \tag{4.1.7b}$$

This p.w. amplitude does not satisfy unitarity. This is a general fact; perturbation theory only verifies *perturbative* unitarity, that is to say, unitarity up to corrections of higher orders. We will see in this example how this works; this will allow us to see explicitly how in this model the rho behaves as a resonance.

First of all we check that f_1 verifies the expected analyticity properties. $f_1^{(s)}(s)$ has a right hand cut, due to that of $\Pi_{\text{ren.}}(s)$ which is the only piece in (4.1.5) which is nonanalytic. From Eq. (4.1.3) we see that it extends from $4\mu^2$ to $+\infty$. The l.h. cut of f_1 comes from the l.h. cut of $F_1^{(u)}(s,t)$. From (4.1.7), the only discontinuity occurs when $\Pi_{\text{ren.}}(u)$ is discontinuous, which happens when $u \geq 4\mu^2$. In terms of s, $\cos\theta$ this condition becomes

$$u - 4\mu^2 = -\tfrac{1}{2}\left[s + 4\mu^2 + (s - 4\mu^2)\cos\theta\right] \geq 0.$$

Therefore, $F^{(u)}(s,t)$ has a discontinuity for s in the range from $-\infty$ to

$$s_\theta = \frac{4\mu^2(\cos\theta - 1)}{1 + \cos\theta}.$$

Because in (4.1.7) we integrate for $\cos\theta$ between -1 and $+1$, it follows that the cut of $f_1^{(u)}(s)$, and hence of $f_1(s)$, runs from $-\infty$ to 0, as was to be expected on general grounds.

4.2. The weak coupling approximation

Let us now make further approximations. If we calculate the rho decay width in our model to lowest order we obtain, after a simple calculation,

$$\Gamma(\rho^+ \to \pi^+\pi^0) = \frac{g_\rho^2 k_\rho^3}{6\pi M^2}, \quad k_\rho = \frac{\sqrt{M^2 - 4\mu^2}}{2}.$$

Putting numbers for the rho mass and width it follows that

$$\frac{g_\rho^2}{16\pi^2} \simeq 0.23$$

so the approximation of considering this quantity to be small is not too bad.

For s physical, and in particular for $s \sim M^2$, the piece $s\Pi_{\text{ren.}}(s)$ in the expression for $f_1^{(s)}(s)$, although of nominal order g_ρ^2 (see (4.1.2)) cannot be neglected; else, $f_1^{(s)}(s)$ would be infinite around $s = M^2$. However, $u\Pi_{\text{ren.}}(u)$ can be neglected to a first approximation in g_ρ^2 in the expression (4.1.7). If we do this, $f_1^{(u)}$ can be easily integrated explicitly and becomes

$$f_1^{(u)}(s) \simeq \frac{4g_\rho^2}{16\pi^2}\frac{2s - 4\mu^2 + \bar{M}^2}{k^2}\left\{1 - \frac{2k^2 + \bar{M}^2}{4k^2}\log\left(1 + \frac{4k^2}{\bar{M}^2}\right)\right\}. \tag{4.2.1}$$

In this approximation the l.h. cut only runs up to $s = 4\mu^2 - M^2$: the discontinuity across the piece $[4\mu^2 - M^2, 0]$ is of order $(g^2/16\pi^2)^2$, and can be neglected (within the model) in a first approximation. With the value found for g_ρ, we expect this to be valid to some 6%.

If we consider the region near $s = \bar{M}^2$, then $|f_1^{(s)}(s)| \sim 1$ while $f_1^{(u)}(s)$ is of order $g^2/16\pi^2$. We can further approximate f_1 by neglecting the whole of $f_1^{(u)}(s)$, and thus write

$$f_1(s) \underset{s\sim\bar{M}^2}{\simeq} f_1^{(s)}(s) = \tfrac{16}{3}\frac{g_\rho^2}{16\pi^2}\frac{k^2}{\bar{M}^2 - s - s\Pi_{\text{ren.}}(s)}. \tag{4.2.2}$$

It is important to notice that this approximation is only valid when $f_1^{(s)}(s)$ is of order unity; otherwise, both s and u channel pieces are of comparable order of magnitude. Another interesting point is that this approximation is unitary and indeed it is very similar to the Breit–Wigner approximation. To see this more clearly, we define the (resonance) mass of the rho as the solution of the equation

$$\bar{M}^2 = M_\rho^2\left\{1 + \frac{g_\rho^2}{16\pi^2}\int_0^1 \mathrm{d}x(1-2x)^2\log\left|\frac{\mu^2 - x(1-x)\bar{M}^2}{\bar{M}^2}\right|\right\}. \tag{4.2.3}$$

From (4.1.3) and (4.1.5) it follows that, in the present approximation, we can identify, for physical s,

$$\cot\delta_1(s) = \mathrm{i} + \frac{6\pi}{k^3 g_\rho^2}\left[\bar{M}^2 - s - s\Pi_{\text{ren}}(s)\right] = \frac{6\pi s^{1/2}}{k^3 g_\rho^2}\left(\bar{M}^2 - s - s\,\mathrm{Re}\,\Pi_{\text{ren}}(s)\right),$$

which, because of (4.2.3), vanishes at $s = M_\rho^2$.

In this model, the effective range function is

$$\Phi_1(s) = \frac{s-4\mu^2}{4}\sqrt{\frac{4\mu^2}{s}-1} + \frac{3\pi}{g_\rho^2}\left[\bar{M}^2 - s - s\Pi_{\rm ren}(s)\right].$$

4.3. Low energy scattering

We now calculate the low energy scattering in the elementary rho model; to be precise, we will evaluate the scattering length, a_1. For this calculation a number of approximations can be made. Although, in the real world, Γ_ρ and μ are very similar, it is believed that they have a different origin. μ^2 is supposed to be proportional to the sum of u and d quark masses, whereas Γ_ρ is related to the QCD parameter Λ. We will thus make a calculation neglecting the u channel contribution and evaluating $\mathrm{Re}\,\Pi$ in leading order in $\log M_\rho^2/\mu^2$. In this approximation we have (cf. Eqs. (2.1.4), (4.1.5))

$$a_1^{(s)} = \frac{g_\rho^2}{12\pi\mu M_\rho^2}\,\frac{1}{1-(4\mu^2/M_\rho^2)\left[1+\dfrac{g_\rho^2}{48\pi^2}\log\dfrac{M_\rho^2}{\mu^2}\right]} \simeq 36\times10^{-3}\,\mu^{-3}. \tag{4.3.1}$$

The experimental value is

$$a_1 = (39.1\pm1.4)\times10^{-3}\,\mu^{-3}.$$

We see that, for such a crude model, the agreement with experiment is quite good; in fact, as we will see in Chapter 8, comparable to what one gets with sophisticated calculations. On the other hand, of course, the model is only valid for the P wave; for example, it gives zero (to order $g_\rho^2/16\pi^2$) for $\pi^0\pi^0$ scattering, although the interaction here is very strong.

4.4. The chiral rho model

The model we have developed for ρ mediated pion interactions is not compatible with chiral symmetry. A model compatible with this has been developed by Gasser and Leutwyler;[9] in it the ρ is coupled through the field strengths, $F_{\mu\nu}^{(a)}$, with a an isospin index, to the pions. The model is rather complicated and can be found in the paper of these authors (Gasser and Leutwyler, 1984; see also Ecker et al. 1989 where it is further developed). This coupling produces a nonrenormalizable interaction (as opposed to the previous rho model, which was renormalizable) so only tree level calculations are, in principle, allowed with it.

In fact, it is possible to make loop calculations with this model, but to get finite results we will have to add extra interactions (and extra coupling constants) every time we go to a higher order in the number of loops taken into account; the model soon loses its predictive power and, in this respect, it is inferior to the nonchiral model we have studied before. Moreover, it cannot satisfy rigorous unitarity (that requires an infinite number of loops), although Dyson resumed versions of it are available in the literature (Guerrero and Pich, 1997). Its main interest lies in providing an *explicit* realization for chiral perturbation theory calculations, and a way to extrapolate these to the resonance region.

We will not give the details of such calculations here, that the interested reader may find in the literature quoted.

[9] In fact, the chiral rho model is much older; see e.g. Coleman, Wess and Zumino (1969) or Weinberg (1968b).

5. The effective range formalism for p.w. amplitudes; resonances (multichannel formalism). Unitarity and form factors; correlators

5.1. General formalism. Eigenphases

The extension of the developments of the previous section to the case where we have several channels open is very simple, provided these channels are all two-particle channels. To a good approximation this is the case for pion-pion scattering up to energies of about $s^{1/2} \simeq 1.3$ GeV.

In the general case, we label the various two-body channels by letters $a, b, \ldots$, each with values $1, 2, \ldots, n$ (for n channels). So, we have the p.w. amplitudes[10] $f^{(l)}_{ab}(s)$ that describe scattering of particles[11] $P_1(b) + P_2(b) \to P_1(a) + P_2(a)$.

As an example, we may have the channels

$$\begin{array}{ll} \pi^+\pi^-, & a = 1 \\ \pi^0\pi^0, & a = 2 \\ K^+K^-, & a = 3 \\ K^0\bar{K}^0, & a = 4. \end{array}$$

This would be simplified to two uncoupled two-channel problems (for isospin 0 and 1) if assuming isospin invariance.

We define the (modulus of the) three-momentum, in channel a, as k_a. Then, the unitarity condition may be written as

$$\mathrm{Im}\, f^{(l)}_{ab}(s) = \frac{\pi}{2s^{1/2}} \sum_c k_c f^{(l)}_{ac}(s) f^{(l)}_{bc}(s)^*, \tag{5.1.1}$$

and we have used time-reversal invariance which implies that

$$f^{(l)}_{ab} = f^{(l)}_{ba}.$$

If we had only one channel, or if there were only diagonal interactions ($f^{(l)}_{ab} = f^{(l)}_a \delta_{ab}$), (5.1.1) would tell us that one can write

$$f^{(l)}_a = \frac{2s^{1/2}}{\pi k_a} \sin\delta^{(l)}_a \mathrm{e}^{\mathrm{i}\delta^{(l)}_a},$$

i.e., Eq. (2.1.2).

[10] We put in this Chapter the angular momentum variable l as an index or superindex, according to convenience. So we write f_l or $f^{(l)}$, $\delta^{(l)}$ or δ_l.

[11] Note the reversed order; this is because the S matrix elements are usually defined by

$$\langle P_1(a), P_2(a)|S|P_1(b), P_2(b)\rangle.$$

To treat the general case it is convenient to use a matrix formalism. Denoting the matrices by boldface letters, we define

$$\mathbf{f}_l = \left(f_{ab}^{(l)}\right), \quad \mathbf{k} = (k_a\delta_{ab}).$$

We will also define the multichannel S-matrix elements,

$$s_{ab}^{(l)}(s) = \frac{2s^{1/2}}{\pi k_a}\delta_{ab} + 2\mathrm{i}f_{ab}^{(l)}(s) \tag{5.1.2a}$$

or, in matrix notation,

$$\mathbf{s}_l = (2s^{1/2}/\pi)\mathbf{k}^{-1} + \mathbf{f}_l. \tag{5.1.2b}$$

If we had uncoupled channels, (2.1.2) would tell us immediately that

$$s_a^{(l)} = \frac{2s^{1/2}}{\pi k}\mathrm{e}^{2\mathrm{i}\delta_a^{(l)}}.$$

To see what the unitarity relations imply in the multichannel case, it is convenient to form the matrix $\mathbf{u}$ with

$$u_{ab} \equiv k_a^{1/2}s_{ab}^{(l)}k_b^{1/2}.$$

After a simple calculation, using (5.1.1) and time reversal invariance, we find that

$$\sum_c u_{ac}^* u_{bc} = \frac{4s}{\pi^2}\delta_{ab}.$$

Therefore, $(\pi/2s^{1/2})\mathbf{u} = \mathbf{D}_l$ is a unitary matrix. We let $\mathbf{C}_l$ be the unitary matrix that diagonalizes it, and denote by $\widetilde{\mathbf{D}}_l$ to the diagonalized matrix, with elements $(\exp 2\mathrm{i}\widetilde{\delta}_a^{(l)})\delta_{ab}$. The $\widetilde{\delta}_a^{(l)}(s)$ are called the *eigenphases*, and are the generalization to the multichannel case of the ordinary phase shifts. We find that we can write:

$$\mathbf{s}_l = \frac{2s^{1/2}}{\pi}\,\mathbf{k}^{-1/2}\mathbf{C}_l\widetilde{\mathbf{D}}_l\mathbf{C}_l^{-1}\mathbf{k}^{-1/2}. \tag{5.1.3}$$

Note that, because of time reversal invariance, the matrix $\mathbf{C}$ may in fact be chosen to be *real*.

Inverting these relations we obtain the general form for the p.w. amplitudes,

$$\begin{aligned}\mathbf{f}_l =& \frac{2s^{1/2}}{\pi}\mathbf{k}^{-1/2}\mathbf{C}_l\widetilde{\mathbf{f}}_l\mathbf{C}_l^{-1}\mathbf{k}^{-1/2},\\ \widetilde{\mathbf{f}}_l =& \begin{pmatrix} \sin\widetilde{\delta}_1^{(l)}\mathrm{e}^{\mathrm{i}\widetilde{\delta}_1^{(l)}} & 0 & \dots & 0\\ 0 & \sin\widetilde{\delta}_2^{(l)}\mathrm{e}^{\mathrm{i}\widetilde{\delta}_2^{(l)}} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \sin\widetilde{\delta}_n^{(l)}\mathrm{e}^{\mathrm{i}\widetilde{\delta}_n^{(l)}}\end{pmatrix}.\end{aligned} \tag{5.1.4}$$

This is the generalization of (2.1.2) to the quasi-elastic multichannel case.

5.2. The K-matrix and the effective range matrix. Resonances

We define the K-matrix, $\mathbf{K}_l$, such that

$$\mathbf{f}_l = \left\{ \mathbf{K}_l^{-1} - \frac{\mathrm{i}\pi}{2s^{1/2}}\mathbf{k} \right\}^{-1}. \tag{5.2.1}$$

In terms of it we can write the matrix $\mathbf{D}_l$ as

$$\mathbf{D}_l = \frac{1 + \mathrm{i}(\pi/2s^{1/2})\,\mathbf{k}^{1/2}\mathbf{K}_l\mathbf{k}^{1/2}}{1 - \mathrm{i}(\pi/2s^{1/2})\,\mathbf{k}^{1/2}\mathbf{K}_l\mathbf{k}^{1/2}}. \tag{5.2.1}$$

The unitarity and symmetry of $\mathbf{D}_l$ in the quasi elastic region means that $\mathbf{K}_l$ will be hermitean and symmetric there, hence it will be *real* across the two particle cuts:

$$\mathbf{K}_l = \mathbf{K}_l^\dagger = \mathbf{K}_l^*.$$

The definition of $\mathbf{K}_l$ does not take into account the behaviour at the thresholds. To do so we define the *effective range* matrix, $\mathbf{\Phi}_l$ by

$$\mathbf{\Phi}_l = \frac{1}{\pi}\mathbf{k}^l\mathbf{K}_l^{-1}\mathbf{k}^l.$$

In terms of it we find

$$\mathbf{f}_l = \frac{1}{\pi}\mathbf{k}^l \left(\mathbf{\Phi}_l - \frac{\mathrm{i}}{2s^{1/2}}\mathbf{k}^{2l+1} \right)^{-1} \mathbf{k}^l, \tag{5.2.2}$$

an obvious generalization of (3.1.5). $\mathbf{\Phi}_l$ is real and symmetric. It is therefore analytic except for the l.h. cut of the $f_{ab}^{(l)}$, and for the r.h. cut that occurs when s is above a true inelastic (multiparticle) threshold, $s > s_{\mathrm{mult.}}$.

Let us now discuss resonances in the multichannel case. It is clear that the eigenstates of the time evolution operator will correspond to the eigenphases, as they are eigenstates of the S-matrix. We will therefore identify resonances with a resonant-like behaviour of the eigenphases: we will say that we have a resonance at $s = M^2$ provided one of the eigenphases crosses $\pi/2$ and varies rapidly there. We will assume that resonances are simple, i.e., only one eigenphase resonates at a given $s = M^2$, and moreover we suppose that M does not coincide with the thresholds. The resonance condition, in eigenchannel r, is then

$$\widetilde{\delta}_r^{(l)}(s = M^2) = \pi/2; \quad \left. \frac{\mathrm{d}\widetilde{\delta}_r^{(l)}(s)}{\mathrm{d}s} \right|_{s=M^2} = \text{maximum}, \tag{5.2.3a}$$

but

$$\widetilde{\delta}_{i\neq r}^{(l)}(s = M^2) \neq \pi/2. \tag{5.2.3b}$$

Let us see what this implies in terms of $\mathbf{\Phi}_l$. From (5.1.4), (5.2.2) we can write

$$\widetilde{\mathbf{f}}_l = 2s^{1/2} \left(\mathbf{C}_l^{-1}\mathbf{k}^{-l-1/2}\mathbf{\Phi}_l\mathbf{k}^{-l-1/2}\mathbf{C}_l - \mathrm{i} \right)^{-1}.$$

Because $\widetilde{\mathbf{f}}_l$ and i are diagonal, so must be $\mathbf{g}_l \equiv \mathbf{C}_l^{-1}\mathbf{k}^{-l-1/2}\mathbf{\Phi}_l\mathbf{k}^{-l-1/2}\mathbf{C}_l$. Recalling again (5.1.4), it follows that its elements are such that

$$(2s^{1/2}g_a^{(l)} - \mathrm{i})^{-1} = \sin\widetilde{\delta}_a^{(l)}\mathrm{e}^{\mathrm{i}\widetilde{\delta}_a^{(l)}},$$

i.e., one can write

$$2s^{1/2}g_a^{(l)} = \cot\widetilde{\delta}_a^{(l)}.$$

The resonance condition then is equivalent (forgetting for the moment the requisite of rapid variation of the derivative of the phase) to the condition

$$g_r^{(l)}(s=M^2)=0; \quad g_{a\neq r}^{(l)}(s=M^2)\neq 0.$$

Therefore, the quantity $\det(\mathbf{g}_l(s))$ has a simple zero at $s=M^2$. Since, for this value of s, the determinants of $\mathbf{k}$, $\mathbf{C}_l$ are finite, we have obtained that the condition of resonant behaviour (above all thresholds) is that the determinant of the effective range matrix,

$$\det(\mathbf{\Phi}_l(s))$$

has a simple zero at $s=M^2$.

We will next incorporate the condition of rapid variation, and calculate the *partial widths*, that generalize the quantity Γ of the one-channel case. Near $s=M^2$ we write

$$\cot\widetilde{\delta}_r^{(l)}(s) \simeq \frac{M^2-s}{M\Gamma}. \tag{5.2.4}$$

The condition of rapid variation is that Γ be small. Next, and using (5.1.4), we have

$$f_{ab}^{(l)} \underset{s\sim M^2}{\simeq} \frac{2s^{1/2}}{\pi}\frac{1}{\sqrt{k_a k_b}}\left\{C_{ar}^{(l)}C_{br}^{(l)}\frac{M\Gamma}{M^2-s-\mathrm{i}M\Gamma}+\sum_{i\neq r}C_{ai}^{(l)}C_{bi}^{(l)}\sin\widetilde{\delta}_i^{(l)}\mathrm{e}^{\mathrm{i}\widetilde{\delta}_i^{(l)}}\right\}. \tag{5.2.5a}$$

We have profited from the unitarity and reality of $\mathbf{C}_l$ to write $\mathbf{C}_l^{-1}=\mathbf{C}_l^{\mathrm{T}}$.

We then define the *partial widths*, Γ_a, and *inelasticity parameters* x_a as

$$\Gamma_a^{1/2} \equiv C_{ar}^{(l)}\Gamma^{(1/2)}; \quad x_a=\Gamma_a/\Gamma.$$

Since the matrix $\mathbf{C}_l$ is orthogonal, one has $\sum_a \Gamma_a=\Gamma$. In terms of the Γ_a we can rewrite (5.2.5a) as

$$f_{ab}^{(l)} \underset{s\sim M^2}{\simeq} \frac{2s^{1/2}}{\pi}\frac{1}{\sqrt{k_a k_b}}\left\{\frac{M\Gamma_a^{1/2}\Gamma_b^{1/2}}{M^2-s-\mathrm{i}M\Gamma}+\sum_{i\neq r}C_{ai}^{(l)}C_{bi}^{(l)}\sin\widetilde{\delta}_i^{(l)}\mathrm{e}^{\mathrm{i}\widetilde{\delta}_i^{(l)}}\right\}. \tag{5.2.5b}$$

Thus we see that in the presence of a resonance all channels show a Breit–Wigner behaviour, plus a background due to the reflection of all the nonresonant eigenphases.

If, for a given channel, $x_a\simeq 1$, then we say that, in this channel, the resonance is *elastic*; if $x_a<1/2$, we say that it is *inelastic*. For elastic resonances, and if the phase is near $\pi/2$ at the resonance, the parameter η of (2.1.4) is related to x by

$$\eta = 2x-1. \tag{5.2.6}$$

In general, when we have a resonance (even in the presence of multiparticle channels) we can write, for a given two-particle channel a,

$$\mathrm{Im}\, f_{aa}^{(l)}(s) \underset{s\sim M^2}{\simeq} \frac{2s^{1/2}}{\pi k_a}\frac{M^2\Gamma^2}{(M^2-s)^2+M^2\Gamma^2}\times \mathrm{BR} \tag{5.2.7}$$

with Γ the total width, and BR the branching ratio into channel a, $\mathrm{BR}=\Gamma_a/\Gamma$.

5.3. Resonance parametrizations in the two-channel case

We will now present explicit formulas for parametrizations of resonances in the important case where only two channels are open. We start by changing a little bit the notation, writing, for obvious reasons, $g_l^{(\pm)}$ for the two eigenvalues of $\mathbf{g}_l$.

We want to present parametrizations that profit from the analyticity of $\mathbf{\Phi}_l$ so that they are not only valid on the resonance; thus, we will write our formulas in terms of $\mathbf{\Phi}_l$. Actually, we will use as parameters the diagonal elements of $\mathbf{\Phi}_l$, $\Phi_{11}^{(l)}(s)$, $\Phi_{22}^{(l)}(s)$, and its determinant, that, because we have a resonance at $s = M^2$, we may write as $\det \mathbf{\Phi}_l(s) = \gamma(s)(s - M^2)$, with $\gamma(s)$ a smooth function (that can in most cases be approximated by a constant).

Next, we express the $g_l^{(\pm)}$ in terms of these parameters. We let Δ and τ be the determinant and trace of $\mathbf{g}_l$. We have, on one hand, and in the physical region for both channels,

$$g_l^{(\pm)} = \frac{\tau \mp \sqrt{\tau^2 - 4\Delta}}{2}; \quad \mathbf{g}_l = \begin{pmatrix} g_l^{(+)} & 0 \\ 0 & g_l^{(-)} \end{pmatrix}; \quad k_1,\, k_2 \geq 0 \tag{5.3.1a}$$

and, on the other,

$$\begin{aligned} \Delta &= \det(\mathbf{g}_l) = (k_1 k_2)^{-2l-1} \det \mathbf{\Phi}_l(s) = (k_1 k_2)^{-2l-1} \gamma(s)(s - M^2), \\ \tau &= \operatorname{Tr} \mathbf{g}_l = k_1^{-2l+1} \Phi_{11}^{(l)} + k_2^{-2l-1} \Phi_{22}^{(l)}. \end{aligned} \tag{5.3.1b}$$

The resonating phase is $\delta_l^{(+)}$ if τ is positive and $\delta_l^{(-)}$ if τ is negative because, from (5.3.1), it follows that Δ vanishes for $s = M^2$.

The mixing matrix $\mathbf{C}_l$ can also be obtained explicitly. One has,

$$\mathbf{C}_l = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}; \quad \cos\theta = \left\{ \frac{k_1^{-2l-1} \Phi_{11}^{(l)} - g_l^{(-)}}{g_l^{(+)} - g_l^{(-)}} \right\}^{1/2}. \tag{5.3.2}$$

5.4. Reduction to a single channel. Weakly coupled channels

We will now consider the case in which one has two channels, but we are interested chiefly on one of them, that we will denote by channel 1. We will further assume that this channel opens before channel 2. Below the opening of channel 2, the formulas reduce to those of one single channel, so we can write (cf. (3.1.5))

$$f_{11}^{(l)} = \frac{1}{\pi} \frac{k_1^{2l}}{\Phi_{\rm el}^{(l)} - \dfrac{\mathrm{i}}{2s^{1/2}} k_1^{2l+1}}. \tag{5.4.1}$$

$\Phi_{\rm el}^{(l)}$ may be expressed in terms of $\mathbf{\Phi}^{(l)}$ using (5.2.2). We define $\kappa_a = \mathrm{i}k_a$ and get,

$$\Phi_{\rm el}^{(l)} = \frac{\dfrac{(-1)^l}{2s^{1/2}} \kappa_2^{2l+1} \Phi_{11}^{(l)} + \det \mathbf{\Phi}^{(l)}}{\dfrac{(-1)^l}{2s^{1/2}} \kappa_2^{2l+1} + \Phi_{22}^{(l)}}. \tag{5.4.2}$$

Before the opening of channel 2, and above the l.h. cut, $\Phi_{\rm el}^{(l)}$ is, as expected, real and analytic.

It is worth noting that Eq. (5.4.2) is still valid above the opening of channel 2, but κ_2 will now be *imaginary*. Because of this some care has to be exercised to identify the quantity $\delta_{11}^{(l)}$. From (2.1.2), which is valid above threshold for channel 1, but below channel 2 threshold we have, using (3.1.5),

$$\cot\delta_{11}^{(l)}(s) = \frac{2s^{1/2}}{k_1^{2l+1}}\,\Phi_{\rm el}^{(l)}(s), \tag{5.4.3}$$

with $\Phi_{\rm el}^{(l)}$ given by (5.4.2). But, because κ_2 becomes imaginary above the opening of channel 2, it follows that $\cot\delta_{11}^{(l)}(s)$ will be complex there. This is of course to be expected; a real $\delta_{11}^{(l)}(s)$ implies strict elastic unitarity.

We next continue with two channels, but now assume that they are weakly coupled. This is made transparent by writing

$$\Phi_{12}^{(l)} \equiv \epsilon_{12},$$

and we will work to lowest nontrivial order in ϵ_{12}. We can write,

$$f_{11}^{(l)} = \frac{1}{\pi}\,\frac{\Phi_{22}^{(l)} - \dfrac{\rm i}{2s^{1/2}}k_2^{2l+1}}{\left(\Phi_{11}^{(l)} - \dfrac{\rm i}{2s^{1/2}}k_1^{2l+1}\right)\left(\Phi_{22}^{(l)} - \dfrac{\rm i}{2s^{1/2}}k_2^{2l+1}\right) - \epsilon_{12}^2}.$$

Expanding to lowest order in the mixing, this becomes

$$f_{11}^{(l)} = \frac{1}{\pi}\,\frac{k_1^{2l+1}}{\Phi_{11}^{(l)} - \dfrac{\rm i}{2s^{1/2}}k_1^{2l+1}}\left\{1 + \frac{\Phi_{11}^{(l)}}{\Phi_{11}^{(l)} - \dfrac{\rm i}{2s^{1/2}}k_1^{2l+1}}\,\frac{\epsilon_{12}^2}{\Phi_{22}^{(l)} - \dfrac{\rm i}{2s^{1/2}}k_2^{2l+1}}\right\} \tag{5.4.4}$$

i.e., like an effective one-channel amplitude,

$$\bar f_{11}^{(l)} = \frac{1}{\pi}\,\frac{k_1^{2l+1}}{\Phi_{11}^{(l)} - \dfrac{\rm i}{2s^{1/2}}k_1^{2l+1}} \tag{5.4.5a}$$

modulated by the factor

$$G_1^{(l)} = 1 + \frac{\Phi_{11}^{(l)}}{\Phi_{11}^{(l)} - \dfrac{\rm i}{2s^{1/2}}k_1^{2l+1}}\,\frac{\epsilon_{12}^2}{\Phi_{22}^{(l)} - \dfrac{\rm i}{2s^{1/2}}k_2^{2l+1}}\,: \tag{5.4.5b}$$

one has,

$$f_{11}^{(l)} = \bar f_{11}^{(l)} G_1^{(l)}. \tag{5.4.5c}$$

In the case in which we have a resonance in each channel, we write

$$\Phi_{11}^{(l)}(s) \simeq (M_1^2 - s)/\gamma_1, \quad \Phi_{22}^{(l)}(s) \simeq (M_2^2 - s)/\gamma_2. \tag{5.4.6}$$

In this case (5.4.4) becomes

$$f_{11}^{(l)} \simeq \frac{1}{\pi}\frac{k_1^{2l+1}\gamma_1}{M_1^2 - s - \mathrm{i}k_1^{2l+1}\gamma_1/2s^{1/2}} \times \left\{1 + \frac{M_1^2 - s}{k_2^{2l+1}\left(M_1^2 - s - \mathrm{i}k_1^{2l+1}\gamma_1/2s^{1/2}\right)}\,\frac{\epsilon_{12}^2\gamma_2 k_2^{2l+1}}{M_2^2 - s - \mathrm{i}k_2^{2l+1}\gamma_2/2s^{1/2}}\right\}. \tag{5.4.7}$$

It is noteworthy that, if the resonances are narrow, and not too near the thresholds, the modulation of the first (M_1) by the second is negligible (of order $\gamma_2\epsilon_{12}^2$) except on top of the second, $s \simeq M_2^2$.

The mixing angle also has a simple expression now:

$$\sin\theta = \frac{(k_1k_2)^{l+1/2}|\epsilon_{12}|^2}{\left|k_2^{2l+1}\Phi_{11}^{(l)} - k_1^{2l+1}\Phi_{22}^{(l)}\right|}. \tag{5.4.8}$$

We note to finish that the coupling of the channels displaces the resonances. Defining them as solutions of the equation

$$\det\mathbf{\Phi}(\widetilde{M}_a^2) = 0, \quad a = 1, 2, \tag{5.4.9a}$$

we see that e.g. for the first we have

$$\widetilde{M}_1^2 = M_1^2 + \frac{\gamma_1\gamma_2\epsilon_{12}^2}{M_2^2 - M_1^2}. \tag{5.4.9b}$$

5.5. Unitarity for the form factors

The expression for the form factor of scalar particles A, $\bar{A}$ (which we consider with electric charge $\pm e$) in the timelike region is defined, for example, in terms of the process

$$e^+e^- \to A\bar{A}.$$

The corresponding matrix element may be written, to lowest order in the electromagnetic interaction, and with the effective photon-hadron interaction $\mathcal{L}_{\text{eff}} = eJ_\mu(x)A^\mu(x)$, as

$$\langle A(p_1)\bar{A}(p_2)|S|e^+(k_1)e^-(k_2)\rangle = \mathrm{i}e^2\frac{1}{(2\pi)^3}\bar{v}(k_1)\gamma_\mu u(k_2)\frac{-\mathrm{i}}{(p_1+p_2)^2} \times(2\pi)^4\delta(k_1+k_2-p_1-p_2)\langle A(p_1)\bar{A}(p_2)|J^\mu(0)|0\rangle,$$

and we recall that the form factor is defined (for spinless particles) as

$$\langle A(p_1)\bar{A}(p_2)|J^\mu(0)|0\rangle = (2\pi)^{-3}(p_1-p_2)^\mu F(s), \quad s = (p_1+p_2)^2.$$

Let us write the S matrix as $S = 1 + \mathrm{i}\mathcal{T}$ so that

$$\langle f|\mathcal{T}|i\rangle = \delta(p_f - p_i)F(i \to f).$$

Unitarity of S implies the relation

$$\mathcal{T} - \mathcal{T}^+ = \frac{1}{\mathrm{i}}\mathcal{T}\mathcal{T}^+.$$

Taking matrix elements, we get

$$\mathrm{Im}\langle A_a(p_1)\bar{A}_a(p_2)|\mathcal{T}|e^+e^-\rangle = \tfrac{1}{2}\langle A_a(p_1)\bar{A}_a(p_2)|\mathcal{T}\mathcal{T}^+|e^+e^-\rangle,$$

and we have assumed that we have several two particle channels, denoted with the index a. Summing now over intermediate states, we find

$$\begin{aligned}&\mathrm{Im}\langle A_a(p_1)\bar{A}_a(p_2)|\mathcal{T}|e^+e^-\rangle\\ &= \sum_b \int \frac{\mathrm{d}^3\mathbf{q}_1}{2q_{10}}\frac{\mathrm{d}^3\mathbf{q}_2}{2q_{20}}\tfrac{1}{2}\langle A_a(p_1)\bar{A}_a(p_2)|\mathcal{T}|A_b(q_1)\bar{A}_b(q_2)\rangle\langle A_b(q_1)\bar{A}_b(q_2)|\mathcal{T}^+|e^+e^-\rangle.\end{aligned}$$

In terms of the form factors and scattering amplitudes, therefore,

$$\begin{aligned}&\mathrm{Im}(p_1-p_2)^\mu F_a(s)\\ &= \tfrac{1}{2}\sum_b \int \frac{\mathrm{d}^3\mathbf{q}_1}{2q_{10}}\frac{\mathrm{d}^3\mathbf{q}_2}{2q_{20}}(q_1-q_2)^\mu F_b^*(s_q)\langle A_a(p_1)\bar{A}_a(p_2)|\mathcal{T}|A_b(q_1)\bar{A}_b(q_2)\rangle\\ &= \tfrac{1}{2}\sum_b \int \frac{\mathrm{d}^3\mathbf{q}_1}{2q_{10}}\frac{\mathrm{d}^3\mathbf{q}_2}{2q_{20}}\delta(q_1+q_2-p_1-p_2)(q_1-q_2)^\mu F_b^*(s_q)F_{ab}(q_1,q_2\to p_1,p_2),\end{aligned}$$

$s_q = (q_1+q_2)^2$. In the c.m., $(p_1-p_2)^0 = 0$, $(p_1-p_2)^i = 2k_i$ with $\mathbf{k}$ the c.m. three-momentum. Considering the spacelike part of above equation (the timelike part is trivial) we find, after simple manipulations,[12]

$$\mathrm{Im}\, F_a(s) = \tfrac{1}{2}\frac{1}{2\mathbf{k}^2}\sum_b F_b^*(s_q)\int \frac{\mathrm{d}^3\mathbf{q}_1}{4q_{10}^2}\,(\mathbf{q}_1\mathbf{p}_1)\delta(2p_{10}-q_{10})F_{ab}.$$

Writing

$$F_{ab} = \sum_l (2l+1)P_l(\cos\theta)f_{ab}^{(l)},$$

$\cos\theta = (\mathbf{q}_1\mathbf{p}_1)/k_ak_b$, $k_a \equiv |\mathbf{p}_1|$, $k_b \equiv |\mathbf{q}_1|$ we finally obtain the expression of unitarity in terms of form factors and p.w. amplitudes:

$$\mathrm{Im}\, F_a(s) = \frac{3\pi}{8s^{1/2}}\sum_b \frac{k_b^2}{k_a}F_b^*(s)f_{ab}^{(1)}(s). \tag{5.5.1}$$

One can diagonalize this. With the formulas for the $f_{ab}^{(1)}$ in terms of the eigenphase shifts, $\widetilde{\delta}_a^{(l)}$, and the diagonal p.w. amplitudes, we find (matrix notation)

$$\mathrm{Im}\,\mathbf{C}^{-1}\mathbf{k}^{3/2}\mathbf{F} = \tfrac{3}{8}\widetilde{\mathbf{f}}^{(1)}\mathbf{C}^{-1}\mathbf{k}^{3/2}\mathbf{F}^*.$$

It follows that the combination

$$\sum_b C_{ba}k_b^{3/2}F_b$$

has a phase equal to $\widetilde{\delta}_a^{(1)}$. For the one channel case this proves the equality of the phases of form factor and p.w. amplitudes with the appropriate quantum numbers; for the electromagnetic form factor, the P wave and for the scalar one, the S0 wave.

[12]We hope there will be no confusion between the form factors, F_a, and scattering amplitudes, $F_{ab} = F_{ab}(s,t)$.

5.6. Unitarity for correlators

We will for definiteness consider a correlator[13] of vector currents (not necessarily conserved), V_μ:

$$\Pi_{\mu\nu}(q) = \mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}q\cdot x}\langle \mathrm{T}V_\mu(x)V_\nu^\dagger(0)\rangle_0 \equiv (-q^2 g_{\mu\nu} + q_\mu q_\nu)\Pi_{\mathrm{tr}}(q^2) + q_\mu q_\nu \Pi_S(q^2), \tag{5.6.1}$$

and we have split it into a transverse component (Π_{tr}) and a scalar one (Π_S). If the current was conserved, $\partial\cdot V = 0$, then $\Pi_S = 0$.

The imaginary part of the correlator is given by the expression

$$\begin{aligned} &I_{\mu\nu}(q) = \operatorname{Im}\Pi_{\mu\nu}(q) = \tfrac{1}{2}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}q\cdot x}\langle [V_\mu(x), V_\nu^\dagger(0)]\rangle_0, \quad q^2 \geq 0; \\ &I_{\mu\nu}(q) = 0, \quad q^2 \leq 0. \end{aligned} \tag{5.6.2a}$$

Inserting a complete sum of states, $\sum_\Gamma |\Gamma\rangle\langle\Gamma|$, this becomes

$$I_{\mu\nu}(q) = \tfrac{1}{2}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}q\cdot x}\sum_\Gamma \langle 0|V_\mu(x)|\Gamma\rangle\langle\Gamma|V_\nu^\dagger(0)|0\rangle.$$

Writing also

$$\langle 0|V_\mu(x)|\Gamma\rangle = \mathrm{e}^{-\mathrm{i}p_\Gamma\cdot x}\langle 0|V_\mu(0)|\Gamma\rangle$$

we get the result

$$I_{\mu\nu}(q) = \tfrac{1}{2}(2\pi)^4 \sum_\Gamma \delta(q - p_\Gamma)\langle 0|V_\mu(0)|\Gamma\rangle\langle 0|V_\nu(0)|\Gamma\rangle^*. \tag{5.6.2b}$$

(Of the two terms in the commutator only the first gives a nonzero result, because necessarily the momentum of Γ, p_Γ, has to be timelike). In particular, (5.6.2b) implies that $I_{\mu\nu}$ is positive definite, i.e., for any p (even complex), $p^{\mu *} I_{\mu\nu} p^\nu \geq 0$. If we write

$$I_{\mu\nu} = (-q^2 g_{\mu\nu} + q_\mu q_\nu)\operatorname{Im}\Pi_{\mathrm{tr}}(q^2) + q_\mu q_\nu \operatorname{Im}\Pi_S(q^2) \tag{5.6.3a}$$

then

$$\operatorname{Im}\Pi_{\mathrm{tr}} \geq 0, \quad \operatorname{Im}\Pi_S \geq 0. \tag{5.6.3b}$$

We will consider two important cases of intermediate states: when $|\Gamma\rangle$ is a single particle state of mass m, any spin, and when it is the state of two spinless particles. In the first case,

$$\sum_\Gamma \to \sum_\lambda \int \frac{\mathrm{d}^3p}{2p_0}|p,\lambda\rangle\langle p,\lambda|$$

and λ is the third component of the spin. Then, and working in the c.m. reference system where $q_0 = \sqrt{q^2} \equiv s^{1/2}$, $\mathbf{q} = 0$,

$$\begin{aligned} I_{\mu\nu}(q) &= \tfrac{1}{2}(2\pi)^4 \sum_\lambda \int \frac{\mathrm{d}^3p}{2p_0}\,\delta(q-p)\langle 0|V_\mu(0)|p,\lambda\rangle\langle 0|V_\nu(0)|p,\lambda\rangle^* \\ &= \frac{(2\pi)^4}{4s^{1/2}}\delta(s^{1/2} - m)\sum_\lambda \frac{\sqrt{2}\,F_\mu(q,\lambda)}{(2\pi)^{3/2}}\frac{\sqrt{2}\,F_\nu^*(q,\lambda)}{(2\pi)^{3/2}}; \end{aligned}$$

[13] We write, generally, $\langle A\ldots B\rangle_0 \equiv \langle 0|A\ldots B|0\rangle$.

we have defined

$$\langle 0|V_\mu(0)|p,\lambda\rangle = \frac{\sqrt{2}}{(2\pi)^{3/2}} F_\mu(p,\lambda). \tag{5.6.4a}$$

If the particle is a pion π^- and V_μ is the weak axial current, $V_\mu = \bar{u}\gamma_\mu\gamma_5 d$, then $F_\mu(q,\lambda)$ is related to the *pion decay constant*, f_π:

$$F_\mu(q,\lambda) = f_\pi p_\mu, \quad f_\pi \simeq 93 \text{ MeV}, \tag{5.6.4b}$$

see Chapter 8. In general we have

$$I_{\mu\nu}(q) = 2\pi\delta(s-m^2)\sum_\lambda F_\mu(q,\lambda)F_\nu(q,\lambda)^*. \tag{5.6.5}$$

For the case of a two-particle intermediate state, with spinless particles,

$$\sum_\Gamma \to \int \frac{\mathrm{d}^3p_1}{2p_{10}}\frac{\mathrm{d}^3p_2}{2p_{20}}\,|p_1p_2\rangle\langle p_1p_2|$$

and then

$$\begin{aligned} I_{\mu\nu}(q) &= \tfrac{1}{2}(2\pi)^4 \int \frac{\mathrm{d}^3p_1}{2p_{10}}\frac{\mathrm{d}^3p_2}{2p_{20}}\,\delta(p-p_1-p_2)\langle 0|V_\mu(0)|p_1p_2\rangle\langle 0|V_\nu(0)|p_1p_2\rangle^* \\ &= \frac{(2\pi)^4}{2s^{1/2}}\int \mathrm{d}^3k\,\delta(s-(p_1+p_2)^2)\langle 0|V_\mu(0)|p_1p_2\rangle\langle 0|V_\nu(0)|p_1p_2\rangle^*; \end{aligned}$$

$\mathbf{k} = \mathbf{p}_1 = -\mathbf{p}_2$. If we assume that the current is conserved, we can express the expectation value of the current in terms of a form factor,[14]

$$\langle 0|V_\mu(0)|p_1,p_2\rangle = \frac{1}{(2\pi)^3}(p_1-p_2)_\mu F(s), \tag{5.6.6}$$

hence

$$I_{\mu\nu}(q) = \frac{|F(s)|^2}{2(2\pi)^2 s^{1/2}}\int \mathrm{d}^3k\,\delta(s-(p_1+p_2)^2)(p_1-p_2)_\mu(p_1-p_2)_\nu.$$

The integral is easiest calculated in the c.m. reference system. Here $(p_1-p_2)_\mu = 2k_\mu$, and we have defined $k_0|_{\text{c.m.}} = 0$. If μ is the mass of the particles in the intermediate state (assumed equal, as they have to be if the current is conserved), then $(p_1+p_2)^2 = 2(\mu^2+\mathbf{k}^2)$. In spherical coordinates,

$$I_{\mu\nu}(q) = \frac{\sqrt{s-4\mu^2)}}{4(2\pi)^2 s^{1/2}}|F(s)|^2\int \mathrm{d}\Omega_{\mathbf{k}}k_\mu k_\nu.$$

The angular integral, returning to an arbitrary reference system is

$$\int \mathrm{d}\Omega_{\mathbf{k}}k_\mu k_\nu = \frac{4\pi}{3s}\left(\frac{s}{4}-\mu^2\right)(-g_{\mu\nu}s+q_\mu q_\nu),$$

so we get the final expression

$$\operatorname{Im}\Pi_{\text{tr}}(s) = \frac{1}{6\pi}\left(\frac{s/4-\mu^2}{s}\right)^{3/2}|F(s)|^2. \tag{5.6.7}$$

[14] If the current is not conserved we will have terms proportional to p_1+p_2 in (5.6.6).

6. Extraction and parametrizations of p.w. amplitudes for $\pi\pi$ scattering. Form factors

6.1. $\pi\pi$ scattering

There is of course no possibility to arrange collisions of *real* pions. One can get information on some phase shifts, at a few energies, from processes such as kaon decays, or from the pion electromagnetic or weak form factors (about which more later). But a lot of, unfortunately not very precise, information comes from peripheral pion production, that we now briefly discuss.

What one does in these types of experiments is to collide pions with protons and produce two pions and either a nucleon, N, or a resonance Δ:

$$\pi p \to \pi\pi N; \quad \pi p \to \pi\pi\Delta.$$

One selects events where the momentum p_π transferred by the incoming pion to the proton is small and thus one can assume that the process is mediated by exchange of a virtual pion (Fig. 6.1.1). The process $\pi p \to \pi\pi\Delta$ is in principle more difficult to analyze than $\pi p \to \pi\pi N$; but the last presents a zero for $p_\pi \sim 0$, thus suppressing it in the more interesting region: both processes are, in consequence, equally well (or equally poorly) suited for extracting $\pi\pi$ scattering data. We then expect that the scattering amplitude for the full process will factorize into the $\pi\pi$ scattering amplitude, with one pion off-shell, $F(s,t;p_\pi^2)$, and the matrix element $\langle H|\phi_\pi|p_\pi\rangle$. Here $H = N,\ \Delta$ and ϕ_π is the pion field operator.

It is clear that the method presents a number of drawbacks. First of all, a model is necessary for the dependence on p_π of $F(s,t;p_\pi^2)$ and $\langle H|\phi_\pi|p_\pi\rangle$. Indeed, a model is required for $\langle H|\phi_\pi|p\rangle$ itself. Secondly, in factorizing the full processes one is neglecting final state interactions between the pions and the N or Δ. These are presumably small, but only rather crude models exist for them.

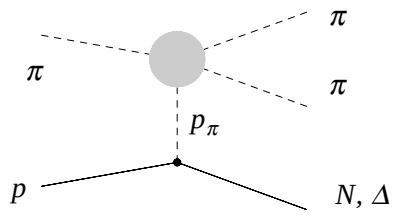

FIGURE 6.1.1.
Diagrams for $\pi p \to \pi\pi N$, Δ.

Another very important problem is that, as soon as inelastic channels become important for $\pi\pi$ scattering, which occurs for $s^{1/2} \sim 1$ GeV for the S wave and for $s^{1/2} \gtrsim 1.4$ GeV for P, D waves, the analysis becomes impossibly complicated: the errors grow very fast.[15] Indeed, above $s^{1/2} \sim 1.5$ GeV it is impossible to disentangle the interesting processes from a number of other ones and, as a consequence, there are no any reliable data. We will discuss this more in Sect. 6.6.

As a consequence of all these difficulties, it happens that the sets of phase shifts one extracts from data present unknown biases and, in particular, are dependent on the models used to perform the fits. This is very clear in the several sets of solutions presented by Protopopescu et al. (1973), Estabrooks and Martin (1974), and in the large errors of the analysis of Hyams et al. (1973) or Grayer et al. (1974). We could have tried to quantify this by introducing systematic errors (for example, the difference between various determinations). This we do in some cases; in others we simply admit that a χ^2/d.o.f. of up to $\sim 2\,\sigma$, with only statistical errors, may be acceptable.

A help out of these difficulties is to use supplementary information from processes like

$$e^+e^- \to \pi^+\pi^-, \quad \tau^+ \to \bar\nu_\tau \pi^+\pi^0, \quad K \to l\bar\nu_l\pi\pi, \quad K \to 2\pi.$$

We will discuss them later later, but note already that this only provides information on the S, P waves at low energy ($s \lesssim 1$ GeV^2). Another possibility is to supplement the experimental information with theory; in Sects. 6.3 to 5 of this chapter we take into account the analyticity properties of p.w. amplitudes to write economical and accurate parametrizations of these; the implementation of other constraints, such as dispersion relations, is left for next chapter.

6.2. Form factors and decays

6.2.1. The pion form factor

The process $e^+e^- \to \pi^+\pi^-$ (Fig. 6.2.1) can, at low energy $t^{1/2} \lesssim 1$ GeV, be related to the pion form factor. We write

$$\frac{\sigma^{(0)}(e^+e^- \to \text{hadrons})}{\sigma^{(0)}(e^+e^- \to \mu^+\mu^-)} = 12\pi\,\mathrm{Im}\,\Pi(t),$$

where Π is hadronic part of the photon polarization function and the superindices (0) mean that we evaluate the so tagged quantities to lowest order in electromagnetic interactions. At low energy this is dominated by the 2π state and we have

$$\mathrm{Im}\,\Pi = \mathrm{Im}\,\Pi_{2\pi}(t) = \frac{1}{48\pi}\left(1 - \frac{4\mu^2}{t}\right)^{3/2} |F_\pi(t)|^2. \tag{6.2.1}$$

The evaluation of the pion form factor is slightly complicated by the phenomenon of $\omega - \rho$ interference. This can be solved by considering only the isospin $I = 1$ component, and adding later the $\omega \to 2\pi$ and interference separately; that is to say, in a first approximation we neglect the breaking of isospin invariance. We will also neglect for now electromagnetic corrections. In this approximation the properties of $F_\pi(t)$ are the following:

[15] In fact, it can be proved (Atkinson, Mahoux and Ynduráin, 1973) that, even if one only has two channels, say, $\pi\pi$ and $\bar K K$, there is no unique solution (at fixed energy) unless one also measured $\pi\pi \to \bar K K$, and $\bar K K \to \bar K K$ as well. That inelastic channels are important for $s^{1/2} \gtrsim 1.4$ GeV is clear by looking at the branching ratios of resonances with higher mass.

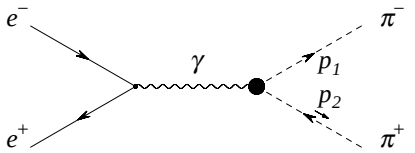

FIGURE 6.2.1. Diagram for $e^+e^- \to \pi^-\pi^+$.

(i) $F_\pi(t)$ is an analytic function of t, with a cut from $4\mu^2$ to infinity.
(ii) On the cut, the phase of $F_\pi(t)$ is, because of unitarity, identical to that of the P wave, $I = 1$, $\pi\pi$ scattering, $\delta_1(t)$, and this equality holds until the opening of the inelastic threshold at $t = s_0$. This we showed in Sect. 5.4, and the property is known as the Fermi–Watson final state interaction theorem.
(iii) For large t, $F_\pi(t) \sim 1/t$. This follows from perturbative QCD.
(iv) $F(0) = 1$.

The inelastic threshold occurs, rigorously speaking, at $t = 16\mu^2$. However, it is an experimental fact that inelasticity is negligible until the quasi-two body channels $\omega\pi$, $a_1\pi$ ... are open. In practice we will take

$$s_0 \simeq 1 \text{ GeV}^2,$$

and fix the best value for s_0 empirically. It will be $s_0 = 1.05^2$ GeV^2, and it so happens that, if we keep close to this value, the dependence of the results of our analysis on s_0 is very slight.

6.2.2. Form factor of the pion in τ decay

Besides the process $e^+e^- \to \pi^+\pi^-$ one can get data on the vector pion form factor from the decay $\tau^+ \to \bar\nu_\tau\pi^+\pi^0$ (Fig. 6.2.2) For this we have to assume isospin invariance, to write the form factor v_1 for τ decay in terms of F_π:

$$v_1 = \tfrac{1}{12}\left(1 - \frac{4\mu^2}{t}\right)^{3/2} |F_\pi(t)|^2, \tag{6.2.2a}$$

where, in terms of the weak vector current $V_\mu = \bar u\gamma_\mu d$, and in the exact isospin approximation,

$$\begin{aligned} \Pi^V_{\mu\nu} &= \left(-p^2 g_{\mu\nu} + p_\mu p_\nu\right)\Pi^V(t) = \mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}p\cdot x}\langle 0|\mathrm{T}V^+_\mu(x)V_\nu(0)|0\rangle; \\ v_1 &= 2\pi\,\mathrm{Im}\,\Pi^V. \end{aligned} \tag{6.2.2b}$$

Eq. (6.2.2) may be verified inserting a complete set of states in the expression for $\mathrm{Im}\,\Pi^V$, and assuming it to be saturated by the states $|\pi^+\pi^0\rangle$; cf. Eq. (5.6.7).

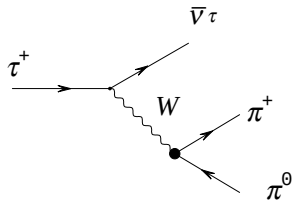

FIGURE 6.2.2
Diagram for $\tau \to \bar{\nu}_\tau \pi^0 \pi^+$.

We next make a few remarks concerning the matter of isospin breaking, due to electromagnetic interactions or the mass difference between u, d quarks, that would spoil the equality (6.2.2a). It is not easy to estimate this. A large part of the breaking, the $\omega \to 2\pi$ contribution and $\omega - \rho$ mixing, may be taken into account explicitly (for the form factor in $\pi^+\pi^-$) with the Gounnaris–Sakurai (1968) method, but this does not exhaust the effects. Eqs. (6.2.2) were obtained neglecting the mass difference $m_u - m_d$ and electromagnetic corrections, in particular the $\pi^0 - \pi^+$ mass difference. We can take the last partially into account by distinguishing between the pion masses in the phase space factor in (6.2.2a). To do so, we write now (6.2.2b) as

$$\Pi^V_{\mu\nu} = \mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}p\cdot x}\langle 0|\mathrm{T}V^+_\mu(x)V_\nu(0)|0\rangle = \left(-p^2 g_{\mu\nu} + p_\mu p_\nu\right)\Pi^V(t) + p_\mu p_\nu \Pi^S; \quad v_1 \equiv 2\pi\,\mathrm{Im}\,\Pi^V. \tag{6.2.3a}$$

We find

$$v_1 = \tfrac{1}{12}\left\{\left[1 - \frac{(m_{\pi^+} - m_{\pi^0})^2}{t}\right]\left[1 - \frac{(m_{\pi^+} + m_{\pi^0})^2}{t}\right]\right\}^{3/2} |F_\pi(t)|^2. \tag{6.2.3b}$$

To compare with the experimentally measured quantity, which involves all of $\mathrm{Im}\,\Pi^V_{\mu\nu}$, we have to neglect the scalar component Π^S. This is reasonable, as it is proportional to $(m_d - m_u)^2$, and thus likely very small. This matter of isospin breaking one thus treats in successive steps. First, we neglect isospin breaking. Then we take it into account by admitting different masses and widths for the resonances ρ^0, ρ^+, including $\omega - \rho$ mixing, and taking into account the difference in phase space, etc. Before doing so, however, we must develop the necessary mathematical tools, which we will do in next chapter.

6.2.3. K_{l4} decay

We now consider the so-called K_{l4} decay (K_{l4} stands for *leptonic four body* decay),

$$K \to l\bar{\nu}_l \pi^+\pi^-,$$

with l an electron or a μ^-. The effective lagrangian for the decay is

$$\mathcal{L}_{\mathrm{int,eff}} = \frac{G_F \sin\theta}{\sqrt{2}}\,\bar{l}\gamma_\mu(1-\gamma_5)\nu_l\,\bar{s}\gamma^\mu(1-\gamma_5)u,$$

where G_F is Fermi's constant, θ the Cabibbo angle, and s, u the field operators for the corresponding quarks. The decay amplitude is then

$$F(K \to l\bar{\nu}_l\pi^+\pi^-) = \frac{G_F \sin\theta}{\sqrt{2}(2\pi)^2}\bar{v}_l\gamma_\mu(1-\gamma_5)u_{\nu_l}F^\mu(s); \tag{6.2.4a}$$

the form factor F^μ is

$$F^\mu(s) = \langle \pi^+(p_+)\pi^-(p_-)|\bar{s}(0)\gamma^\mu\gamma_5 u(0)|K\rangle, \quad s = (p_+ + p_-)^2. \tag{6.2.4b}$$

If we expand F^μ into a scalar (F_S) and a vector piece, F_P (P for P wave),

$$F^\mu = (p_+^\mu + p_-^\mu)F_S + (p_+^\mu - p_-^\mu)F_P, \tag{6.2.5}$$

then one can, with an argument like that of Sect. 5.5, show that

$$\arg F_S(s) = \delta_0^{(0)}(s), \quad \arg F_P(s) = \delta_1(s). \tag{6.2.6}$$

It follows that, by measuring the differential decay rate

$$\frac{\mathrm{d}\Gamma(K \to l\bar{\nu}_l\pi^+\pi^-)}{\mathrm{d}s\mathrm{d}\Omega_{\mathbf{p}}},$$

with $\mathbf{p} = \mathbf{p}_+|_{\text{c.m.}}$, we can separate the contributions from $|F_S|^2$, $|F_P|^2$ and the interference piece, $F_S^* F_P = |F_S|\,|F_P|\cos[\delta_0^{(0)}(s) - \delta_1(s)]$, and thus get the difference of phases $\delta_0^{(0)}(s) - \delta_1(s)$. This provides very important information on low energy $\pi\pi$ scattering, particularly since, in this process, both pions are on their mass shell.

6.2.4. The $K \to 2\pi$ decays

If we denote by H_W to the weak interaction hamiltonian, we will consider the matrix elements related to the decays $K^+ \to \pi^+\pi^0$, $K_S \to \pi^+\pi^-$ and $K_S \to \pi^0\pi^0$:

$$\begin{aligned}
\langle \pi^+\pi^0|H_W|K^+\rangle &= -\langle 2,1|H_W|K^+\rangle, \\
\langle \pi^+\pi^-|H_W|K_S\rangle &= -\sqrt{\tfrac{1}{3}}\,\langle 2,0|H_W|K_S\rangle - \sqrt{\tfrac{2}{3}}\,\langle 0,0|H_W|K_S\rangle, \\
\langle \pi^0\pi^0|H_W|K_S\rangle &= \sqrt{\tfrac{2}{3}}\,\langle 2,0|H_W|K_S\rangle - \sqrt{\tfrac{1}{3}}\,\langle 0,0|H_W|K_S\rangle.
\end{aligned} \tag{6.2.7}$$

Here the labels in the $\langle I, I_3|$ refer to isospin and third component thereof. From the Fermi–Watson final state interaction theorem, it follows that

$$\langle I, I_3|H_W|K\rangle = \left|\langle I, I_3|H_W|K\rangle\right| \mathrm{e}^{\mathrm{i}\delta_0^{(I)}(m_K^2)}, \tag{6.2.8}$$

and, therefore, measuring the three decays provides a determination of the difference of phase shifts

$$\delta_0^{(0)}(m_K^2) - \delta_0^{(2)}(m_K^2). \tag{6.2.9}$$

A precise analysis requires considerations of isospin violation, especially by electromagnetic interactions,[16] that shift the phase by some 4°. The old experimental determinations gave (see, e.g. Pascual and Ynduráin, 1974)

$$\delta_0^{(0)}(m_K^2) - \delta_0^{(2)}(m_K^2) = 58.0 \pm 4.6^\circ, \tag{6.2.10a}$$

while more modern determinations of the kaon decays (Aloisio et al., 2002; Gatti, 2003) have led to the numbers

$$\begin{aligned}
\delta_0^{(0)}(m_K^2) - \delta_0^{(2)}(m_K^2) &= 48.5 \pm 2.6; \\
\delta_0^{(0)}(m_K^2) - \delta_0^{(2)}(m_K^2) &= 47.8 \pm 2.8.
\end{aligned} \tag{6.2.10b}$$

[16] Belavin and Navodetsky (1968); Nachtmann and de Rafael (1969); Cirigliano, Donoghue and Golowich (2000).

6.3. The P wave

We present in this and the following two sections of the present chapter parametrizations of the S, P, D and F waves in $\pi\pi$ scattering that follow from the theoretical requirements we have discussed in previous chapters, and which agree with *experimental* data (we will also say a few words on G waves). To check that the scattering amplitude that one obtains in this way is consistent with dispersion relations or the Froissart–Gribov representation will be done in the following chapter. When neglecting isospin violations we will take the Gasser–Leutwyler convention of approximating the pion mass by $M_\pi = m_{\pi^\pm}$.

6.3.1. The P wave in the elastic approximation

We will consider first the P wave for $\pi\pi$ scattering, because it is obtained at low energy with a method different from those used for the other waves. We start thus considering the region of energies where the inelasticity is below the 2% level; say, $s_0 \leq 1.1$ GeV^2. We will neglect for the moment isospin invariance violations due to e.m. interactions or the $u - d$ quark mass difference. This implies, in particular, neglecting the ω and ϕ interference effects.

We may use the analyticity properties of $\Phi_1(s)$ to write a simple parametrization of $\Phi_1(s)$, hence of $\delta_1(s)$. An effective range expansion is not enough, as it only converges in the region $|s-4M_\pi^2| < 0$ (Fig. 3.1.2). To take fully advantage of the analyticity domain, shown in Fig. 2.1.1, the simplest procedure is to make a conformal mapping of the cut plane into the unit disk (Fig. 6.3.1) by means of the transformation[17]

$$w = \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}}. \tag{6.3.1}$$

One can then expand $\Phi_1(s)$ in powers of w, and, reexpressing w in terms of s, the expansion will be convergent over all the cut s-plane. Actually, and because we know that the P wave resonates at $s = M_\rho^2$, it is more convenient to expand not $\Phi_1(s)$ itself, but $\psi(s)$ given by

$$\Phi_1(s) = (s - M_\rho^2)\psi(s)/4; \tag{6.3.2a}$$

so we write

$$\psi(s) = \{B_0 + B_1 w + \cdots\}. \tag{6.3.2b}$$

In terms of $\Phi_1(s)$ we find the expression for the phase shift, keeping two terms in the expansion,

$$\cot\delta_1(s) = \frac{s^{1/2}}{2k^3}(M_\rho^2 - s)\left[B_0 + B_1\frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}}\right]; \tag{6.3.3}$$

M_ρ, B_0, B_1 are free parameters to be fitted to experiment. In terms of Φ_1, ψ we have, for the rho width,

$$\Gamma_\rho = \frac{2k_\rho^3}{M_\rho^2\psi(M_\rho^2)}, \quad k_\rho = \tfrac{1}{2}\sqrt{M_\rho^2 - 4M_\pi^2}, \tag{6.3.4a}$$

[17] This type of parametrization presents a number of advantages with respect to less efficient ones used in the literature. The gain obtained by taking into account the correct analyticity properties is enormous; see the Appendix B here for a discussion and an explicit example, and Pišut (1970) for other examples and applications to $\pi\pi$ scattering. Moreover, the physical meaning of, say, (6.3.3) is very clear: B_0 gives the normalization, and B_1 is related to the average intensity of the l.h. cut and the inelastic cut.

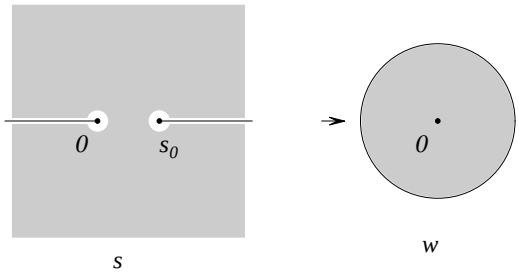

FIGURE 6.3.1. The mapping $s \to w$.

and the scattering length, a_1, is

$$a_1 = \frac{1}{4M_\pi \Phi_1(4M_\pi^2)} = \frac{1}{M_\pi \psi(4M_\pi^2)}. \tag{6.3.4b}$$

The values $B_0 = \text{const.}$, $B_{i\geq 1} = 0$ would correspond to a perfect Breit–Wigner. Actually, it is known that the ρ deviates from a pure Breit–Wigner and for a precision parametrization two terms, B_0 and B_1, have to be kept in (6.3.3). Note that the parametrization holds not only on the physical region $4M_\pi^2 \leq s \leq s_0$, but on the unphysical region $0 \leq s \leq 4M_\pi^2$ and also over the whole region of the complex s plane with $\text{Im}\, s \neq 0$. The parametrization given now is the one that has less biases, in the sense that no model has been used: we have imposed only the highly safe requirements of analyticity and unitarity, depending only on causality and conservation of probability.

The best values for our parameters are actually obtained from fits to the pion form factor, that we will discuss in Sect. 7.2. Including systematic experimental errors in the fits, and fitting also the value $a_1 = (38 \pm 3) \times 10^{-3}\, M_\pi^{-3}$ for the scattering length we have,

$$B_0 = 1.071 \pm 0.007, \quad B_1 = 0.18 \pm 0.05; \quad M_\rho = 773.5 \pm 0.85 \text{ MeV}\,. \tag{6.3.5a}$$

The corresponding values for the width of the ρ and for the scattering length and effective range parameter are

$$\begin{aligned} &a_1 = (38.6 \pm 1.2) \times 10^{-3} M_\pi^{-3}, \quad b_1 = (4.47 \pm 0.29) \times 10^{-3} M_\pi^{-5}; \\ &\Gamma_\rho = 145.5 \pm 1.1 \text{ MeV}\,. \end{aligned} \tag{6.3.5b}$$

Although the values of the experimental $\pi\pi$ phase shifts were *not* included in the fit, the phase shifts that (6.3.5a) implies are en very good agreement with them, as shown in Fig. 6.3.2.

Eqs. (6.3.5) above were evaluated with an average of information on the two channels that contain the $I = 1$ P wave, $\pi^+\pi^-$ (dominated by the ρ^0) and $\pi^0\pi^+$, dominated by the ρ^+. The

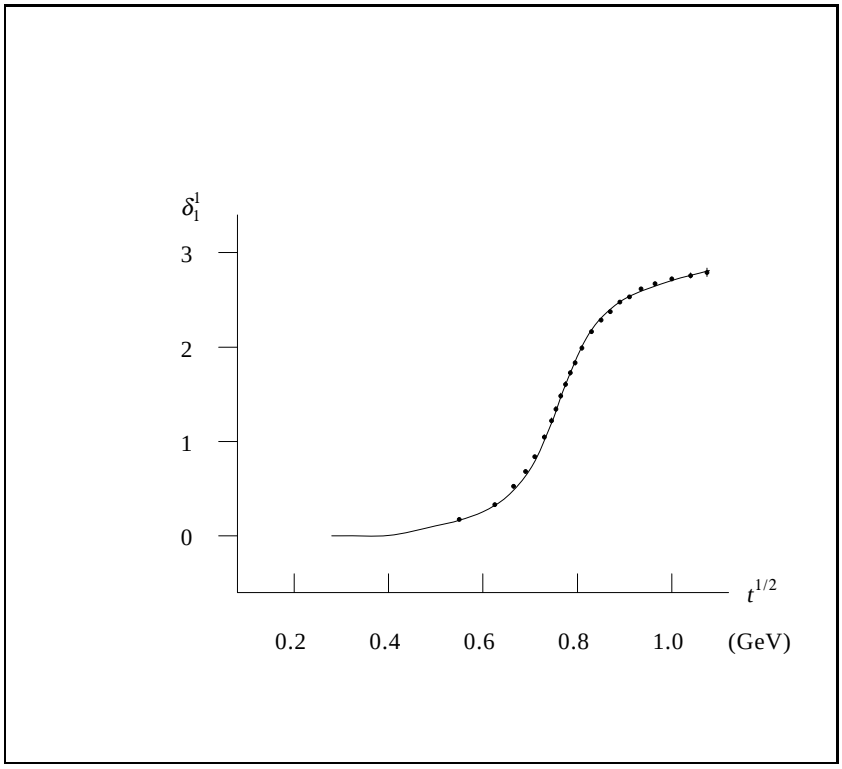

FIGURE 6.3.2 The phase shifts of solution 1 from Protopopescu et al. (1973) (the dots, with errors of the size of the dots) compared with the prediction with the parameters (6.3.5a), described by the solid line. We emphasize that this solid line is *not* a fit to the data of Protopopescu et al., but is obtained from the pion form factor.

values for a pure ρ^0 ($\pi^+\pi^-$) are slightly different; we find

$$\begin{aligned} &B_0 = 1.065 \pm 0.007, \quad B_1 = 0.17 \pm 0.05, \quad M_{\rho^0} = 773.1 \pm 0.6, \\ &\Gamma_{\rho^0} = 147.4 \pm 1.0 \text{ MeV}, \end{aligned} \tag{6.3.5c}$$

and a_1, b_1 do not change appreciably. However, this last feature occurs only because the fit was made including the constraint $a_1 = (38 \pm 3) \times 10^{-3}\, M_\pi^{-3}$; see Sect. 9.5 for more on this. Eqs. (6.3.5) provide an estimate of the importance of isospin breaking.

6.3.2. The ρ and weakly coupled inelastic channels: $\omega - \rho$ interference

Because of the different masses of the u, d quarks, isospin invariance is broken and there is a nonzero probability of transition between $\pi^+\pi^-$ in isospin 1 and isospin 0 states: hence, a small –but nonzero– mixing of the ρ and ω resonances.

To study this phenomenon a popular approximation is that of Gounnaris and Sakurai (1968). A consistent treatment requires a two-channel analysis. We denote by channel 1 to the P wave

isospin 1 $\pi^+\pi^-$ state, and channel 2 will be a P wave isospin zero 3π state. To be fully rigorous, we would have to set up a three-body formalism for the last; but we will simply take this into account replacing the two body by three body phase space for the ω. Using now Eqs. (5.4.4) to (5.4.7) we write

$$\begin{aligned} f_{11}^{(1)} =& \frac{1}{\pi}\frac{k_1^3\gamma_1}{M_\rho^2 - s - \mathrm{i}k_1^3\gamma_\rho(s)/2s^{1/2}} \\ &\times\left\{1 + \frac{M_\rho^2 - s}{k_\omega^3(s)\left(M_\rho^2 - s - \mathrm{i}k_1^{2l+1}\gamma_\rho(s)/2s^{1/2}\right)}\,\frac{\epsilon_{12}^2\gamma_\omega k_2^3}{M_\omega^2 - s - \mathrm{i}k_\omega^3(s)\gamma_\omega/2s^{1/2}}\right\}. \end{aligned} \tag{6.3.7}$$

Here we still have

$$k_1 = \tfrac{1}{2}\sqrt{s - 4M_\pi^2} \tag{6.3.8a}$$

but for $k_\omega(s)$ we have to take the value following from three-body phase space. Because the interference effect is only important near $s = M_\omega^2$, a reasonable approximation for it is to take k_ω constant: this is the model of Gounnaris and Sakurai (1968). The model is completed if we take a constant width for the ω, justified in view of its narrowness, but a full effective range formula for the ρ:

$$\gamma_\omega = \Gamma_\omega^2/2f_\omega(M_\omega^2), \quad \gamma_\rho(s) = 1/\bar{\Phi}(s), \tag{6.3.8b}$$

with $\bar{\Phi}(s)$ given by a parametrization like (6.3.3). The effect of this modulation is a shoulder above the ρ that may be seen in e.g. the pion form factor (cf. Fig. 7.2.1).

6.3.3. The P wave for $1\,\mathrm{GeV} \le s^{1/2} \le 1.42\,\mathrm{GeV}$

In the range 1 GeV $\le s^{1/2} \le$ 1.3 GeV one is sufficiently far away from thresholds to neglect their influence (the coupling to $\bar{K}K$ is negligible) and, moreover, the inelasticity is reported small: according to Protopopescu et al. (1973) and Hyams et al. (1973), below the 7% level. A purely empirical parametrization that agrees with the data in this references up to 1.2 GeV, within errors, is given by a modulated ρ tail,

$$\begin{aligned} \delta_1(s) =& \,\mathrm{arc\ cot}\,\frac{\eta\,(M_\rho - s)}{M_\rho\Gamma_\rho} - \epsilon\left(1 - \frac{4m_K^2}{s}\right)^{3/2}, \\ \eta =& \,0.75 \pm 0.10, \quad \epsilon = 0.08 \pm 0.02. \end{aligned} \tag{6.3.9}$$

and the second term takes into account the effects of the inelasticity.

For larger $s^{1/2}$, (6.3.9) is incompatible with the properties of the P wave as measured in the analysis of Hyams et al. (1973), or in e^+e^- annihilations, where a highly inelastic resonance,that we here denote by ρ', occurs around 1450 MeV. An alternate parametrization for the imaginary part of the p.w. amplitude that takes this into account is obtained by adding to the imaginary part produced by (6.3.9) the inelastic piece

$$\begin{aligned} \mathrm{Im}\, f_{1;\mathrm{inel}}(s) =& \frac{2s^{1/2}}{\pi k}\frac{\mathrm{BR}\times M_{\rho'}^2\Gamma^2[k/k(M_{\rho'}^2]^6}{(s - M_{\rho'}^2)^2 + M_{\rho'}^2\Gamma^2[k/k(M_{\rho'}^2]^6}; \\ M_{\rho'} =& \,1.45\ \mathrm{MeV}, \quad \Gamma = 310\ \mathrm{MeV}, \quad \mathrm{BR} \simeq 0.15. \end{aligned} \tag{6.3.10}$$

The value of BR could vary by 50%. For more details, see Appendix A.

6.4. The S waves

6.4.1. Parametrization of the S wave for $I = 2$

We consider two sets of experimental data. The first, that we will denote by "Hoogland A", corresponds to solution A in the paper by Hoogland et al. (1977), who use the reaction $\pi^+ p \to \pi^+\pi^+ n$; and the second set, denoted by "Losty," corresponds to that from the work of Losty et al. (1974), who analyze instead $\pi^- p \to \pi^-\pi^-\Delta$. We will not consider the so-called solution B in the paper of Hoogland et al. (1977); while it produces results similar to the other two, its errors are clearly underestimated. We will also not include in the fit the data of Cohen et al. (1973); it may be biased at low energy because it is obtained from scattering on neutons bound in deuterium. Neverteless, our fits go nicely over these experimental points. The result of Losty et al and Hoogland et al. (also those of Cohen et al.) represent a substantial improvement over previous ones; since they produce two like charge pions, only isospin 2 contributes, and one gets rid of the large isospin zero S wave and P wave contamination. However, they still present the problem that one does not have scattering of real pions.

For isospin 2, there is no low energy resonance, but $f_0^{(2)}(s)$ presents the feature that a zero is expected (and, indeed, confirmed by the fits) in the region $0 < s < 4M_\pi^2$. If we neglected this and wrote

$$\cot\delta_0^{(2)}(s) = \frac{2s^{1/2}}{2k}\,\frac{B_0 + B_1 w(s)}{4};$$
$$w = \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}}, \quad s_0 = (1.450\ \text{GeV})^2,$$

then we could fit the data with the parameters

$$B_0 = -1.87, \quad B_1 = 5.56.$$

We have a not too bad χ^2 /d.o.f. $= 13.8/(14-2)$ but the expansion has poor convergence properties as, in most of the region, $|B_1 w|$ is rather larger than $|B_0|$. The corresponding value of the scattering length would be $a_0^{(2)} = -0.16\, M_\pi^{-1}$, way too large (that a naive fit gives a scattering length of this order has been known for a long time; see Prokup et al., 1974). Clearly, we have to take the zero of the partial wave into account.

The zero of $f_0^{(2)}(s)$ is related to the so-called Adler zeros (see Chapter 9) and, to lowest order in chiral perturbation theory, occurs at $s = 2z_2^2$ with $z_2 = M_\pi$. In view of this, in a first fit we extract the zero (leaving its value as a *free* parameter) and write

$$\cot\delta_0^{(2)}(s) = \frac{s^{1/2}}{2k}\,\frac{M_\pi^2}{s - 2z_2^2}\Big\{B_0 + B_1 w(s)\Big\}. \tag{6.4.1a}$$

The quality of the fit improves substantially: we get χ^2 /d.o.f. $= 8.0/(14-3)$ and a second order term such that $|B_1 w| < |B_0|$. The parameters are now

$$B_0 = -116 \pm 5.6, \quad B_1 = -127 \pm 9, \qquad z_2 = 145 \pm 21\ \text{MeV}\,. \tag{6.4.1b}$$

$w(s)$ is as for the D2 wave, Eq. (6.4.1b).

In the fit (6.4.1) we have not considered experimental data above 0.97 GeV. The result for the scattering length,

$$a_0^{(2)} = (-0.061 \pm 0.023)\, M_\pi^{-1}. \tag{6.4.1c}$$

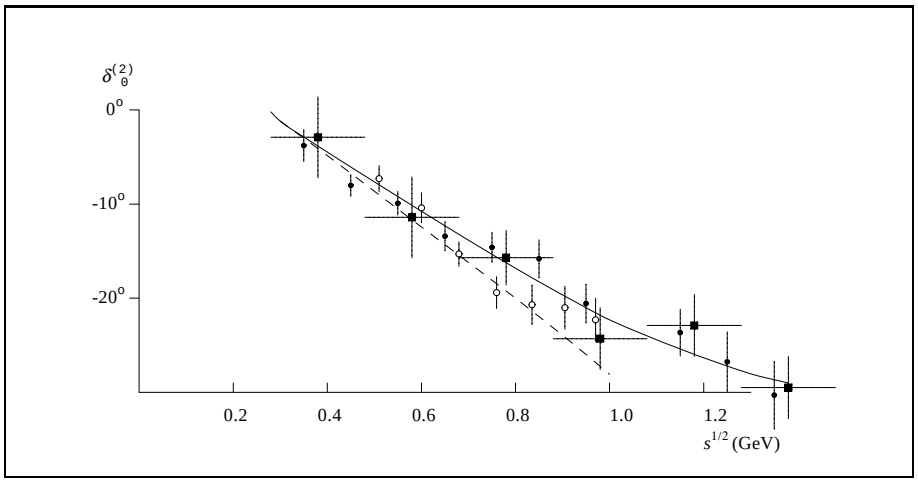

FIGURE 6.4.1. The $I = 2$, S-wave phase shifts corresponding to (6.4.1), with experimental points from Losty et al. (1974) (open circles), Hoogland et al. (1977), solution A (black dots) and Cohen et al. (1973) (crosses). The dashed line is the S2 phase of Colangelo, Gasser and Leutwyler (2001).

is compatible (within $\sim 1\,\sigma$), as we will see, with the values suggested by chiral perturbation theory;[18] and this agreement is satisfactory also in another respect: the value for z_2 which the fit returns, $z_2 = 145 \pm 21$ MeV, comprises the value expected from second order chiral perturbation theory that gives $z_2 = 131$ MeV.

One can improve on this fit by using forward dispersion relations, in the form of the Olsson sum rule, see Subsect. 7.4.3. We moreover fix $z_2 = M_\pi$ and fit all experimental data, up to $s^{1/2} = 1350$ MeV. One finds slightly different parameters:

$$\begin{aligned} \cot\delta_0^{(2)}(s) &= \frac{s^{1/2}}{2k}\,\frac{M_\pi^2}{s-2z_2^2}\left\{B_0 + B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}\right\}; \\ s_0^{1/2} &= 1.45\ \text{GeV}; \quad \chi^2/\text{d.o.f.} = 17.2/(19-2). \\ B_0 &= -\,118 \pm 2.5, \quad B_1 = -105 \pm 2.5, \quad z_2 = 139.57\ \text{MeV [fixed]}. \end{aligned} \tag{6.4.2a}$$

The value of the scattering length which this implies,

$$a_0^{(2)} = (-0.0422 \pm 0.0022)\, M_\pi^{-1},$$

is between what is obtained with the help of Roy equations by different groups:

$$\begin{aligned} a_0^{(2)} &= (-0.0444 \pm 0.0010)\, M_\pi^{-1} \quad \text{(Colangelo, Gasser ad Leutwyler, 2001)}, \\ a_0^{(2)} &= (-0.0382 \pm 0.0038)\, M_\pi^{-1} \quad \text{(Descotes et al., 2002)}, \\ a_0^{(2)} &= (-0.0343 \pm 0.0036)\, M_\pi^{-1} \quad \text{(Kamiński, Leśniak and Loiseau (2003))}. \end{aligned}$$

[18] The discrepancy in the central value is likely due to a systematic bias of the experimental data. In fact, if we also include in the fit the data of the Cern-Munich collaboration, in the version of Estabrooks and Martin (1974), the central value of $a_0^{(2)}$ becomes $0.46\, M_\pi^{-1}$.

We have, for a precision representation, to include the inelasticity as determined in the experiments of Cohen et el. (1973) and Losty et al. (1974). We will take a purely empirical fit,

$$\eta_0^{(2)}(s) = 1 - c(1 - M_{\rm eff}^2/s)^{3/2}, \quad c = 0.28 \pm 0.12. \tag{6.4.2b}$$

The fit is good, χ^2 /d.o.f. $= 4.2/(5-1)$, and this formula is supposed to hold for $s^{1/2} \geq 0.96$ GeV.

We will take (6.4.2) to be valid up to 1.42 GeV.

6.4.2. Parametrization of the S wave for $I = 0$

The S wave with isospin zero is by far the most difficult to parametrize. Here we have a very broad enhancement, variously denoted as ϵ, σ, f_0, around $s^{1/2} \equiv M_\sigma \sim 800$ MeV; we will use the name σ. We will not discuss here whether this enhancement is a *bona fides* resonance; we merely remark that in all experimental phase shift analyses $\delta_0^{(0)}(s)$ crosses 90° somewhere between 600 and 900 MeV. (This is *not* enough to class the object as a resonance. For example, the derivative $\mathrm{d}\delta_0^{(0)}(s)/\mathrm{d}s$ is more a minimum than a maximum at M_σ).

There is also a possible resonance, which used to be called S^* and is now denoted by $f_0(980)$, and another resonance (which was called ϵ' in the seventies), labeled as $f_0(1370)$ in the Particle Data Tables, with a mass around 1.37 GeV. Moreover,we expect a zero of $f_0^{(0)}(s)$ (Adler zero), hence a pole of the effective range function $\Phi_0^{(0)}(s)$, for $s = z_0^2$ with z_0^2 in the region $0 < s < 4M_\pi^2$. In fact, chiral perturbation theory suggests that this zero is located at $z_0^2 = \frac{1}{2}M_\pi^2$ but, as we will discuss in Subsect. 9.3.5, one cannot trust the accuracy of this prediction, unlike what happened for the $I = 2$ zero, z_2.

We can distinguish two energy regions: below $s_0^{1/2} = 2m_K$ we are under the $\bar{K}K$ threshold. Between $s_0^{1/2}$ and $s^{1/2} \sim 1.2$ there is a strong coupling between the $\bar{K}K$ and $\pi\pi$ channels and the analysis becomes very unstable, because there is little information on the process $\pi\pi \to \bar{K}K$ and even less on $\bar{K}K \to \bar{K}K$. We will not treat this case here in any detail; the interested reader may find details and references in Ynduráin (1975), Aguilar-Benítez et al. (1978). We will merely present, in the next subsection, an empirical fit in the region of energies around and above 1 GeV, and we will now concentrate our efforts in the low energy region.

Below the $\bar{K}K$ threshold we can write a one-channel formula:

$$\cot\delta_0^{(0)}(s) = \frac{2s^{1/2}}{k}\,\Phi_0^{(0)}(s). \tag{6.4.4}$$

To parametrize $\Phi_0^{(0)}$ we have, as stated, a difficult situation, from the theoretical as well as from the experimental point of view. From the first, and because of the strong coupling of the $\bar{K}K$ channel above $s = 4m_K^2$, it is essential to take into account the presence of the associated cut. Moreover, and to reproduce correctly the low energy region data, the Adler zero cannot be neglected: we must necessarily use a complicated parametrization.

On the experimental side the situation is still a bit confused, although it has cleared up substantially in the last years. The experimental information we have on this S0 wave is of three kinds: from phase shift analysis in collisions $\pi p \to \pi\pi N, \Delta$; from the decay K_{l4}; and from the decay $K_{2\pi}$ (Subsects. 6.2.3, 6.2.4). The last gives the value of the combination $\delta_0^{(0)} - \delta_0^{(2)}$ at $s^{1/2} = m_K$; the decay K_{l4} gives $\delta_0^{(0)} - \delta_1$ at low energies, $s^{1/2} \lesssim 380$ MeV. If using the more recent $K_{2\pi}$ information (Aloisio et al., 2002) together with the $I = 2$ phase obtained in the previous subsection, this gives the $I = 0$ phase

$$\delta_0^{(0)}(m_K^2) = 43.3^\circ \pm 2.3^\circ\,. \tag{6.4.6}$$

The change is substantial from the previous experimental values that implied (Pascual and Ynduráin, 1974)

$$\delta_0^{(0)}(m_K^2) = 51^\circ \pm 8^\circ .$$

From the various phase shift analyses one concludes that there is *not* a unique solution if fitting only $\pi\pi$ data; one can get an idea of the uncertainties in old analyses by having a look at Fig. 3.3.6 in the book by Martin, Morgan and Shaw (1975) or realizing that the values of the scattering length $a_0^{(0)}$ that the various experimental fits (Protopopescu et al., 1973; Hyams et al., 1973; Grayer et al. 1974) gave varied in the range

$$0.1 \leq a_0^{(0)} \leq 0.9\ M_\pi^{-1}.$$

Today one can improve substantially on this thanks to the appearance of K_{l4} decay data and to use of consistency conditions, but, as we will see, the situation is not as satisfactory as for the P wave.

We will here consider here the following set of data to be fitted. First of all we take the low energy data from K_{l4} decay (Rosselet et al., 1977; Pislak et al., 2001).[19] Then we impose the value of $\delta_0^{(0)}(m_K^2)$ in (6.4.6).

The main virtue of these data is that they refer to pions on their mass shell; but, unfortunately, this is not sufficient to stabilize the fit at high energy, $s^{1/2} \gtrsim 0.8$ GeV. For this we have to add further data:

$$\begin{aligned} &\delta_0^{(0)}(0.870^2\ \text{GeV}^2) = 91 \pm 9^\circ\,; \quad \delta_0^{(0)}(0.910^2\ \text{GeV}^2) = 99 \pm 6^\circ\,; \\ &\delta_0^{(0)}(0.935^2\ \text{GeV}^2) = 109 \pm 8^\circ\,; \quad \delta_0^{(0)}(0.965^2\ \text{GeV}^2) = 134 \pm 14^\circ\,. \end{aligned} \tag{6.4.7a}$$

These points are taken from solution 1 of Protopopescu et al. (1973) (both with and without modified moments), with the error increased by the difference between this and solution 3 data in the same reference. These data points have the rare virtue of agreeing, within errors, with the results of other experimental analyses. Their inclusion is essential; if we omit them, the fits would produce results at total variance with experimental information above $s^{1/2} = 0.5$ GeV. We will also include in the fit the data, at similar energies, of Grayer et al. (1974):

$$\begin{aligned} &\delta_0^{(0)}(0.912^2\ \text{GeV}^2) = 103 \pm 8^\circ\,; \quad \delta_0^{(0)}(0.929^2\ \text{GeV}^2) = 112.5 \pm 13^\circ\,; \\ &\delta_0^{(0)}(0.952^2\ \text{GeV}^2) = 126 \pm 16^\circ\,; \quad \delta_0^{(0)}(0.970^2\ \text{GeV}^2) = 141 \pm 18^\circ\,. \end{aligned} \tag{6.4.7b}$$

The central values are obtained averaging the three solutions given by Grayer et al., and the error is calculated adding quadratically the statistical error of the highest point, the statistical error of the lowest point (for each energy) and the difference between the central value and the farthest point. Moreover, we add three points between 0.8 and 0.9 GeV obtained averaging the s-channel solution of Estabrooks and Martin (1974), which consistently provides the less biased data (see for example for the D0 wave, fig. 6.4.1), and solution 1 of Protopopescu et al. (1973), which represent

[19] As a technical point, we mention that we have increased by 50% the error in the point at highest energy, $s^{1/2} = 381.4$ MeV, from the K_{e4} compilation of Pislak et al. (2001), whose status is dubious; the experimental value represents an average over a long energy range that extends to the edge of phase space.

two extremes. The error is obtained adding the difference between these two in quadrature to the largest statistical error. In this way we obtain the numbers,

$$\begin{aligned}&\delta_0^{(0)}(0.810^2\ \text{GeV}^2) = 88 \pm 6^\circ\,; \quad \delta_0^{(0)}(0.830^2\ \text{GeV}^2) = 92 \pm 7^\circ\,;\\ &\delta_0^{(0)}(0.850^2\ \text{GeV}^2) = 94 \pm 6^\circ\,.\end{aligned} \tag{6.4.7c}$$

We will not add points at lower energies ($0.5\ \text{GeV} \le s^{1/2} \le 0.8\ \text{GeV}$); the difference among the values found for the phases in different experiments is such that no meaningful value could be given for the errors.

For the theoretical formulas we consider two basic possibilities. We impose the Adler zero at $s = \frac{1}{2}M_\pi^2$ (no attempt is made to vary this), and a resonance with mass M_σ, a free parameter. Then we map the s plane cut along the left hand cut ($s \le 0$) and the $\bar{K}K$ cut, writing

$$\cot\delta_0^{(0)}(s) = \frac{s^{1/2}}{2k}\,\frac{M_\pi^2}{s-\frac{1}{2}M_\pi^2}\,\frac{M_\sigma^2 - s}{M_\sigma^2}\psi(s), \tag{6.4.8}$$

and

$$\psi(s) = \big[B_0 + B_1 w(s) + B_2 w(s)^2\big]; \quad w(s) = \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}}, \quad s_0 = 4m_K^2$$

(we have taken $m_K = (0.496\,\text{GeV})$). The complicated structure of this wave requires two or three parameters B_0, B_1 and B_2 (besides M_σ) for an acceptable fit.

This parametrization does not represent fully the coupling of the $\bar{K}K$ channel and, indeed, the corresponding phase shift deviates somewhat from experiment at the upper energy range ($s^{1/2} > 0.96$ GeV; see Fig. 6.4.3). We can, alternatively, try to use the reduction to one channel of the two channel formulas (5.4.1,2) and write

$$\cot\delta_0^{(0)}(s) = \frac{s^{1/2}}{2k}\psi_{\text{el}}(s), \quad \psi_{\text{el}}(s) = \frac{\frac{\kappa_2}{2s^{1/2}}\,\Phi_{11}(s) + \det\mathbf{\Phi}}{\frac{\kappa_2}{2s^{1/2}} + \Phi_{22}(s)} \tag{6.4.9a}$$

where $\kappa_2 = \frac{1}{2}\sqrt{4m_K^2 - s}$. We take a linear approximation for the Φ_{ii}, and a constant for Φ_{12}, requiring a zero of $\det\mathbf{\Phi}$ at $s = M_{f_0}^2$, and we allow M_{f_0} to vary between 1 and 1.4 GeV. So we write,

$$\begin{aligned}&\Phi_{11}(s) = \alpha_1 + \beta_1 s, \quad \Phi_{22}(s) = \alpha_2 + \beta_2 s,\\ &\det\mathbf{\Phi} = (\alpha_1 + \beta_1 s)(\alpha_2 + \beta_2 s) - (\alpha_1 + \beta_1 M_{f_0}^2)(\alpha_2 + \beta_2 M_{f_0}^2).\end{aligned} \tag{6.4.9b}$$

This represents correctly the $\bar{K}K$ cut, but does not allow for the Adler zero or produce a dynamical left hand cut. Therefore we expect reliability of (6.4.9) near $4m_K^2$, but poor description near threshold, which is indeed the case. We do not try to combine the two parametrizations as this would lead to a hopeless tangle due to the large number of parameters and also to the appearance of left hand cut of $\bar{K}K$, that the Φ_{ij} inherit (but which must cancel for ψ_{el}).

Let us now turn to the results of the fits. First of all, we note that the inclusion of the value of the phase at $s = m_K^2$ is essential; for example, if we had not included it we would have found

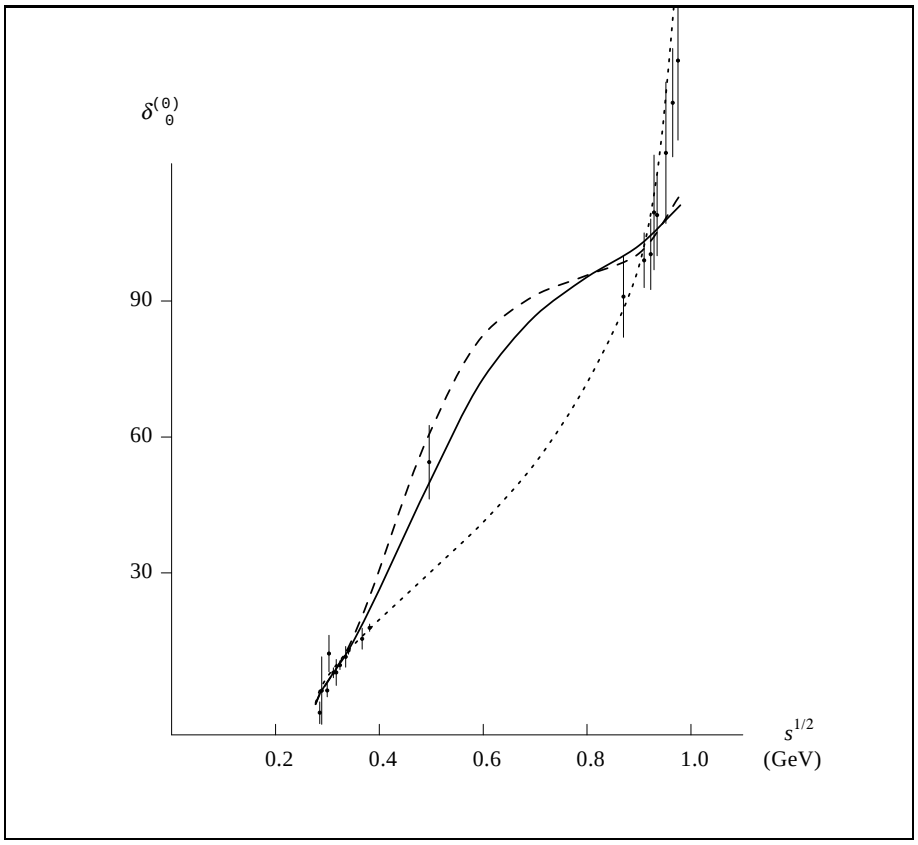

FIGURE 6.4.2. The $I = 0$, S-wave phase shifts corresponding to (6.4.10) (dashed line) and what one would get with the old value $\delta_0^{(0)}(M_K^2) = 49^\circ \pm 5^\circ$, continuous line. Also shown are the points, at low energies, from the K_{l4} experiments, the point from $K_{2\pi}$ decay (the more recent value), and the high energy data of Protopopescu et al. and Grayer et al. included in the fits.

the following minimum:[20]

$$\begin{aligned} &B_0 = 46.87 \pm 0.68, \quad B_1 = 92.72 \pm 1.47, \quad B_2 = 60.59 \pm 3.24, \\ &M_\sigma = 874 \pm 30 \text{ MeV}; \\ &a_0^{(0)} = (0.274 \pm 0.024)\, M_\pi^{-1}; \quad \delta_0^{(0)}(m_K^2) = 30^\circ\,. \end{aligned} \tag{6.4.10}$$

The χ^2 /d.o.f. $= 20.0/(20-4)$ is reasonable; the value of $\delta_0^{(0)}(m_K^2)$ is not.

If we impose $\delta_0^{(0)}(m_K^2)$ as given in (6.4.6) we find quite different results. With only two B_is,

[20] That using only phase shifts data there are two alternate possibilities for the intermediate energy S0 wave was recognized already by, e.g., Estabrooks and Martin (1974); see also the textbook of Martin, Morgan and Shaw (1975) for a discussion.

we have what we will call the *B2 Solution*,

$$\cot\delta_0^{(0)}(s) = \frac{s^{1/2}}{2k}\,\frac{M_\pi^2}{s-\frac{1}{2}M_\pi^2}\,\frac{M_\sigma^2-s}{M_\sigma^2}\,\left\{B_0 + B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}\right\};$$

$$B_0 = 21.04, \quad B_1 = 6.62, \quad M_\sigma = 782 \pm 24 \text{ MeV}; \quad \frac{\chi^2}{\text{d.o.f.}} = \frac{15.7}{19-3}. \tag{6.4.11a}$$

$$a_0^{(0)} = (0.230 \pm 0.010) \times M_\pi^{-1}; \quad \delta_0^{(0)}(M_K) = 41.0^\circ \pm 2.1^\circ;$$

this fit we take to be valid for $s^{1/2} \leq 0.96$ GeV. The errors of the B_i are strongly correlated; uncorrelated errors are obtained if replacing the B_i by the parameters x, y with

$$B_0 = y - x; \quad B_1 = 6.62 - 2.59x; \quad y = 21.04 \pm 0.75, \quad x = 0 \pm 2.4. \tag{6.4.11b}$$

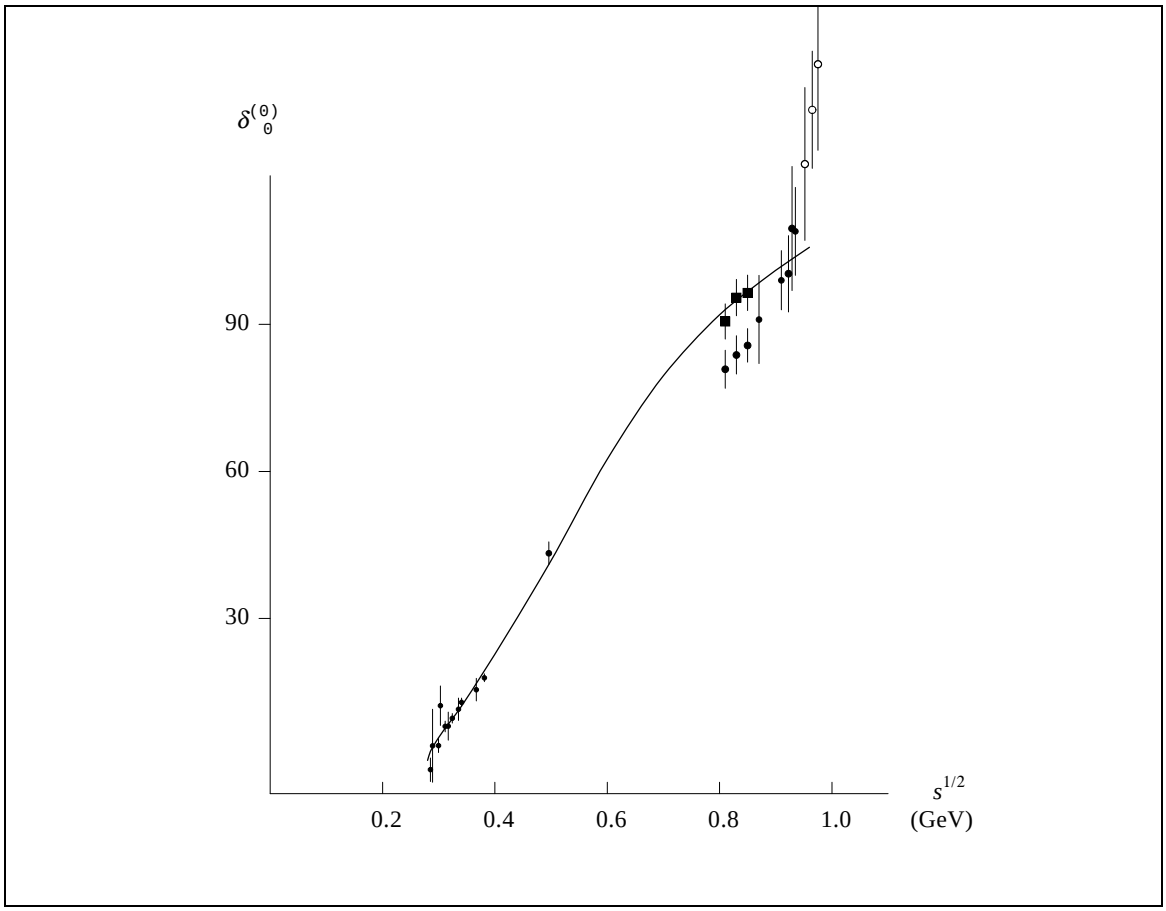

FIGURE 6.4.3. The $I = 0$, S-wave phase shifts corresponding to Eq. (6.4.11) (continuous line). Also shown are the points from K_{l4} and $K_{2\pi}$ decays, and the high energy data of Protopopescu et al. (black dots), Grayer et al. (open circles), and the s-channel solution of Estabrooks and Martin (black squares) included in the fits.

If we allow for an extra parameter a new minimum appears:

$$\begin{aligned}
\cot\delta_0^{(0)}(s) &= \frac{s^{1/2}}{2k}\,\frac{M_\pi^2}{s-\frac{1}{2}M_\pi^2}\,\frac{M_\sigma^2-s}{M_\sigma^2} \\
&\quad\times\left\{B_0+B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}+B_2\left[\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}\right]^2\right\}; \\
s_0^{1/2} &= 2M_K;\quad \chi^2/\text{d.o.f.} = 11.1/(19-4). \\
M_\sigma &= 806\pm 21,\ B_0 = 21.91\pm 0.62,\ B_1 = 20.29\pm 1.55,\ B_2 = 22.53\pm 3.48; \\
a_0^{(0)} &= (0.226\pm 0.015)\ M_\pi^{-1}.
\end{aligned} \tag{6.4.12}$$

This, that we may denote by *B3 Solution*, is something between (6.4.11) and the solution of Colangelo, Gasser and Leutwyler (2001), which indeed is comprised inside the errors of (6.4.12).

We next say a few words on results using (6.4.9). The quality of the fit is substantially lower than all the fits given in (6.4.11,12). Although we expect (6.4.9) to reproduce better the high energy range, the lack of correct left-hand cut structure clearly disrupts the lower range. Thus, for the fit not imposing $\delta_0^{(0)}(m_K^2)$ we find a χ^2/d.o.f. of 45/(15-4), certainly excessive; so we stick to (6.4.11).

It is difficult to give reasons to prefer any of the two sets of parameters (6.4.11,12). Both give essentially identical results for the Olsson and Froissart–Gribov sum rules, but (6.4.11) looks more appealing in that the convergence properties of the conformal expansion are clearly superior to those of (6.4.12). For these reasons we will only give results using (6.4.11).

The value of the scattering length that (6.4.11) gives compares well with the recent value of Descotes et al. (2002) who impose also the Roy equations, and less so with the solution of Colangelo, Gasser and Leutwyler, 2001:

$$\begin{aligned}
a_0^{(0)} &= (0.230\pm 0.010)\times M_\pi^{-1}, \quad \text{[Eq. (6.4.11)]} \\
a_0^{(0)} &= (0.228\pm 0.012)\times M_\pi^{-1}, \quad \text{[Descotes et al., 2002]} \\
a_0^{(0)} &= (0.220\pm 0.005)\times M_\pi^{-1}, \quad \text{[Colangelo, Gasser and Leutwyler, 2001]} \\
a_0^{(0)} &= (0.224\pm 0.013)\times M_\pi^{-1}, \quad \text{[Kamiński, Leśniak and Loiseau, 2003]}.
\end{aligned} \tag{6.4.13}$$

6.4.3. The $I = 0$ S wave between 960 MeV and 1420 MeV

As we have already commented, the description of pion-pion scattering above the $\bar{K}K$ threshold requires a full two-channel formalism. To determine the three independent components of the effective range matrix $\mathbf{\Phi}$, Φ_{11}, Φ_{22} and Φ_{12}, one requires measurement of three cross sections. Failing this, one gets an indeterminate set, which is reflected very clearly in the wide variations of the effective range matrix parameters in the energy-dependent fits of Protopopescu et al. (1973) and Hyams et al. (1973), Grayer et al. (1974).

The raw data themselves are also incompatible; Protopopescu et al. find a phase shift that flattens above $s^{1/2} \simeq 1.04$ GeV, while that of Hyams et al. or Grayer et al. continues to grow. This incompatibility is less marked if we choose the solution with modified higher moments by

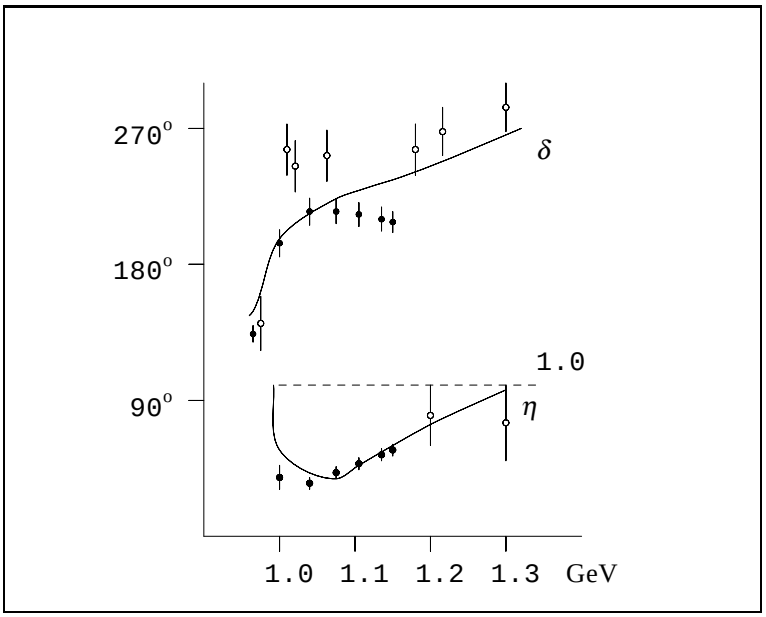

FIGURE 6.4.4. Fits to the $I = 0$, S-wave phase shift and inelasticity from 960 to 1300 MeV. Data from solution 1 of Protopopescu et al. (1973) (black dots) and Grayer et al. (1974) (open circles).

Protopopescu et al. (Table XIII there). The inelasticities are more compatible among the various determinations, although the errors of Protopopescu et al. appear to be underestimated.

In spite of this it is possible to give a reasonable semi-phenomenological fit to $\delta_0^{(0)}$ and $\eta_0^{(0)}$, defined as in (2.1.4). We write

$$\cot\delta_0^{(0)}(s) = c_0\,\frac{(s-M_\sigma^2)(M_f^2-s)}{M_f^2 s^{1/2}}\,\frac{|k_2|}{k_2^2},\quad k_2=\frac{\sqrt{s-4m_K^2}}{2} \tag{6.4.14a}$$

and

$$\eta_0^{(0)} = 1-\left(c_1\frac{k_2}{s^{1/2}}+c_2\frac{k_2^2}{s}\right)\,\frac{M'^2-s}{s}. \tag{6.4.14b}$$

In the first, c_0 and M_σ are free parameters and we fix $M_f = 1320$ MeV. In (6.4.14b), the free parameters are c_1, c_2 and we adjust M' to get the inelasticity agreeing with the central value given by Hyams et al. (1973) on the $f_0(1370)$. We choose to fit the data points of solution 1 of Protopopescu et al. above $\bar{K}K$ threshold, plus two values at 1.2 and 1.3 GeV of Hyams et al. for the inelasticity. For the phase shift, more conflictive as there is clear incompatibility between the two sets of experiments, we include the seven values of Protopopescu et al. for $s^{1/2} \geq 965$ MeV, and another seven points of Grayer et al. (1974), in the same range. The errors of these data have been evaluated as for (6.4.7). We find,

$$\begin{aligned} &c_0 = 1.36\pm0.05,\quad M_\sigma = 802\pm11\ \text{MeV};\quad \chi^2/\text{dof} = 36.2/(14-2)\\ &c_1 = 6.7\pm0.17,\quad c_2 = -17.6\pm0.8;\quad \chi^2/\text{dof} = 7.7/(8-2). \end{aligned} \tag{6.4.14c}$$

The errors for c_0, M_σ correspond to *three* standard deviations, since we have a χ^2 /d.o.f. $\simeq 3$. The fit (6.4.14c) presents the nice feature that the value of M_σ coincides, *grosso modo*, with what we found below $\bar{K}K$ threshold. The qualitative features of the fits may be seen in Fig. 6.4.4, where the incompatibility of the data of both sets of experiments is apparent.

6.5. The D, F and G waves

6.5.1. Parametrization of the $I = 2$ D wave

For isospin equal 2, there are no resonances in the D wave (or, indeed, in any other wave), at least at low energies. This is an experimental fact that can be understood theoretically by recalling that one cannot have $I = 2$ with a quark-antiquark state.

We would only expect important inelasticity when the channels $\pi\pi \to \rho\rho$ $\pi\pi \to \rho\pi\pi$ open up, so we will take

$$w(s) = \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}}, \quad s_0 = 1.45^2 \text{ GeV}^2 \sim 4M_\rho^2. \tag{6.5.1}$$

But life is complicated: a pole term is necessary to get an acceptable fit down to low energy since we expect $\delta_2^{(2)}$ to change sign near threshold. The experimental measurements (Losty et al., 1974; Hoogland et al., 1977) give negative and small values for the phase above some 500 MeV, while we will see that chiral perturbation calculations (Sect. 9.4) and the Froissart–Gribov representation[21] (Sect. 7.5) indicate a positive scattering length, $a_2^{(2)} \simeq (2.2 \pm 0.2) \times 10^{-4}\, M_\pi^{-5}$.

If we want a parametrization that applies down to threshold, we must incorporate this zero of the phase shift. So we write

$$\cot \delta_2^{(2)}(s) = \frac{s^{1/2}}{2k^5}\, [B_0 + B_1 w(s)]\, \frac{{M_\pi}^4 s}{4({M_\pi}^2 + \Delta^2) - s} \tag{6.5.2a}$$

with Δ a free parameter and

$$w(s) = \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}}, \quad s_0 = 1450 \text{ MeV}\,.$$

Moreover, we impose the value for the scattering length that follows from the Froissart–Gribov representation, $a_2^{(2)} = (2.22 \pm 0.33) \times 10^{-4}$, in units of M_π (see below, Sect. 7.5).

We first perform a fit up to $s^{1/2}$; we do not include in it the data of Cohen et al. (1973). Since these are obtained from scattering off bound neutrons (in deuterium) they are more liable to systematic errors at low energy and, in fact, if we included them the resulting effective range parameter $b_2^{(2)}$ would be far from its expected value (see below). We only include two parameters B_0, B_1 (*Solution B2*); we get a mediocre fit, χ^2 /d.o.f. $= 53/(16 - 3)$, and the values of the parameters are[22]

$$B_0 = (2.30 \pm 0.17) \times 10^3, \quad B_1 = -267 \pm 750, \quad \Delta = 103 \pm 11 \text{ MeV}; \quad s^{1/2} \lesssim 1.2 \text{ GeV}\,. \tag{6.5.2b}$$

[21] An interesting feature of the Froissart–Gribov calculation is that the structure of $\delta_2^{(2)}$, in particular the zero near threshold, was in fact *predicted* from it (Palou and Ynduráin, 1974).

[22] This fit, due to Peláez and Ynduráin (2003) corrects an error of the previous version of the present paper.

Doubtlessly the incompatibilities between the experimental data (which is obvious from a look at Fig. 6.5.1), probably related to those for the S2 wave, preclude a better fit.

The fit returns a good value for the scattering length, and also for the effective range parameter, $b_2^{(2)}$:

$$a_2^{(2)} = (2.20 \pm 0.16) \times 10^{-4}\, {M_\pi}^{-5}; \quad b_2^{(2)} = (-5.75 \pm 1.26) \times 10^{-4}\, {M_\pi}^{-7}, \tag{6.5.2c}$$

to be compared with what we will get from the Froissart–Gribov representation (Sect. 7.5),

$$a_2^{(2)} = (2.22 \pm 0.33) \times 10^{-4}\, {M_\pi}^{-5}; \quad b_2^{(2)} = (-3.34 \pm 0.24) \times 10^{-4}\, {M_\pi}^{-7}. \tag{6.5.2c}$$

For once, the value of $a_2^{(2)}$ is more accurate than the value following from the Froissart–Gribov calculation; the value of $b_2^{(2)}$differs by less than $2\,\sigma$ from the expected one.

To get the parameters for the region above 1.0 GeV, we take simply a quadratic fit. We find,

$$\begin{aligned} \delta_2^{(2)}(s) &= (-0.051 \pm 0.003) + a\left(\frac{s}{1\,\mathrm{GeV}^2} - 1\right) + b\left(\frac{s}{1\,\mathrm{GeV}^2} - 1\right)^2; \\ a &= -0.081 \pm 0.033, \quad b = 0.042 \pm 0.005; \quad s \geq 1.0\ \mathrm{GeV}\,. \end{aligned} \tag{6.5.2d}$$

We can add inelasticity to the D2 wave by assuming that it is something between zero and what one has for the S2 wave (Subsect. 6.5.1). So we would write,

$$\begin{aligned} \eta_2^{(2)}(s) &= 1 - c(1 - M_{\mathrm{eff}}^2/s)^{3/2}, \quad M_{\mathrm{eff}} = 0.96\ \mathrm{GeV}, \quad c = 0.12 \pm 0.12; \\ s^{1/2} &\geq 0.96\ \mathrm{GeV}\,. \end{aligned} \tag{6.5.2e}$$

We should add that it is possible to get a reasonable fit to data at all energies, with a formula like (6.5.2a), but we require *four* parameters B_i (*Solution B4*). Including also the data of Cohen et al. (1973) one gets,

$$\begin{aligned} &B_0 = (1.94 \pm 0.14) \times 10^3, \quad B_1 = (10.15 \pm 1.3) \times 10^3, \quad B_2 = (18.68 \pm 2.4) \times 10^3, \\ &B_3 = (-31.04 \pm 5.5) \times 10^3; \qquad \Delta = 218 \pm 22\ \mathrm{MeV}\,. \end{aligned} \tag{6.5.3a}$$

The errors here correspond to $3\,\sigma$. One has χ^2 /d.o.f. $= 57/(25-5)$ and the fit returns the values of the low energy parameters

$$a_2^{(2)} = (2.04 \pm 0.5) \times 10^{-4}\, M_\pi^{-5}, \quad b_2^{(2)} = (1.6 \pm 0.3) \times 10^{-4}\, M_\pi^{-7}. \tag{6.5.3b}$$

The large values of the parameters $B - i$, and the incompatibility of the three data sets, makes one suspect that the corresponding minimum is spureous.

6.5.2. Parametrization of the $I = 0$ D wave

The D wave with isospin 0 in $\pi\pi$ scattering presents two resonances below 1.7 GeV: the $f_2(1270)$ and the $f_2(1525)$, that we will denote respectively by f_2, f_2'. Experimentally, $\Gamma_{f_2} = 185 \pm 4$ GeV and $\Gamma_{f_2'} = 76 \pm 10$ GeV. The first, f_2, couples mostly to $\pi\pi$, with small couplings to KK $(4.6 \pm 0.5\,\%)$, 4π $(10 \pm 3\,\%)$ and $\eta\eta$. The second couples mostly to $2K$, with a small coupling to $\eta\eta$ and 2π, respectively $10 \pm 3\,\%$ and $0.8 \pm 0.2\,\%$. This means that the channels $\pi\pi$ and KK are essentially decoupled: they only connect indirectly, so it is not very profitable to set up a multiple channel calculation. To a 15% accuracy we may neglect inelasticity up to $s_0 = 1.42^2\,\mathrm{GeV}^2$. The formulas are like those for the P wave; we will discuss them presently.

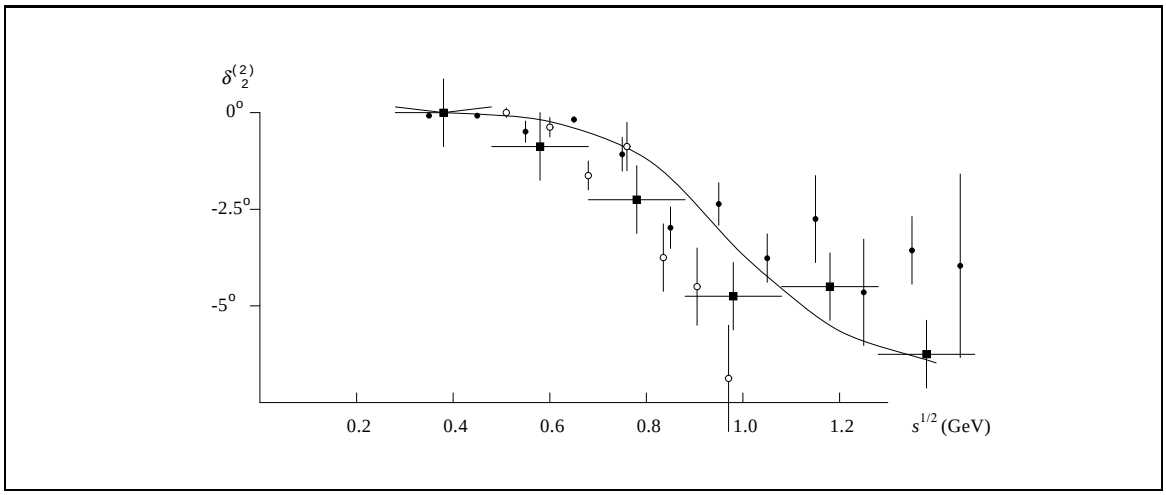

FIGURE 6.5.1. Fits to the $I = 2$, D-wave phase shift. Also shown are the data points of Losty et al., 1974 (open circles), from solution A of Hoogland et al., 1977 (black dots) and from Cohen et al. (1973) (crosses).

There are not many experimental data on the D wave which, at accessible energies, is small. So, the compilation of $\delta_2^{(0)}$ phase shifts of Protopopescu et al. (1973) covers only the range $810 \leq s^{1/2} \leq 1150$ MeV. In view of this, it is impossible to get accurately the D wave scattering lengths, or indeed any other low energy parameter, from this information. We give here a parametrization whose use lies in that it represents with reasonable accuracy the data, something that will be useful later on. We write

$$\cot \delta_2^{(0)}(s) = \frac{s^{1/2}}{2k^5} \left(M_{f_2}^2 - s\right) M_\pi^2 \, \psi(s), \quad \psi(s) = B_0 + B_1 w(s) + \cdots , \tag{6.5.4a}$$

and

$$w = \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}}, \quad s_0^{1/2} = 1.43 \text{ GeV} .$$

We take the data of Protopopescu et al. (1973) and consider the so-called "solution 1", with the two possibilities given in Table VI and Table XIII (with modified higher moments). These data cover the range mentioned before, $s^{1/2} = 0.810$ GeV to 1.150 GeV. The problem with these data points is that they are contaminated, for $s \gtrsim 1.1$ GeV^2, by the bias of the S wave with $I = 0$ in the same region, whose values there are quite incompatible with those of other experiments (see Subsect. 6.5.3). For this reason we perform two fits: either including or excluding the data points for $s^{1/2} \geq 1.075$ GeV. In both cases we present results only for the version with modified higher moments (Table XIII in Protopopescu et al., 1973) as they are the ones that show better compatibility with other experiments. We also impose the fit to the width of the f_2 resonance, with the condition $\Gamma_{f_2} = 185 \pm 10$ MeV. We find,

$$\frac{\chi^2}{\text{d.o.f.}} = \frac{46.9}{14 - 2}, \quad B_0 = 20.16, \quad B_1 = 19.48, \quad \text{[All points]};$$
$$\Gamma_{f_2} = 213 \text{ MeV}, \quad a_2^{(0)} = 17 \times 10^{-4} \, M_\pi^{-5}$$

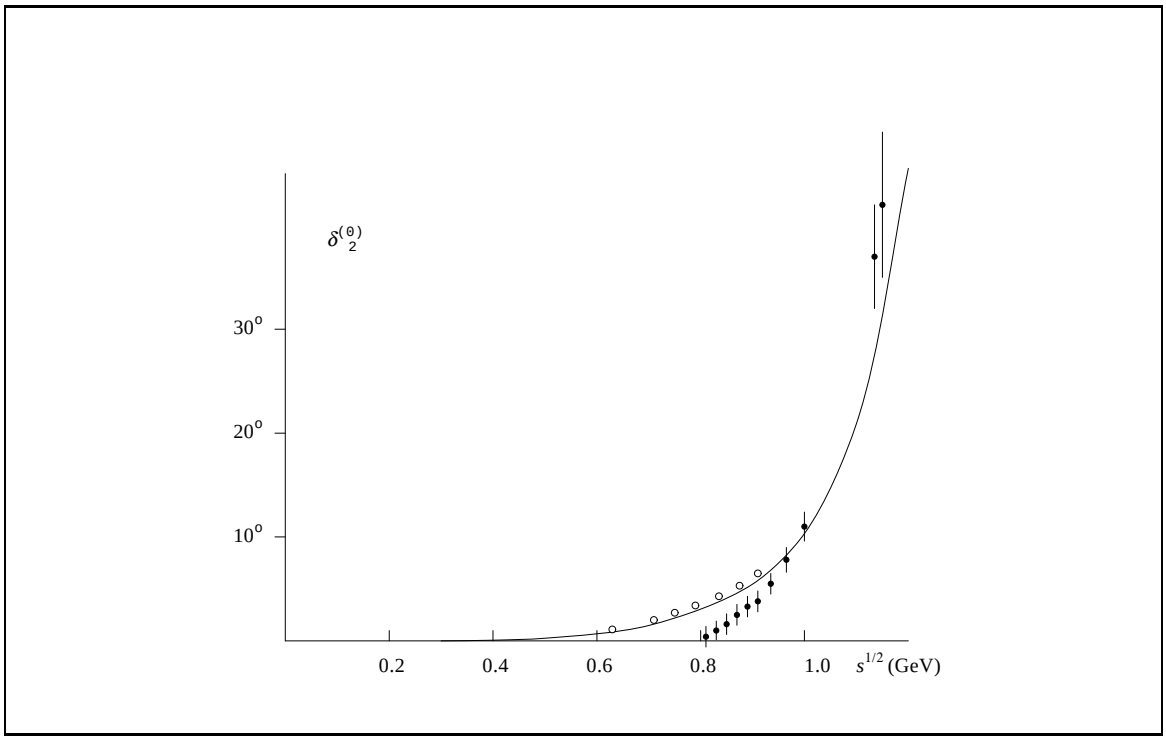

FIGURE 6.5.2. Fits to the $I = 0$, D-wave phase shift. Also shown are the data points from solution 1 of Protopopescu et al. 1973 (black dots) and some data of Estabrooks and Martin 1974 (open circles).

and

$$\frac{\chi^2}{\text{d.o.f.}} = \frac{20.5}{10-2}, \quad B_0 = 23.95, \quad B_1 = 18.91, \quad [\text{Points for } s^{1/2} < 1.075 \text{ GeV}];$$
$$\Gamma_{f_2} = 187 \text{ MeV}, \quad a_2^{(0)} = 11 \times 10^{-4}\, M_\pi^{-5}.$$

The drastic decrease of the χ^2/d.o.f. when eliminating the higher energy points signals clearly their biased character.[23] However, the parameters of the fits are reasonably stable, no doubt because we have imposed the correct width of the resonance f_2. We can therefore take as our best result an average of the two determinations, with half their difference as an estimated error:

$$B_0 = 22.1 \pm 1.9, \quad B_1 = 19.2 \pm 0.3 \tag{6.5.4b}$$

and this corresponds to

$$\Gamma_{f_2} = 200 \pm 13 \text{ MeV}, \quad a_2^{(0)} = (14 \pm 3) \times 10^{-4}\, M_\pi^{-5},$$

[23] We remark again that the χ^2/d.o.f. is less poor than it looks at first sight, as it only takes into account statistical errors, while systematic ones are certainly is large as these.

reasonably close to their experimental values, the second as deduced from the Froissart–Gribov representation (cf. Sect. 7.6):

$$a_2^{(0)} = (18.1 \pm 0.4) \times 10^{-4}\, M_\pi^{-5}, \quad b_2^{(0)} = (-3.60 \pm 0.25) \times 10^{-4}\, M_\pi^{-7}$$

(we also give the effective range parameter from the same source).

An alternate possibility is to include the scattering length, as deduced from the Froissart–Gribov representation, in the fit, which completely stabilizes the results. Moreover, we take into account the inelasticity iteratively. We write

$$\cot \delta_2^{(0)}(s) = \frac{s^{1/2}}{2k^5}\,(M_{f_2}^2 - s)\, {M_\pi}^2\, \psi(s), \quad \psi(s) = B_0 + B_1 w(s) + \cdots \tag{6.5.5a}$$

and

$$w(s) = \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}}, \quad s_0 = 1430 \text{ MeV}; \quad M_{f_2} = 1275.4 \text{ MeV}\,.$$

We find, fitting also the points of Estabrooks and Martin (1974),

$$\frac{\chi^2}{\text{d.o.f.}} = 74/(21-2), \quad B_0 = 22.4 \pm 0.1, \quad B_1 = 23.3 \pm 3.0;$$

The very poor χ^2/d.o.f. is obviously due to the strong bias of the data of Protopopescu et al., 1973, clearly seen in Fig. 6.5.2.

Above values of the B_i would give $\Gamma_{f_2} = 196 \pm 6$ MeV. We will then take this solution to be valid up to $\bar{K}K$ threshold; on it, we join the solution to a new one, for which we impose the f_2 width; we get

$$B_0 = 22.5 \pm 0.1, \quad B_1 = 28.5 \pm 3.6.$$

Therefore, we have

$$B_0 = \begin{cases} 22.4 \pm 0.1, & s < 4M_K^2, \\ 22.5 \pm 0.1, & s > 4M_K^2 \end{cases}; \qquad B_1 = \begin{cases} 23.3 \pm 3.0, & s < 4M_K^2, \\ 28.5 \pm 3.6, & s > 4M_K^2. \end{cases} \tag{6.5.5b}$$

We then take into account the inelasticity by writing

$$\eta_2^{(0)}(s) = \begin{cases} 1, & s < 4M_K^2, \\ 1 - 2 \times \epsilon_f\, \dfrac{k_2(s)}{k_2(M_{f_2}^2)}, & \epsilon_f = 0.131 \pm 0.015; \quad s > 4M_K^2. \end{cases} \tag{6.5.5c}$$

$k_2 = \sqrt{s/4 - M_K^2}$. We have fixed the coefficient ϵ_f fitting the inelasticities of Protopopescu et al., 1973, and the experimental inelasticity of the f_2; the overall χ^2/d.o.f. of this fit is ~ 1.8. The fit returns the values

$$\begin{aligned} a_2^{(0)} &= (18.4 \pm 7.6) \times 10^{-4} \times M_\pi^{-5}, \quad b_2^{(0)} = (-7.9^{+4.1}_{-11.0}) \times 10^{-4} \times M_\pi^{-7}; \\ \Gamma_{f_2} &= 185 \pm 5 \text{ MeV}\,. \end{aligned}$$

One could try to improve the fit by adding an extra term, $B_2 w^2$, and requiring also the value of $b_2^{(0)}$ to agree with the chiral perturbation theory value (or with that obtained from the Froissart–Gribov representation, see below). In fact, we prefer to keep the larger errors given above; the values of the inelasticities and low energy parameters are compatible at the 1.5 σ level with experimental information, and we feel that the improvement obtained by diminishing the errors would be made at the cost of reliability.

6.5.3. The F wave

The experimental situation for the F wave is somewhat confused. According to Protopopescu et al. (1973) it starts negative (but compatible with zero at the $2\,\sigma$ level) and becomes positive around $s^{1/2} = 1$ GeV. Hyams et al. (1973) and Grayer et al. (1974) report a positive $\delta_3(s)$ when it differs from zero (above $s^{1/2} = 1$ GeV). In both cases the inelasticity is negligible up to, at least, $s^{1/2} = 1.5$ GeV.

The corresponding scattering length may be calculated with the help of the Froissart–Gribov representation and one finds (Sect. 7.6)

$$a_3 = (6.00 \pm 0.07) \times 10^{-5}\, M_\pi^{-7}.$$

It could in principle be possible that $\delta_3(s)$ changes sign *twice*, once near threshold and once near $s^{1/2} = 1$ GeV. However, we disregard this possibility and write, simply,

$$\cot\delta_3(s) = \frac{s^{1/2}}{2k^7}\left\{B_0 + B_1 w(s)\right\} M_\pi^6, \quad w(s) = \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}}, \tag{6.5.6a}$$

with $s_0^{1/2} = 1.5$ GeV, and *impose* (6.5.5).

It is to be understood that this parametrization provides only an empirical representation of the available data, and that it may not be reliable except at very low energies, where it is dominated by the scattering length, and for $s^{1/2} \sim 1$ GeV. We find

$$\frac{\chi^2}{\text{d.o.f.}} = \frac{5.7}{7-2}, \quad B_0 = (1.07 \pm 0.03) \times 10^5, \quad B_1 = (1.35 \pm 0.03) \times 10^5. \tag{6.5.6b}$$

For $s^{1/2} \geq 1.1$ GeV, however, the effect of the $\rho_3(1690)$ resonance should be included; see Peláez and Ynduráin (2003) and Appendix A.8 here for an explicit expression valid for $\text{Im}\, f_3$.

6.5.4. The G waves

The experimental information on the G waves is very scarce. For the wave G2, we have six nonzero values for $\delta_4^{(2)}$, two from Cohen et al. (1973) and four from Losty et al. (1974); they are somewhat incompatible. We then fit the data separately, with a scattering length formula; we write

$$\cot\delta_4^{(2)}(s) = \frac{s^{1/2} M_\pi^8}{2k^9}\, B.$$

If we fit he data of Cohen et al. (1973) we find $B = (-9.5 \pm 2.7) \times 10^6$, while from Losty et al. (1974) we get $B = (-0.6 \pm 0.1) \times 10^6$. Fitting both sets together we find $B = -7.8 \times 10^6$, and a very poor $\chi^2\,/\text{d.o.f.} = 30/(6-1)$. Enlarging the resulting error so to cover $6\,\sigma$ we obtain our best result,

$$\cot\delta_4^{(2)}(s) = \frac{s^{1/2} M_\pi^8}{2k^9}\, B, \quad B = (-7.8 \pm 3.3) \times 10^6. \tag{6.5.7}$$

This formula can only be considered as an empirical fit, valid for a limited range, $0.9 \leq s^{1/2} \leq 1.5$ GeV. In fact, from the Froissart–Gribov representation it follows that the G2 scattering length is *positive*,

$$a_4^{(2)} = (4.5 \pm 0.2) \times 10^{-6}\, M_\pi^{-9}$$

while (6.5.7) would give a negative value.

For the G0 wave, the situation is worse: there are no experimental data on the phase shift. All we know is the existence of a very inelastic resonance with mass around 2 GeV. An effective value for the imaginary part of the corresponding partial wave may be found in Appendix A.9. At low energy we can use a scattering length approximation with

$$a_4^{(0)} = (8.0 \pm 0.2) \times 10^{-6}\, M_\pi^{-9}.$$

6.6. On experimental phase shifts in the range 1.4 GeV $\simeq s^{1/2} \simeq$ 2 GeV

As we mentioned in Sect. 6.1, we expect that, as soon as the center of mass kinetic energy in a reaction, $E_{\rm kin}$, increases beyond 1 GeV, inelastic processes become more and more important with increasing energy, so much so that, for $E_{\rm kin} \gtrsim 1.2$ GeV, they should dominate elastic ones. This is easily understandable in the QCD, ladder version of the Regge picture, as discussed in Sect. 2.4; and indeed, it is verified experimentally in the hadronic processes πN, KN and NN, $\bar{N}N$ where, for $E_{\rm kin} > 1.2$ GeV, the elastic cross section is smaller than the inelastic one and, for $E_{\rm kin} > 1.5$ GeV, the elastic cross sections are a third or less than the total cross sections. There is no reason to imagine that $\pi\pi$ scattering would follow a different pattern. In fact, the experimental results on $\pi\pi$ cross sections at high energies (like e.g., those of Robertson, Walker and Davis, 1973) have checked unambiguously all these features.

In this case in which inelastic cross sections are large, and again as mentioned in Sect. 6.1, it can be proved theoretically that there is not a unique solution to the phase shift analysis: some sets of ηs and δs may fit the data; but so would others.

In spite of this, the Cern-Munich experiments[24] have produced a set of phase shifts and inelasticities which go up to $s^{1/2} \simeq 2$ GeV, which have been used in several theoretical analyses. Unfortunately, these phase shifts are likely to diverge more and more from reality as $s^{1/2} = E_{\rm kin} + 2M_\pi$ beomes larger and larger than (say) 1.5 GeV. This is suggested, besides the theoretical reasons mentioned in Sect. 6.1, because the Cern-Munich phase shifts and inelasticities clearly contradict a number of physical properties related to their (lack of) inelasticity: we will here mention a few.

First of all, the inelasticities $(1 - \eta_l^{(I)})$ for all the waves in the Cern-Munich results remain small in the range 1.6 GeV to 2 GeV. However, as we have remarked above, one would expect dominant inelastic cross sections there. For a given wave, equality of elastic and inelastic cross sections occurs for $\eta_l^{(I)} = \cos 2\delta_l^{(I)}$ (cf. (2.1.4b)), and the condition to have the inelastic cross section much larger than the elastic one is $\eta_l^{(I)} \ll \cos 2\delta_l^{(I)}$.[25] This inequality is not satisfied by any of the Cern–Munich phases and elasticity parameters, except for the F wave on the $\rho_3(1690)$ resonance. The Cern–Munich elastic cross sections are larger or comparable to the inelastic ones up to $s^{1/2} = 2$ GeV and, what is worse, their inelastic cross sections, alone of all hadronic cross sections, *decrease* when the kinetic energy grows from 1 GeV to 1.7 GeV; for e.g., $\pi^+\pi^-$ scattering this is clearly shown in Fig. 7 in the paper of Hyams et al. (1973).

Secondly, the combination of δ and η for both P and S0 waves at an energy around 1.8 GeV is incompatible with what QCD implies for the electromagnetic and scalar form factors of the pion.

[24] Hyams et al., (1973); Grayer et al. (1974).

[25] For the P and D0 waves, this suggests that one should have $\delta_1 > \pi$, $\delta_2^{(0)} > \pi$ near 2 GeV. As we show below, there is extra evidence for the first inequality (violated by the Cern–Munich phase) from a different source.

In fact, as we will show in Sect. 7.2, the Brodsky–Farrar counting rules for these form factors imply that their phases $\delta(t)$ behave like

$$\delta(t) \underset{t\to\infty}{\simeq} \pi\left(1+\frac{\log s/\hat{t}}{\log\log s/\hat{t}}\right), \quad s \gg \Lambda^2; \quad \hat{t}\sim\Lambda^2 \tag{6.6.1}$$

(Λ is the QCD parameter). One may take (6.6.1) to hold for $s \gtrsim 3$ GeV^2 ($s^{1/2} > 1.6$ GeV). If one had negligible inelasticity for these waves somewhere in the region 1.6 GeV $\lesssim s^{1/2} \lesssim 2$ GeV, as the Cern–Munich data seem to imply, form factors and partial waves should have the same phase at such energies, and thus the same behaviour (6.6.1) should hold for $\delta_1(s)$, $\delta_0^{(0)}(s)$. But the phases the Cern–Munich experiment gives clearly contradict (6.6.1) around 1.8 GeV. For example, the Cern–Munich phase δ_1 stays consistently *below* π, while (6.6.1) implies that it should be *above*. We have already seen evidence on this coming from inelasticity inequlalities above.

Thirdly, and as happens for πN, KN, NN and even γN, $\gamma\gamma$, scattering, we would expect a levelling off of the total cross section for $E_{\rm kin} > 1.3$ GeV. However, the Cern–Munich total cross section decreases roughly like $1/s$ up to 2 GeV. This is because P, D0 phases are $\sim\pi$, and $\delta_3 \sim 0$, in the range $E_{\rm kin} > 1.3$ GeV, while the corresponding parameters $\eta_l^{(I)}$ are near unity there: the Cern–Munich scattering amplitude is almost exclusively S waves for $s^{1/2} > 1.55$ GeV.

Fourthly, we would expect large isospin S2 and D2 waves as we approach the 2ρ thresold. However, these waves are essentially ignored in the Cern–Munich analysis. Thus, besides the general problem for the cross sections we have individual problems for each of the S0, S2, P and D2 phases.

Finally, both the Regge picture and the experimental cross sections for all hadronic processes indicate that the number of waves that contribute effectively to the imaginary part (say) of the scattering amplitudes grows with the kinetic energy as $E_{\rm kin}/\Lambda$ for $E_{\rm kin}$ upwards of 1 GeV. We thus expect 2 to 3 waves (for fixed isospin) at $E_{\rm kin} \sim 1$ GeV, and almost double this, 3 to 5 waves at $E_{\rm kin} \sim 1.7$ GeV. In fact, for $\pi\pi$ scattering at this energy, the contribution of the F wave is as large as that of the P wave, the D0 wave is as large as the S0 wave, and the contribution of the D2 wave is as large or larger than that of the S2 wave: the partial wave series with only two waves per isospin channel is not convergent. The approximations that neglect all higher waves at such energies have another reason for being irrealistic.

All these arguments indicate that, in particular, the inelasticities of the Cern–Munich phase shifts are much underestimated beyond ~ 1.5 GeV. It is possible that the results presented by the Cern–Munich group fit the *elastic* $\pi\pi$ cross section, but, because they feature insufficient inelasticity, they certainly misrepresent the *total* cross section. They must therefore lead to a distorted imaginary part of the $\pi\pi$ scattering amplitude. It is thus not surprising that Pennington (1975), Ananthanarayan et al. (2001) and Colangelo, Gasser and Leutwyler (2001), who fix their Regge parameters by balancing them above 2 GeV with phase shifts below 2 GeV, get incorrect values for the first.

We would like to emphasize that what has been said should not be taken as implying criticism of the Cern–Munich experiment which, for $s^{1/2} \lesssim 1.4$ GeV, produced what are probably the best determinations of phase shifts and inelasticities. Above 1.4 GeV, they did what they could: it is for theorists to realize that this was not enough to produce acceptable phase shifts and inelasticities at these higher energies.

7. Analiticity: dispersion relations and the Froissart–Gribov representation. Form factors: the Omnès–Muskhelishvili method

7.1. The Omnès–Muskhelishvili method

In the analysis of the pion form factors we have the following situation: we have information on the phase of a quantity, F, and, in some cases, know *experimentally* its modulus. We would like to translate this into a general parametrization of the quantity. This last problem was first solved by Muskhelishvili (1958) and later applied to the physical case by Omnès (1958). We turn to this method.

7.1.1. The full Omnès–Muskhelishvili problem

We want to find the most general representation for a function, $F(t)$, of which we know that it is analytic in the complex t plane, cut from $t = 4\mu^2$ to ∞, assuming that we know its phase on the cut,

$$\arg F(t) = \delta(t), \quad 4\mu^2 \leq t. \tag{7.1.1}$$

This is the so-called (full) Omnès–Muskhelishvili problem. Note that we do *not* take the principal value of the argument here, except for s near threshold; we have to assume δ to be continuous, so it could go above 2π at high energy.

First of all, it is clear that, unless we have further information on F, the solution to this equation is highly nonunique. For, if $F_0(t)$ is a solution to (7.1.1), then any

$$\mathrm{e}^{at} F_0(t), \quad \text{or} \quad \mathrm{e}^{a\mathrm{e}^{bt}} F_0(t), \ \ldots$$

would also be a solution. Fortunately, in the physically interesting cases we have information on the growth of $F(t)$ at large t that precludes such functions. For example, and as already discussed, the Brodsky–Farrar counting rules imply that, for meson form fators,

$$F(t) \underset{t\to\infty}{\lesssim} \frac{\text{Const.}}{-t\log^{\nu}(-t)}. \tag{7.1.2}$$

We will restrict our analysis to the case where $\delta(t)$ is Hölder continuous (Muskhelishvili, 1958). We will also, in this subsection, assume that the phase has a finite, positive limit as $t \to \infty$:

$$\delta(t) \underset{t\to\infty}{\to} \delta(\infty), \quad \delta(\infty) > 0. \tag{7.1.3}$$

In fact, (7.1.2) implies $\delta(\infty) = \pi$, so it will turn out that (7.1.3) is really the condition that is relevant for physical applications; so we assume it.

To solve our problem the first step is to form the auxiliary function

$$J(t) = \exp \frac{t}{\pi} \int_{4\mu^2}^{\infty} \mathrm{d}s \, \frac{\delta(s)}{s(s - t - \mathrm{i}0)}. \tag{7.1.4}$$

We will assume that $F(0) = 1$; otherwise, we would consider the function $F(t)/F(0)$. From (7.1.4) two properties of J are immediately obvious: $J(0) = 1$ and J has no zero in the complex plane (the last because, due to the continuity of δ, the integral in the exponent is finite).

It is easy to verify that the function J has the same analyticity properties and the same phase as F. For example, using the relation $1/(x \pm \mathrm{i}0) = \mathrm{P.P.}(1/x) \mp \mathrm{i}\pi\delta(x)$, we have

$$\frac{t}{\pi}\int_{4\mu^2}^{\infty} \mathrm{d}s\, \frac{\delta(s)}{s(s-t-\mathrm{i}0)} = \frac{t}{\pi}\,\mathrm{P.P.}\int_{4\mu^2}^{\infty} \mathrm{d}s\, \frac{\delta(s)}{s(s-t)} + \mathrm{i}\delta(t). \tag{7.1.5a}$$

At large t, the real part of the integral above dominates over its imaginary part and we find

$$\frac{t}{\pi}\int_{4\mu^2}^{\infty} \mathrm{d}s\, \frac{\delta(s)}{s(s-t-\mathrm{i}0)} \underset{t\to\infty}{\simeq} -\frac{\delta(\infty)}{\pi}\log|t|. \tag{7.1.5b}$$

In view of this last relation we obtain the behaviour,

$$J(t) \underset{t\to\infty}{\simeq} |t|^{-\delta(\infty)/\pi}. \tag{7.1.6}$$

Next step is to form the function $G(t)$, defined by

$$F(t) = G(t)J(t).$$

Because J never vanishes, $G(t)$ is, at least, analytic in the same domain as $F(t)$. Moreover, since J and F have the same phase on the cut, it follows that $G(t)$ is real on the cut. According to the theorem of Painlevé, this implies that G is also analytic on the cut, hence $G(t)$ is analytic in the whole t plane, i.e., it is an entire function.

It is now that the growth condition (7.1.2) enters. The only entire functions that do not grow exponentially (or faster) in some direction are the polynomials. Hence, (7.1.2) implies that $G(t) = P_N(t)$, where $P_N(t)$ is a polynomial of degree N: we have found the general representation

$$F(t) = P_N(t)J(t). \tag{7.1.7}$$

Now, which polynomials are allowed depends on the value of $\delta(\infty)$. We will simplify the discussion by assuming that $\delta(\infty) = n\pi$, n an integer; the interested reader may find information on other situations in the text of Muskhelishvili (1958). On comparing (7.1.6) and (7.1.2) it follows immediately that $N = n - 1$. Thus, in the case (that will turn out to be the more interesting one for us here) in which $n = 1$, the function J is actually the most general solution to the problem:

$$F(t) = J(t) \quad [\delta(\infty) = \pi]. \tag{7.1.8}$$

7.1.2. The incomplete Omnès-Muskhelishvili problem

In the physically relevant cases we do not know $\delta(t)$ for all t, but only up to a certain s_0, typically the energy squared at which inelastic channels start becoming important. We will make the calculations for the form factor F of spinless particles with mass μ, so the results will be directly applicable to pions.

The idea for the treatment of this case is to extend $\delta(t)$ to the full t range, in an appropriate manner, so as to reduce the problem to the previous one. Let us call $\delta_{\rm eff}(t)$ to this extension, so that $\delta_{\rm eff}(t) = \delta(t)$ for $t \leq s_0$. We then form

$$J_{\rm eff}(t) = \exp \frac{t}{\pi} \int_{4\mu^2}^{\infty} \mathrm{d}s \, \frac{\delta_{\rm eff}(s)}{s(s-t-\mathrm{i}0)} \tag{7.1.9a}$$

and define G by

$$F(t) = G(t) J_{\rm eff}(t). \tag{7.1.9b}$$

Because now $\delta_{\rm eff}(t)$ equals $\delta(t)$ only for $4\mu^2 \leq t \leq s_0$, $G(t)$ will not be analytic on the whole t plane, but will retain a cut from $t = s_0$ to ∞. G will be an unknown function, that will have to be obtained from a model or fitted to experiment. Because of this, we have interest to have it as smooth as possible, so that a few terms will represent it. Since discontinuities of $\delta_{\rm eff}(t)$ will generate infinities of $J_{\rm eff}(t)$, and of $G(t)$, we must choose a smooth continuation of $\delta(t)$ above $t = s_0$. Moreover, if we do not want to have a G growing without limit for large t, we have to construct a $J_{\rm eff}(t)$ that decreases at infinity like $F(t)$. These conditions are fulfilled if we simply define

$$\delta_{\rm eff}(t) = \begin{cases} \delta(t), & t \leq s_0; \\ \pi + [\delta(s_0) - \pi] \, \dfrac{s_0}{t}, & t \geq s_0. \end{cases} \tag{7.1.10}$$

In this case the piece from s_0 to ∞ in the integral in Eq. (7.1.9a) can be performed explicitely and we get

$$\begin{aligned} F(t) \; &= G(t) J_{\rm eff}(t) = G(t) \mathrm{e}^{1-\delta(s_0)/\pi} \left(1 - \frac{t}{s_0}\right)^{[1-\delta(s_0)/\pi] s_0/t} \left(1 - \frac{t}{s_0}\right)^{-1} \\ &\times \exp\left\{ \frac{t}{\pi} \int_{4\mu^2}^{s_0} \mathrm{d}s \, \frac{\delta(s)}{s(s-t-\mathrm{i}0)} \right\}. \end{aligned} \tag{7.1.11}$$

If we knew that $F(t)$ behaves exactly as $1/t$, for $t \to \infty$, it would follow that its phase has to tend to π at infinity. More generally, if one has, as in (7.1.2),

$$F(t) \underset{t\to\infty}{\simeq} \frac{\text{Const.}}{-t \log^{\nu}(-t)},$$

then this implies

$$\delta(t) \underset{t\to\infty}{\simeq} \pi \left\{ 1 + \nu \frac{\log\log t}{\log t} \right\}.$$

Therefore, $\delta_{\rm eff}$ in (7.1.10) may be then considered as a linear interpolation (in t^{-1}) for $\delta(t)$ between s_0 and infinity, and $G(t)$ may be interpreted as giving the correction to this.

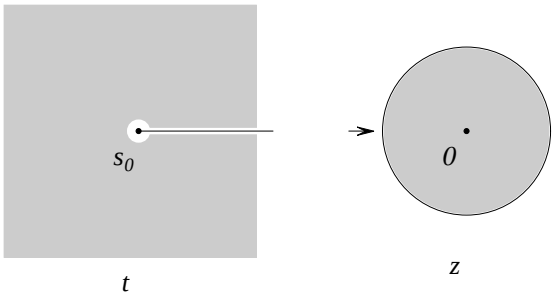

FIGURE 7.1.1 The mapping $t \to z$.

It only remains to write a general parametrization of $G(t)$ compatible with its known properties. To do so, we map the cut t plane into the unit disk in the variable z (Fig. 7.1.1),

$$z = \frac{\frac{1}{2}\sqrt{s_0} - \sqrt{s_0 - t}}{\frac{1}{2}\sqrt{s_0} + \sqrt{s_0 - t}}. \tag{7.1.12}$$

The most general G is analytic inside this disk, so we can write a Taylor expansion for it, which conveniently we set in the form

$$G(t) = 1 + A_0 + c_1 z(t) + c_2 z(t)^2 + c_3 z(t)^3 + \cdots \tag{7.1.13a}$$

This expansion that will be uniformly convergent for all t inside the cut plane. We can implement the condition $G(0) = 1$, necessary to ensure $F(0) = 1$ order by order, by putting

$$A_0 = -\left[c_1 z_0 + c_2 z_0^2 + c_3 z_0^3 + \cdots\right], \quad z_0 \equiv z(t = 0) = -1/3. \tag{7.1.13b}$$

We remark in passing that since, inside the unit circle, one has $|z| \leq 1$, it follows that to every finite order in the expansion (7.1.13a), $G(t)$ is bounded in the t plane. Hence $F(t)$ and $J_{\text{eff}}(t)$ have the same asymptotic behaviour, as desired.

We end this section with a simple example that shows clearly the desirability of expanding a function which, like G, is regular at the frontier of the domain of analyticity (which happens because we were careful to extrapolate δ without introducing singularities and keeping the correct asymptotic behaviour). Consider the three series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{(A)};$$

$$\log(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n \quad \text{(B)};$$

$$\int_0^z \mathrm{d}t\, \frac{\log(1-t)}{t} = \sum_{n=1}^{\infty} \frac{1}{n^2} z^n \quad \text{(C)}.$$

The first has a pole, the second a logarithmic singularity and the third is regular at the edge of the convergence disk. The first series diverges there, the second is conditionally convergent at all points except at $z = 1$, and the third is convergent even at the edge of the disk. This pattern is general.

7.2. Application to the pion form factors of the Omnès-Muskhelishvili method

7.2.1. The electromagnetic form factor

The application to the pion form factors of the formalism presented in the previous section is straightforward as, indeed, it was tailored for precisely this case. We start with the electromagnetic form factor. The function $\delta(t)$ is now the P wave phase in $\pi\pi$ scattering, that we have denoted by $\delta_1(t)$. If we consider $\pi^+\pi^-$ scattering, then we have experimental information on $|F(t)|$ from $e^+e^- \to \pi^+\pi^-$ and, at $t < 0$, we can use data on $F(t)$ from πe^- scattering. If we take $\pi^+\pi^0$, then the information, at positive t, comes from the decay $\tau^+ \to \bar{\nu}_\tau \pi^+\pi^0$. We may parametrize $\delta_1(t)$ as in (6.3.3); as for $G(t)$, we take two terms in (7.1.13) and write

$$G(t) = 1 + c_1 \left[\frac{\frac{1}{2}\sqrt{s_0} - \sqrt{s_0 - t}}{\frac{1}{2}\sqrt{s_0} + \sqrt{s_0 - t}} + \frac{1}{3} \right] + c_2 \left[\left(\frac{\frac{1}{2}\sqrt{s_0} - \sqrt{s_0 - t}}{\frac{1}{2}\sqrt{s_0} + \sqrt{s_0 - t}} \right)^2 - \frac{1}{9} \right], \tag{7.2.1}$$

c_1, c_2 free parameters. We remark that, although there are only two free parameters, this is because we have imposed the condition $G(0) = 1$; the expansion (7.2.1) gives correctly the first three terms.

The quality of the fits, with only five free parameters (B_0, B_1, M_ρ; c_0, c_1) is remarkable, as can be seen in the accompanying figures 7.2.1,2; the χ^2 is, including systematic and statistical errors, χ^2 /d.o.f. = 213/204 (the $\omega - \rho$ interference effect was treated with the Gounnaris–Sakurai method).

Because we are interested not only on (relatively) rough estimates, but aim at pinning down fine details of isospin breaking as well, we will spend some time presenting the results. These results have been obtained in the course of the work reported by de Trocóniz and Ynduráin (2002), but not all of them have been published.

We consider the following types of fits. Firstly, we may take into account $\pi^+\pi^-$ form factor data (in the spacelike as well as the timelike regions) and data from τ decay into $\nu\pi^+\pi^0$. Isospin breaking is incorporated by using the correct phase space for each case, and allowing for different masses and widths for ρ^0, ρ^+; but the function $G(t)$, whose cut only starts at $t \sim 1$ GeV^2, is assumed isospin independent. This produces the best results for the hadronic contributions to the $g - 2$ of the muon and for the mass and width of the ρ:

$$\begin{aligned} M_{\rho^0} &= 772.6 \pm 0.5 \text{ MeV}, \quad \Gamma_{\rho^0} = 147.4 \pm 0.8 \text{ MeV}; \\ M_{\rho^+} &= 773.8 \pm 0.6 \text{ MeV}, \quad \Gamma_{\rho^+} = 147.3 \pm 0.9 \text{ MeV}. \end{aligned} \tag{7.2.2}$$

However, for our purposes here it is more interesting to consider two other possibilities. We may use only $\pi^+\pi^-$ data (possibility A) or we may use both $\pi^+\pi^-$ and $\pi^+\pi^0$ data, *neglecting isospin breaking effects*, in particular with the same ρ parameters (possibility B); this would then represent a kind of isospin averaged result. The departure of A from B will be a measure of isospin breaking effects.

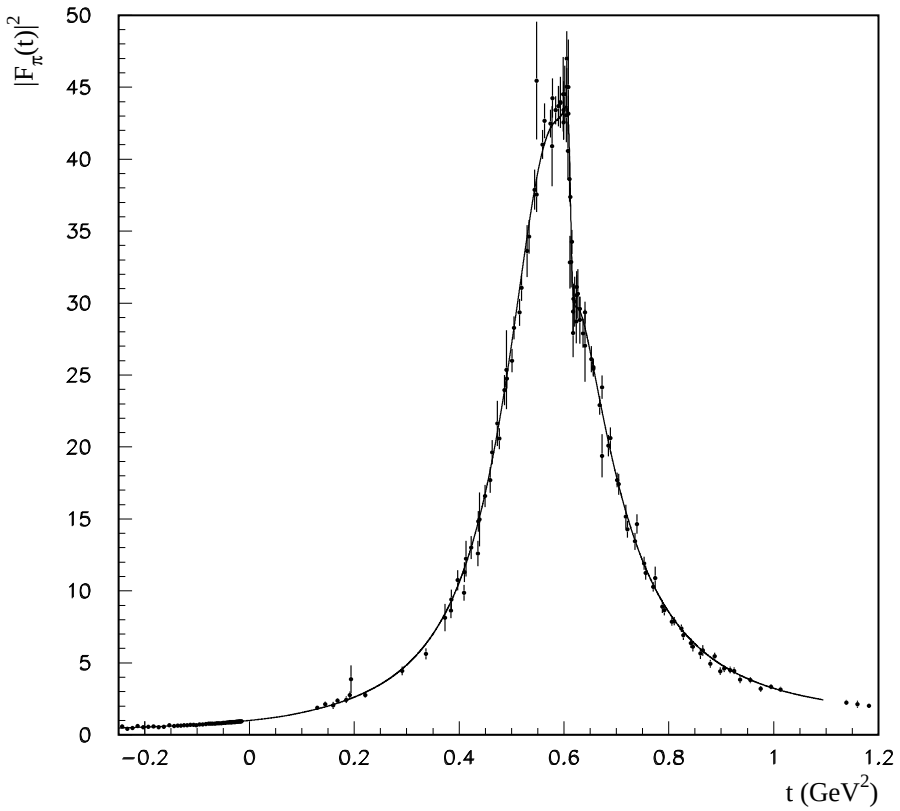

FIGURE 7.2.1. Plot of the fit to $|F_\pi(t)|^2$, timelike and spacelike data. The theoretical curve actually drawn is that obtained by fitting also τ data, but the curve obtained fitting only e^+e^- and πe data could not be distinguished from that drawn if we plotted it.

For the parameters B_i, c_i we get, in case A, and without including systematic experimental errors,[26]

$$\begin{aligned} c_1 &= 0.19 \pm 0.04, \quad c_2 = -0.15 \pm 0.10, \\ B_0 &= 1.070 \pm 0.006, \quad B_1 = 0.28 \pm 0.06 \end{aligned} \tag{7.2.3a}$$

[26]The experimental numbers are from Barkov et al. (1985), Akhmetsin et al. (1999), Amendolia et al. (1986), Anderson et. al, 2000, Barate et al. (1997) and Ackerstaff et al. (1999). There appears to be an inconsistency between the old an new versions of Akhmetsin et al. (1999), related to whether the radiative corrections have been fully incorporated. The fit given in the present paper uses the old set of data; we have checked that replacing them by the new one leaves the values of the parameters B_i, c_i, M_ρ essentially unchanged.

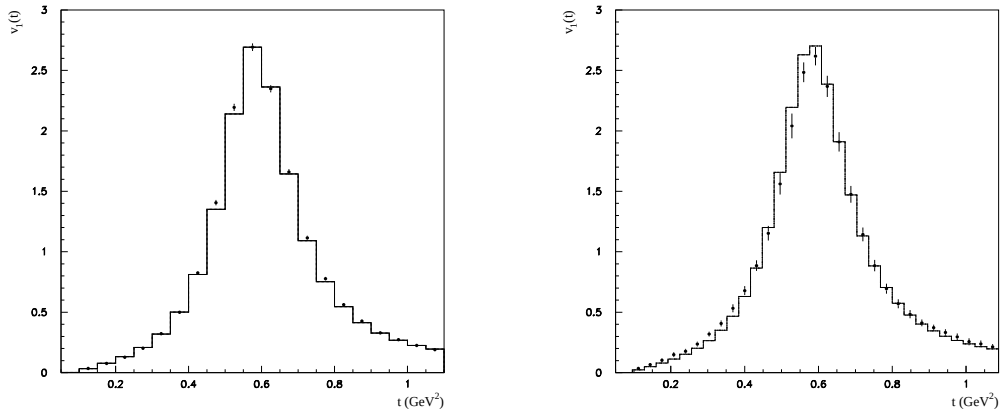

FIGURE 7.2.2 Plot of the fits to $v_1(t)$ (histograms), and data from τ decay (black dots).
Left: Aleph data. Right: Opal data. The theoretical values (histograms) are results of the *same* calculation, with the same parameters, so the differences between the two merely reflect the variations between the two experiments. We do not include the result of the fit to the third existing set of data (Anderson et al., 2000), which is much like the ones depicted here.

and, in case B,

$$\begin{aligned} c_1 = 0.23 \pm 0.01,\ c_2 = -0.16 \pm 0.03, \\ B_0 = 1.060 \pm 0.005,\ B_1 = 0.24 \pm 0.04. \end{aligned} \tag{7.2.3b}$$

If we included systematic errors we would obtain the values already reported in Subsect. 6.3.1, Eqs. (6.3.5).

Another question that has to be taken into account is the relative normalization of the various experiments. This is particularly important for the P wave $\pi\pi$ parameters, and for the slope (quadratic charge radius, $\langle r_\pi^2 \rangle$) and second derivative (c_π) of the electromagnetic pion form factor, with these last two defined by

$$F_\pi^2(t) \underset{t\to 0}{\simeq} 1 + \tfrac{1}{6}\langle r_\pi^2 \rangle t + c_\pi t^2. \tag{7.2.4}$$

What happens is that, as is clear from Fig. 7.2.2, there is a small but systematic difference between Opal and Aleph data and, as shown in de Trocóniz and Ynduráin (2002), the spacelike data on $F_\pi(t)$ do not agree well with the theoretical curve unless one takes into account systematic normalization effects.

In view of this, we present two sets of values for each quantity. In the first, the various experiments are fitted including only *statistical* errors. In the second set we repeat the fit, including *systematic* normalization effects.

It is in principle unclear which of the two sets of results is to be preferred; however, we will see later (Sect. 7.5.3) that analyticity, in the form of the Froissart–Gribov representation, favours evaluations with systematic normalization errors taken into account. The results of all three procedures are presented in the following tables:

$\pi^+\pi^-$ only	Only stat. errors	With normalization errors
$10^3 \times a_1$	$(42 \pm 2)\ M_\pi^{-3}$	$(39 \pm 2)\ M_\pi^{-3}$
$10^3 \times b_1$	$(4.5 \pm 0.4)\ M_\pi^{-5}$	$(4.5 \pm 0.3)\ M_\pi^{-5}$
$\langle r_\pi^2 \rangle$ (fm^2)	0.433 ± 0.002	0.426 ± 0.003
c_π (GeV^{-4})	3.58 ± 0.04	3.49 ± 0.06

TABLE IA

$\pi^+\pi^-$ & $\pi^+\pi^0$	Only stat. errors	With normalization errors
$10^3 \times a_1$	$(40.6 \pm 1.4)\ M_\pi^{-3}$	$(38.6 \pm 1.2)\ M_\pi^{-3}$
$10^3 \times b_1$	$(4.18 \pm 0.43)\ M_\pi^{-5}$	$(4.47 \pm 0.29)\ M_\pi^{-5}$
$\langle r_\pi^2 \rangle$ (fm^2)	0.438 ± 0.003	0.435 ± 0.003
c_π (GeV^{-4})	3.64 ± 0.05	3.56 ± 0.04

TABLE IB

The lack of dependence of a_1 on the procedure used to obtain it is a bit fictitious as the fits were obtained including the constraint $a_1 = (38 \pm 3) \times 10^{-3}\, M_\pi^{-3}$. If we had not included it, the fits would have yielded values as high as $(43 \pm 3) \times 10^{-3}\, M_\pi^{-3}$, whose central value is difficult to reconcile with $\pi\pi$ scattering data.

The parameters given above are the ones that, in particular, produce the excellent *prediction* of P wave phase shifts shown in Fig. 6.3.2, as well as the precise values of some of the chiral parameters $\bar{l}_i$ that we will give in Sect. 9.4.

7.2.2. The scalar form factor and radius of the pion

An important quantity in chiral perturbation theory calculations is the quadratic scalar radius of the pion. To define it, we start with the pion scalar form factor, F_S, given by

$$\langle \pi(p_1)|m_u \bar{u}(0)u(0) + m_d \bar{d}(0)d(0)|\pi(p_2)\rangle = F_S(t), \quad t = (p_1 - p_2)^2. \tag{7.2.5}$$

As t goes to zero we write

$$F_S(t) \underset{t\to 0}{\simeq} F_S(0)\Big\{1 + \tfrac{1}{6} t \langle r^2_{\mathrm{S},\pi}\rangle\Big\}. \tag{7.2.6}$$

One can obtain $\langle r_S^2 \rangle$ in two ways. Theoretically, we may relate it to meson masses and decay

constants, using chiral perturbation theory:

$$\begin{aligned}\langle r^2_{\mathrm{S},\pi}\rangle &= \frac{6}{m_K^2-M_\pi^2}\left(\frac{f_K}{f_\pi}-1\right)+\delta_3;\\ \delta_3 &= -\frac{1}{64\pi^2 f_\pi^2}\,\frac{1}{m_K^2-M_\pi^2}\left\{6(2m_K^2-M_\pi^2)\log\frac{m_K^2}{M_\pi^2}+9m_\eta^2\log\frac{m_\eta^2}{M_\pi^2}\right.\\ &\quad\left.-2(m_K^2-M_\pi^2)\left(10+\tfrac{1}{3}\frac{M_\pi^2}{m_\eta^2}\right)\right\}.\end{aligned}\tag{7.2.7a}$$

The details may be found in Gasser and Leutwyler (1985b) who using this obtain

$$\langle r^2_{\mathrm{S},\pi}\rangle = 0.55\pm 0.15\ \mathrm{fm}^2\quad(\mathrm{GL}),\tag{7.2.7b}$$

and the error measures the dependence of the result on the estimated higher order effects.

Alternatively, Donoghue, Gasser and Leutwyler (1990) calculate $\langle r^2_{\mathrm{S},\pi}\rangle$ from experiment, using a dispersive method based on the Omnès–Muskhelishvili procedure, to give what is presented as an accurate number:

$$\langle r^2_{\mathrm{S},\pi}\rangle = 0.61\pm 0.04\ \mathrm{fm}^2\quad(\mathrm{DGL}).\tag{7.2.7b}$$

This calculation, however, neglects inelastic (in particular, 4π) states and gives an overoptimistic treatment of quasielastic $\bar{K}K$ contributions, which are much worse known that what these authors think. The high energy contributions they take are also suspect: the central value may be biased, and the error in (7.2.7b) is certainly underestimated.

We will here give a brief account of a calculation with the Omnès–Muskhelishvili method; since its application it is very similar to that for the electromagnetic form factor, we may skip details. First of all, we remark that unitarity implies that, for $4M_\pi^2\le t\le s_0$, the phase of $F_S(t)$ equals the phase of the S0 wave in $\pi\pi$ scattering, $\delta_0^{(0)}(t)$. Here s_0 is the energy squared where inelastic contributions begin to be nonnegligible; in our case, this happens at $\bar{K}K$ threshold, so $s_0=4m_K^2$. We will assume a behaviour of the scalar form factor similar to that of the electromagnetic one, as follows from the counting rules, Eq. (2.2.20). From it, it follows that, if we denote by $\delta(t)$ to the phase of $F_S(t)$, one must have

$$\delta(t)\underset{t\to\infty}{\simeq}\pi\left\{1+\nu\frac{\log\log t/\hat{t}}{\log t/\hat{t}}\right\}.\tag{7.2.8}$$

Given this condition, δ determines uniquely F_S: one has

$$F_S(t)=F_S(0)\exp\left\{\frac{t}{\pi}\int_{4M_\pi^2}^\infty \mathrm{d}s\,\frac{\delta(s)}{s(s-t)}\right\}.\tag{7.2.9}$$

From this we get a simple sum rule for the square radius $\langle r_S^2\rangle$ corresponding to $F_S(t)$:

$$\langle r_S^2\rangle=\frac{6}{\pi}\int_{4M_\pi^2}^\infty \mathrm{d}s\,\frac{\delta(s)}{s^2}.\tag{7.2.10}$$

We will split $\langle r^2_{\mathrm{S},\pi}\rangle$ as follows:

$$\langle r^2_{\mathrm{S},\pi}\rangle = Q_J(s_0)+Q_\Phi(s_0)+Q_G(s_0).\tag{7.2.11}$$

Here Q_J is the piece in (7.2.10) coming from the region where we know δ,

$$Q_J(s_0) \equiv \frac{6}{\pi}\int_{4M_\pi^2}^{s_0} \mathrm{d}s\, \frac{\delta(s)}{s^2}; \quad \delta(s) = \delta_0^{(0)}(s). \tag{7.2.12a}$$

Q_Φ is obtained defining, as in (7.1.10), an effective phase that interpolates linearly (in t^{-1}) between the values of $\delta(t)$ at s_0 and ∞: we write

$$\delta_{\rm eff}(t) \equiv \pi + \left[\delta_0^{(0)}(s_0) - \pi\right]\frac{s_0}{t}, \tag{7.2.13b}$$

and then set

$$Q_\Phi(s_0) \equiv \frac{6}{\pi}\int_{s_0}^{\infty} \mathrm{d}s\, \frac{\delta_{\rm eff}(s)}{s^2}. \tag{7.2.13c}$$

Finally, Q_G corrects for the difference between δ and $\delta_{\rm eff}$:

$$Q_G(s_0) \equiv \frac{6}{\pi}\int_{s_0}^{\infty} \mathrm{d}s\, \frac{\delta(s) - \delta_{\rm eff}(s)}{s^2}. \tag{7.2.13d}$$

Q_J, Q_Φ are known; Q_G has to be fitted or estimated. The decomposition (7.2.11) is equivalent to decomposing F_S as a product, as we did for F_π:

$$\begin{aligned} F_S(t) &= F_S(0) J_S(t) \Phi_S(t) G_S(t); \\ J_S(t) &= \exp\left\{\frac{t}{\pi}\int_{4M_\pi^2}^{s_0} \mathrm{d}s\, \frac{\delta_0^{(0)}(s)}{s(s-t)}\right\}, \\ \Phi_S(t) &= \mathrm{e}^{1-\delta_0^{(0)}(t_0)/\pi}\left(1-\frac{t}{t_0}\right)^{[1-\delta_0^{(0)}(t_0)/\pi]t_0/t}\left(1-\frac{t}{t_0}\right)^{-1} \end{aligned} \tag{7.2.14a}$$

(we have integrated explicitly $\delta_{\rm eff}$), and $G_S(t)$ is defined by

$$G_S(t) = \exp\left\{\frac{t}{\pi}\int_{s_0}^{\infty} \mathrm{d}s\, \frac{\delta(s) - \delta_{\rm eff}(s)}{s(s-t)}\right\}. \tag{7.2.14b}$$

What we know about $G(t)$ is that $G(0) = 1$, and that it is analytic except for a cut $s_0 \le t < \infty$. Unlike for the case of the electromagnetic form factor, however, now we do not have experimental information on F_S to which we could fit G_S, so one has to rely on models or approximations for it.

Q_J is easily evaluated with the parametrizations of Sect. 6.5; likewise, one can get Q_Φ using the value $\delta_0^{(0)}(4m_K^2) = 3.14 \pm 0.52$, which comprises all experimental determinations. We find,

$$Q_J = 0.465 \pm 0.05 \text{ fm}^2,\ Q_\Phi = 0.237 \pm 0.02 \text{ fm}^2; \quad Q_J + Q_\Phi = 0.70 \pm 0.06 \text{ fm}^2. \tag{7.2.15}$$

This equation should be interpreted a *lower bound* on $\langle r_{\mathrm{S},\pi}\rangle$; it assumes that the phase of $F_S(s)$ does not increase for s beyond $\bar{K}K$ threshold, while from (7.2.8) we expect $\delta(s)$ to increase somewhat before decreasing to its asymptotic value, $\delta(\infty) = \pi$. We have therefore found the result,

$$\langle r^2_{\mathrm{S},\pi}\rangle \ge 0.70 \pm 0.06 \text{ fm}^2. \tag{7.2.16}$$

To get a value for $\langle r^2_{\mathrm{S},\pi}\rangle$ we need an estimate for G_S or, alternatively, for the phase $\delta(t)$ between $\bar{K}K$ threshold and the asymptotic region, say $s \simeq 2$ GeV^2. We will not present here the details, that the reader may find in Ynduráin (2003). One gets,

$$\langle r^2_{\mathrm{S},\pi}\rangle = 0.75 \pm 0.07 \text{ fm}^2. \tag{7.2.17}$$

7.2.3. The mixed $K\pi$ scalar form factor

The mixed $K\pi$ scalar form factor and quadratic radius are defined by

$$\begin{aligned}\langle\pi(p)|(m_s-m_q)\bar{q}s(0)|K(p')\rangle &= (2\pi)^{-3}f_{K\pi}(t),\ q=u,\,d;\\ f_{K\pi}(t) &\underset{t\to 0}{\simeq} f_{K\pi}(0)\Big\{1+\tfrac{1}{6}\langle r^2_{\mathrm{S},K\pi}\rangle\, t\Big\}.\end{aligned} \tag{7.2.18a}$$

To lowest order in chiralperturbation theory (Chapter 9),

$$f_{K^0\pi^+}(0)=M_K^2-M_\pi^2,\quad f_{K^+\pi^0}(0)=\sqrt{2}(M_K^2-M_\pi^2). \tag{7.2.18b}$$

The mixed quadratic scalar radius $\langle r^2_{\mathrm{S},K\pi}\rangle$ can be evaluated in terms of its phase, $\delta(t)$, which, when we can neglect inelasticity, equals the phase shift for the S wave $K\pi$ scattering with isospin $\frac{1}{2}$, $\delta_0^{(1/2)}(t)$:

$$\langle r^2_{\mathrm{S},K\pi}\rangle = \frac{6}{\pi}\int_{(M_\pi+m_K)^2}^{\infty}\mathrm{d}t\,\frac{\delta(t)}{t}^2,\qquad \delta(t)=\delta_0^{(1/2)}(t)\text{ for }t\le s_0; \tag{7.2.19}$$

cf. (7.2.10). Experimentally,

$$\langle r^2_{\mathrm{S},K\pi}\rangle = 0.31\pm 0.06\ \mathrm{fm}^2. \tag{7.2.20}$$

We will write, as for the pion radius,

$$\langle r^2_{\mathrm{S},K\pi}\rangle = Q_J+Q_\Phi+Q_G. \tag{7.2.21}$$

For the low energy piece we consider two possibilities. First, we assume the phase $\delta(t)$ to be given, for $t^{1/2}\le 1.5$ GeV, by the resonance $K^*(1430)$, whose properties we take from the Particle Data Tables. Its mass is $M_*=1412\pm 6$ MeV, and its width $\Gamma_*=294\pm 23$ MeV; we neglect its small inelasticity ($\sim 7\%$). We write a Breit–Wigner formula for the phase:

$$\cot\delta_0^{(1/2)}(t)=\frac{t^{1/2}}{2q}(1-s/M_*^2)B_0,\quad q=\frac{\sqrt{[s-(M_K-M_\pi)^2][s-(M_K+M_\pi)^2]}}{2s^{1/2}}$$

and $B_0=2q(M_*^2)/\Gamma_*=4.15\pm 0.35$. We take $s_0^{1/2}=1.5$ GeV, and then we have

$$Q_J=0.050\pm 0.025,\quad Q_\Phi=0.087\pm 0.001;\qquad Q_J+Q_\Phi=0.137\pm 0.03. \tag{7.2.22a}$$

This means that Q_G is large; in fact, on comparing with the experimental value, Eq. (7.2.20), we find

$$Q_G=0.175\pm 0.03. \tag{7.2.22b}$$

The sum of Q_J and Q_Φ substantially *underestimates* the value of the mixed scalar square radius: the true phase $\delta(t)$ of the form factor would have to go on growing a lot before setting to the asymptotic regime (7.2.8). The size of the phase necessary to produce the large Q_G required appears excessive.

An alternate possibility is the existence of a lower energy resonance (or enhancement; we denote it by κ), *below* the $K^*(1430)$, which some analyses suggest,[27] with $M_\kappa\sim 1$ GeV MeV and

[27] From a theoretic analysis, Oller, Oset and Peláez (1999) give $M_\kappa\simeq 1.01$ GeV; the experimental analysis of Aitala et al. (2002) gives $M_\kappa=0.80$ GeV, $\Gamma_\kappa=400\pm 100$ MeV. Note that (7.2.23) should be interpreted as an *effective* parametrization; the experimental phase does not cross 90° at $t=M_\kappa^2$.

$\Gamma_\kappa = 400 \pm 100$ MeV. In this case, we approximate the low energy phase, $s \leq s_0 = 1$ GeV^2, by writing

$$\cot \delta_0^{(1/2)}(t) = \frac{t^{1/2}}{2q}(1 - s/M_\kappa^2)B_\kappa, \quad B_\kappa = 1.8 \pm 0.5 \tag{7.2.23}$$

and find

$$Q_J = 0.07 \pm 0.03, \quad Q_\Phi = 0.18 \pm 0.006; \qquad Q_J + Q_\Phi = 0.25 \pm 0.04, \tag{7.2.24a}$$

which reproduces well the experimental number with a small Q_G, compatible with zero:

$$Q_G \sim 0.06 \pm 0.07. \tag{7.2.24c}$$

It would thus seem that the experimental data on $\langle r^2_{\mathrm{S},K\pi}\rangle$ supports the existence of this κ enhancement.

7.3. Dispersion relations and Roy equations

A possible way to improve the quality of the analysis of experimental data is to use what are known as *dispersion relations*, either at fixed t or in the form of the so-called Roy equations. We start with the first.

7.3.1. Fixed t dispersion relations

The analyticity properties of $F(s,t)$, as discussed in Sect. 2.1, imply that we can write a Cauchy representation for it, fixing t and allowing s to be complex. Starting with $s \to s + \mathrm{i}\epsilon$, s positive and $\epsilon > 0$, $\epsilon \to 0$, we have

$$F(s + \mathrm{i}\epsilon, t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \mathrm{d}s' \, \frac{A_s(s',t)}{s' - (s + \mathrm{i}\epsilon)} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \mathrm{d}s' \, \frac{A_u(s',t)}{s' - u}.$$

Here $A_s(s',t) = (1/2\mathrm{i})\{F(s' + \mathrm{i}\epsilon, t) - F(s' - \mathrm{i}\epsilon, t)\}$ is the so-called *absorptive* part of the scattering amplitude across the right hand cut, which actually equals $\mathrm{Im}\, F(s',t)$. A_u is the corresponding quantity connected with the left hand cut. Taking $\epsilon = 0$ above we find a relation between the *dispersive* part of F, $D(s,t)$, which coincides with its real part, and the $A_{s,u}$. For s physical this reads

$$\mathrm{Re}\, F(s,t) = D(s,t) = \frac{1}{\pi}\,\mathrm{P.P.}\int_{4\mu^2}^{\infty} \mathrm{d}s' \, \frac{A_s(s',t)}{s' - s} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \mathrm{d}s' \, \frac{A_u(s',t)}{s' - u} \tag{7.3.1}$$

(P.P. denotes Cauchy's principal part of the integral). This is the fixed t dispersion relation.

Actually, and because, in many cases, the $A(s,t)$ grow with s, (7.3.1) is divergent. This is repaired by *subtractions*; that is to say, writing the Cauchy representation not for F itself, but for $F(s,t)/(s - s_1)$ where s_1 is a convenient subtraction point, usually taken to coincide with a threshold. This introduces a constant in the equations (the value of $F(s,t)$ at $s = s_1$). Rewriting our equations with the appropriate subtraction incorporated is a technical problem, that we leave for the reader to take into account; for the important case of forward dispersion relations we will perform explicitly the subtractions in next subsection.

Let us rewrite the dispersion relation in a form such that we separate out the high energy contribution. We have

$$D(s,t) = \frac{1}{\pi}\,\mathrm{P.P.}\int_{4\mu^2}^{s_0} \mathrm{d}s'\,\frac{A_s(s',t)}{s'-s} + \frac{1}{\pi}\int_{4\mu^2}^{s_0} \mathrm{d}s'\,\frac{A_u(s',t)}{s'-u} + V(s,t;s_0) \tag{7.3.2a}$$

and

$$V(s,t;s_0) = \frac{1}{\pi}\int_{s_0}^{\infty} \mathrm{d}s'\,\frac{A_s(s',t)}{s'-s} + \int_{s_0}^{\infty} \mathrm{d}s'\,\frac{A_u(s',t)}{s'-u}; \tag{7.3.2b}$$

we are assuming $s < s_0$. Both D and the A may be written in terms of the *same* set of phase shifts by expanding them in partial waves:[28]

$$A(s,t) = \frac{2s^{1/2}}{\pi k}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\sin^2\delta_l(s), \tag{7.3.3a}$$

$$D(s,t) = \frac{2s^{1/2}}{\pi k}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\cos\delta_l(s)\sin\delta_l(s). \tag{7.3.3b}$$

These equations provide *constraints* for the phase shifts provided one knows (or has a reliable model) for the high energy term, $V(s,t;s_0)$. They enforce analyticity and $s \leftrightarrow u$ crossing symmetry, but not $s \leftrightarrow t$ or $t \leftrightarrow u$ crossing. This is very difficult to implement completely, as it would require analytical continuation, but a partial verification is possible through the Froissart–Gribov representation that we will discuss in Sect. 7.5.

7.3.2. Forward dispersion relations

By far the more important case of dispersion relations is that in which we take $t = 0$ (*forward dispersion relations*), which we discuss now in some detail.

Let us denote by $F_{\mathrm{o}}(s,t)$ to a scattering amplitude which is odd under the exchange of $s \leftrightarrow u$, and by $F_{\mathrm{e}}(s,t)$ to an even one. An example of the first is the amplitude corresponding to isospin $I_t = 1$ in the t channel,

$$F^{(I_t=1)} = \tfrac{1}{3}F^{(I_s=0)} + \tfrac{1}{2}F^{(I_s=1)} - \tfrac{5}{6}F^{(I_s=2)}. \tag{7.3.4}$$

Examples of even amplitudes are those for $\pi^0\pi^0 \to \pi^0\pi^0$, $\pi^0\pi^+ \to \pi^0\pi^+$:

$$\begin{aligned} F_{0+} &\equiv F(\pi^0\pi^+ \to \pi^0\pi^+) = \tfrac{1}{2}F^{(I_s=1)} + \tfrac{1}{2}F^{(I_s=2)}, \\ F_{00} &\equiv F(\pi^0\pi^0 \to \pi^0\pi^0) = \tfrac{1}{3}F^{(I_s=0)} + \tfrac{2}{3}F^{(I_s=2)}. \end{aligned} \tag{7.3.5a}$$

In terms of isospin in the t channel we have

$$F_{0+} = \tfrac{1}{3}F^{(I_t=0)} - \tfrac{1}{3}F^{(I_t=2)}, \quad F_{00} = \tfrac{1}{3}F^{(I_t=0)} + \tfrac{2}{3}F^{(I_t=2)}. \tag{7.3.5b}$$

Because there are three isospin states for pions, the three amplitudes $F^{(I_t=1)}$, $F_{\pi^0\pi^+}$ and $F_{\pi^0\pi^0}$ form a complete set.

[28] We are actually simplifying a little; (7.3.3) should take into account the isospin structure of s and u channels, which the reader may find in e.g. the text of Martin, Morgan and Shaw (1976), or one can consider that we are studying $\pi^0\pi^+$ or $\pi^0\pi^0$ scattering, for which s and u channels are identical. Also, we are assuming that there is no appreciable inelasticity below s_0.

For odd amplitudes we may profit from the antisymmetry to write a Cauchy representation for $F_{\rm o}(s,0)$ and obtain

$$\mathrm{Re}\, F_{\rm o}(s,0) = \frac{2s-4\mu^2}{\pi}\ \mathrm{P.P.}\int_{4\mu^2}^{\infty} \mathrm{d}s'\, \frac{\mathrm{Im}\, F_{\rm o}(s',0)}{(s'-s)(s'+s-4\mu^2)}. \tag{7.3.6}$$

The integral is convergent. As discussed in Sect. 2.4, Regge theory implies, for $F_{\rm o}(s,t) = F^{(I_t=1)}(s,t)$, the behaviour

$$F_{\rm o}(s,t) \underset{s\to\infty}{\simeq} C s^{\alpha_\rho(0)+\alpha'_\rho t}, \qquad \alpha_\rho(0)\simeq 0.52, \quad \alpha'_\rho \simeq 1\,\mathrm{GeV}^{-2}.$$

For even amplitudes we have to *subtract*, i.e., consider combinations $[F_{\rm e}(s,t)-F_{\rm e}(\hat{s},t)]/(s-\hat{s})$, where $\hat{s}$ is a convenient energy squared, usually taken in the range $0<\hat{s}\le 4\mu^2$. Two popular choices are the $s\leftrightarrow u$ symmetric point, $\hat{s}=2\mu^2$, and threshold, $\hat{s}=4\mu^2$. We then get the equations, respectively,

$$\begin{aligned}\mathrm{Re}\, F_{\rm e}(s,0) &= F_{\rm e}(2\mu^2,0)\\ &\quad + \frac{(s-2\mu^2)^2}{\pi}\,\mathrm{P.P.}\int_{4\mu^2}^{\infty}\mathrm{d}s'\,\frac{\mathrm{Im}\, F_{\rm e}(s',0)}{(s'-2\mu^2)(s'-s)(s'+s-4\mu^2)}\end{aligned} \tag{7.3.7a}$$

and

$$\begin{aligned}\mathrm{Re}\, F_{\rm e}(s,0) &= F_{\rm e}(4\mu^2,0)\\ &\quad + \frac{s(s-4\mu^2)}{\pi}\,\mathrm{P.P.}\int_{4\mu^2}^{\infty}\mathrm{d}s'\,\frac{(2s'-4\mu^2)\,\mathrm{Im}\, F_{\rm e}(s',0)}{s'(s'-4\mu^2)(s'-s)(s'+s-4\mu^2)}.\end{aligned} \tag{7.3.7b}$$

These integrals are convergent; the behaviour expected from Regge theory is now

$$F_{\rm e}(s,t) \underset{s\to\infty}{\simeq} C s^{1+\alpha'_P t}, \quad \alpha'_P \simeq 0.11\ \mathrm{GeV}^{-2}.$$

Actually, the convergence of (7.3.7) may be proved to follow in a general local field theory.

For a variety of other types of forward dispersion relations, see the article of Morgan and Pišut (1970) or the text of Martin, Morgan and Shaw (1976).

7.3.3. The Roy equations

Eqs. (7.3.2), (7.3.3) look rather cumbersome. Roy (1971) remarked that they appear simpler if we project (7.3.2) into partial waves: one finds the *Roy equations*

$$\cos\delta_l(s)\sin\delta_l(s) = \sum_{l'=0}^{\infty}\int_{4\mu^2}^{s_0}\mathrm{d}s'\, K_{ll'}(s,s')\sin^2\delta_{l'}(s) + V_l(s;s_0). \tag{7.3.8}$$

Here the kernels $K_{ll'}$ are known and the V_l are the (still unknown) projections of V.

Eq. (7.3.8) is valid in the simplified case we are considering here, i.e., without subtractions. If we had subtractions, the fixed t dispersion relations would acquire an extra term, a function $g(t)$. This may be eliminated, using crossing symmetry, in favour of the S wave scattering lengths. Eq. (7.3.8) would be modified accordingly.

It should be clear that there is no physics ingredient entering the Roy equations that is not present in the fixed t dispersion relation, plus partial wave expansions; (7.3.8) is strictly equivalent to the pair (7.3.2b) and (7.3.3). (In fact, there is some loss of information when using the Roy

equations: in (7.3.2b) we can also require agreement between the integral and the real part, at high energies, using also Regge theory to evaluate the last).

Roy equations became fashionable in the early seventies, but were soon abandoned; not only high energy physicists found other, more interesting, fish to fry, but it soon became obvious that they produced no better results than dispersion relations and a straightforward phase shift analysis in which one parametrizes the δ_l in a way compatible with analyticity. There are several reasons why this is so. First of all (7.3.8) (say) are highly nonlinear integral and matrix equations, and it is not clear that a solution to them exists *for a general* set of V_l. Solutions are known to exist in some favorable cases; but this constitutes the second problem: there are too many of them.[29] In fact, Atkinson (1968) proved a long time ago that, for any arbitrary $V(s,t;s_0)$ such that it is sufficiently smooth and decreasing at infinity, one can obtain by iteration a solution not only of the Roy equations, but of the full Mandelstam representation and compatible with inelastic unitarity for *all* s as well. Therefore, the solutions to the equations (7.3.8) are ambiguous in an unknown function; only the fact that the phase shifts fit experiment really constrains the solution. Indeed, it was found in the middle seventies that solutions of the Roy equations with suspiciously small errors simply reflected the prejudice as to what is a *reasonable* $V(s,t;s_0)$ and about which sets of experimental phase shifts one ought to fit. Recently, the Roy equations have been resuscitated thanks to the appearance of new experimental data that allow more meaningful constraints.

From a practical point of view, the Roy equations present two further drawbacks (with respect to the method of parametrizations based on the effective range formalism, plus straight dispersion relations). First, they mix various waves and, hence, transmit uncertainties of (say) the S-waves to other ones, and they require information on the medium and high energy regions ($s \gtrsim 1$ GeV^2) where the mixing of $\pi\pi$ with channels such as $\bar{K}K$ is essential. Second, in the integrals in the r.h. side in (7.3.8) we have to project over partial waves, hence integrate with Legendre polynomials which, for l larger than 1, oscillate and thus create unstabilities, which are difficult to control, for the D and higher waves.

Note, however, that this should not be taken as criticism of the use of Roy's equations, that provide a useful tool to analyze $\pi\pi$ scattering. In the present review, however, we prefer to concentrate on other methods: we leave the implementation of the Roy equations (and of fixed t dispersion relations, except in a few simple cases) outside the scope of these notes. The interested reader may consult the classic papers of Basdevant, Froggatt and Petersen (1972, 1974), Pennington (1975) or, more recently, the very comprehensive articles of Colangelo, Gasser and Leutwyler (2001), Ananthanarayan et al. (2001) and Descotes et al. (2002).

7.4. Evaluation of the forward dispersion relation for $\pi\pi$ scattering

As examples of application of forward dispersion relations we will evaluate here (7.3.7a) for the scattering $\pi^0\pi^+ \to \pi^0\pi^+$ and $\pi^0\pi^0 \to \pi^0\pi^0$, subtracted at $s = 4M_\pi^2$, and the Olsson sum rule,

[29] As a simple example, consider the toy model in Chapter 4. One can add to the interaction with the rho a new term: either an interaction with a scalar field, $g_\sigma \vec{\pi}\,\vec{\pi}\sigma$, or a quartic interaction, $\lambda_1(\vec{\pi}\,\vec{\pi})^2 + \lambda_2(\vec{\pi}\times\vec{\pi})^2$. Both are renormalizable field theoretical models, therefore they will satisfy unitarity, Roy's equations, crossing sum rules and the whole kit-and-caboodle as accurately as one may wish by going to high enough orders in the coupling. Moreover, they fit reasonably well the P wave and, by tuning the parameters g_ρ and g_σ, λ_i one can get any desired values for $a_0^{(0)}$, $a_0^{(2)}$; yet the two models give very different scattering amplitudes: one does, and the other does not have a scalar resonance. The statement, found at times in the literature, that the Roy equations plus the S wave scattering lengths fix the low energy scattering amplitude is plain nonsense. Fit to experiment is essential!

connected with the $F^{(I_t=1)}$ amplitude.

7.4.1. The Olsson sum rule

We will first consider the forward dispersion relation for the odd amplitude under $s \leftrightarrow u$, $F^{(I_t=1)}$, given in (7.3.6). At $s = 4M_\pi^2$, we have $F^{(I_t=1)} = (8M_\pi/\pi)(\frac{1}{3}a_0^{(0)} - \frac{5}{6}a_0^{(2)})$ and we find the so-called Olsson (1967) sum rule,

$$2a_0^{(0)} - 5a_0^{(2)} = D_{\rm Ol.}, \quad D_{\rm Ol.} = 3M_\pi \int_{4M_\pi^2}^{\infty} {\rm d}s \, \frac{{\rm Im}\, F^{(I_t=1)}(s,0)}{s(s-4M_\pi^2)}. \tag{7.4.1}$$

If we take $2a_0^{(0)} - 5a_0^{(2)}$ from (6.4.11) and (6.4.1c) we get

$$2a_0^{(0)} - 5a_0^{(2)} = (0.691 \pm 0.042) \times M_\pi^{-1}. \tag{7.4.2}$$

On the other hand, evaluating the dispersive integral also with the parametrizations (6.4.11) and (6.4.1), the remaining waves as given in Sect. 6, and the high energy contribution ($s^{1/2} \geq 1.42$ GeV) with the Regge formulas of Subsect. 7.3.4, we find $D_{\rm Ol.} = (0.659 \pm 0.020) \times M_\pi^{-1}$. Although the results are compatible within errors, the central values are certainly displaced. One may argue that this displacement is due to a bias of $a_0^{(2)}$, the only important quantity that was obtained fitting data with one pion off mass shell. If we accordingly repeat the fit to the S2 wave, *including* fulfillment of the Olsson sum rule into the fit, we find the parameters for this wave reported in (6.4.2). With them we have (7.4.2) replaced by

$$2a_0^{(0)} - 5a_0^{(2)} = (0.671 \pm 0.023)\, M_\pi^{-1}. \tag{7.4.3a}$$

For the dispersive integral we then find (we now present the results in detail)

$$\begin{array}{cc} \text{PY, dispersive} & \\ 0.431 \pm 0.016 & [\text{PY: S, P, } \ s^{1/2} \leq 0.82 \text{ GeV}] \\ 0.148 \pm 0.004 & [\text{Rest, } s^{1/2} \leq 1.42 \text{ GeV (incl., D, F below 0.82 GeV)}] \\ 0.073 \pm 0.010 & [\text{Regge, } \rho \ \ s^{1/2} \geq 1.42 \text{ GeV}] \\ 0.010 \pm 0.003 & [\text{Regge, Bk; } s^{1/2} \geq 1.42 \text{ GeV}] \\ 0.664 \pm 0.018 & [\text{Total, disp.}] \end{array} \tag{7.4.3b}$$

We call "Rest" to the contributions of the D, F waves below 1.42 GeV, plus the S, P waves between 0.82 and 1.42 GeV. Of this "Rest", the largest contribution comes from the D0 and P waves. The piece labeled "Regge, Bk" is a background to the rho Regge pole; see Peláez and Ynduráin (2003).

For future reference, we give the results we would have obtained using, for the S, P waves at energies below 0.82 GeV, the phase shifts of Colangelo, Gasser and Leutwyler (2001), that we denote by CGL, or those of Descotes et al. (2002). For the first, we have

$$2a_0^{(0)} - 5a_0^{(2)} = (0.663 \pm 0.007) \times M_\pi^{-1} \quad [\text{CGL}], \tag{7.4.4a}$$

$$D_{\rm Ol.} = (0.632 \pm 0.014) \times M_\pi^{-1}. \tag{7.4.4b}$$

We will later discuss the reasons for this mismatch. For the ones of Descotes et al. (2002),

$$2a_0^{(0)} - 5a_0^{(2)} = (0.646 \pm 0.031) \times M_\pi^{-1} \quad \text{[Descotes et al.]}, \tag{7.4.5a}$$

$$D_{\text{Ol.}} = (0.666 \pm 0.010) \times M_\pi^{-1}, \tag{7.4.5b}$$

and the error in the second only takes into account the error in the Regge contribution. The phase shifts of Descotes et al. are perfectly compatible with the Olsson sum rule and standard Reggeistics.

7.4.2. $\pi^0\pi^0$

Next, we consider the forward dispersion relation for $\pi^0\pi^0$ scattering, subtracted at the symmetric point $2M_\pi^2$. We have, with $F_{00}(s)$ the forward $\pi^0\pi^0$ amplitude,

$$F_{00}(4M_\pi^2) = F_{00}(2M_\pi^2) + D_{00}, \quad D_{00} = \frac{4M_\pi^4}{\pi} \int_{4M_\pi^2}^{\infty} \mathrm{d}s \, \frac{\operatorname{Im} F_{00}(s)}{s(s-2M_\pi^2)(s-4M_\pi^2)}. \tag{7.4.6}$$

In a first approximation we neglect the dispersive integral, and then get the approximate sum rule

$$\frac{8M_\pi}{3\pi}\left(a_0^{(0)} + 2a_0^{(2)}\right) = F_{00}(4M_\pi^2) \simeq F(2M_\pi^2) \simeq \tfrac{2}{3} f_0^{(0)}(2M_\pi^2).$$

The $\pi^0\pi^0$ amplitude contains, in the S wave, an $I = 2$ component. This we will fix as given by (6.4.2). Likewise, we fix the D waves as given by the parametrizations of Sect. 6.4. Finally, for the S wave with $I = 0$, we take the parameters of (6.4.11). Then $F_{00}(4\mu^2) = 0.093$ and $F_{00}(2\mu^2) = 0.072$, reasonably close.

A more precise test requires that we evaluate D_{00}. At high energy, i.e., for $s^{1/2} \geq 1420$ MeV, we use the Regge expression for $\operatorname{Im} F_{00}$. The amplitude for exchange of isospin 2 is evaluated as in Peláez and Ynduráin (2003). The bulk of the contribution to D_{00} is that of the S wave with $I = 0$:

D_{00}	Contribution
37.84×10^{-3}	S0
3.12×10^{-3}	S2
0.65×10^{-3}	D0
0.06×10^{-3}	D2
0.046×10^{-3}	[Regge, Pomeron $s^{1/2} \geq 1.42$ GeV]
0.006×10^{-3}	[Regge, P'; $s^{1/2} \geq 1.42$ GeV]
0.005×10^{-3}	[Regge, $I = 2$; $s^{1/2} \geq 1.42$ GeV]
41.73×10^{-3}	[Total]

(7.4.7a)

This is to be compared with

$$\begin{aligned} F_{00}(4M_\pi^2, 0) - F_{00}(2M_\pi^2, 0) &= [(123.6 \pm 8.5) - (79.1 \pm 7.7)] \times 10^{-3} \\ &= (44.5 \pm 11.4) \times 10^{-3} : \end{aligned} \tag{7.4.7b}$$

we find full overlap.

7.4.3. $\pi^0\pi^+$

We write $F_{0+}(s) \equiv F_{\pi^0\pi^+}(s,0)$ and so we have

$$F_{0+}(4M_\pi^2) = \frac{4M_\pi}{\pi} a_0^{(2)} = F_{0+}(2M_\pi^2) + D_{0+}, \tag{7.4.8a}$$

where the dispersive integral is

$$D_{0+} = \frac{4M_\pi^4}{\pi} \int_{4M_\pi^2}^{\infty} \mathrm{d}s \, \frac{\mathrm{Im}\, F_{0+}(s)}{s(s-2M_\pi^2)(s-4M_\pi^2)}. \tag{7.4.8b}$$

Before making a detailed evaluation we will make a quantitative one. Because the scattering lengths both for S2 and P waves are very small, and the *imaginary* parts of the amplitudes are (at low energy) proportional to the scattering lengths squared, we can, in a first approximation, neglect D altogether. Moreover, for $F_{0+}(2M_\pi^2)$, the S2 wave is very near its zero. If we therefore neglect it we have the approximate sum rule

$$\frac{4M_\pi}{\pi} a_0^{(2)} \simeq 3 f_1(2M_\pi^2). \tag{7.4.9}$$

Using the parametrization for the P wave (Subsect. 6.3.1) which, it will be remembered, converges down to the left hand cut, $s = 0$, we find $3f_1(2M_\pi^2) = -0.0742$ and thus the scattering length $a_0^{(2)} = -0.058\ M_\pi^{-1}$, a very reasonable number, agreeing with what we deduced from $\pi\pi$ scattering data and, to a 20%, with what is expected in chiral perturbation theory.

We next proceed to a more accurate evaluation, for which we use the same input as in the previous subsection. The calculation is now more precise because D_{0+} is dominated by the P wave, very well known. We find, for the dispersive evaluation,

D_{0+}	Contribution
2.339×10^{-3}	S2
7.928×10^{-3}	P
0.044×10^{-3}	D2
0.007×10^{-3}	F
0.046×10^{-3}	[Regge, Pomeron $s^{1/2} \geq 1.42$ GeV]
0.006×10^{-3}	[Regge, P'; $s^{1/2} \geq 1.42$ GeV]
-0.002×10^{-3}	[Regge, $I = 2$; $s^{1/2} \geq 1.42$ GeV]
10.32×10^{-3}	[Total]

(7.4.10a)

On the other hand, using directly the explicit parametrizations for the partial wave amplitudes in Sect. 2, which are valid at and below threshold (provided $s > 0$) one has

$$\begin{aligned} F_{0+}(4M_\pi^2, 0) - F_{0+}(2M_\pi^2, 0) &= [(-53.7 \pm 2.8) + (67.9 \pm 2.6)] \times 10^{-3} \\ &= (14.2 \pm 4.0) \times 10^{-3}. \end{aligned} \tag{7.4.10b}$$

This is within less than 1σ from (7.4.10a).

The fulfillment of the dispersion relations with the values of the parameters we found in this and the previous Subsections is then, for $\pi^0\pi^+$, $\pi^0\pi^0$, and the Olsson sum rule very satisfactory;

but perhaps the more impressive feature of the calculations is how little imposing the fulfillment of the dispersion relation affects the values of the parameters.[30] Those obtained from the fits to data, respecting the appropriate unitarity and analyticity requirements, wave by wave, are essentially compatible with the dispersion relations.

7.5. The Froissart–Gribov representation and low energy P, D, F wave parameters

7.5.1. Generalities

A reliable method to obtain the P and, especially, D and higher scattering lengths and effective range parameters, which incorporates simultaneously s, u and t crossing symmetry, is the Froissart (1961)–Gribov (1962) representation, to which we now turn. This method of analysis was developed long ago by Palou and Ynduráin (1974) where, in particular, a rigorous proof of the validity of the equations (7.5.3,4) below may be found and, especially, by Palou, Sánchez-Gómez and Ynduráin (1975), where a complete calculation of higher waves and effective range parameters was given. Also in the last reference the method is extended to evaluate the scattering lengths for the processes $\bar{K}K \to \pi\pi$. The interest of the representation is that it ties together s, u and t channel quantities, without need of singular extrapolations.

Consider a $\pi\pi$ scattering amplitude, $F(s,t)$, symmetric or antisymmetric under the exchange $s \leftrightarrow u$, such as $\pi^0\pi^0$ or $\pi^0\pi^+$ (symmetric), or the amplitude with isospin 1 in the t channel (antisymmetric). We may project $F(s,t)$ into the lth partial wave in the $t-$channel, which is justified for $t \leq 4\mu^2$. We have,

$$f_l(t) = \tfrac{1}{2}\int_{-1}^{+1} \mathrm{d}\cos\theta_t\, P_l(\cos\theta_t) F(s,t). \tag{7.5.1a}$$

Here $\cos\theta_t = 1 + 2s/(t-4\mu^2)$ is the t channel scattering angle. We then write a dispersion relation, in the variable s:

$$F(s,t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \mathrm{d}s'\, \frac{\operatorname{Im} F(s',t)}{s'-s} + u \text{ channel}. \tag{7.5.1b}$$

We have not written subtractions that, for $l = 1$ and higher do not alter anything, and we also have not written explicitly the u-channel contribution; it simply multiplies by 2 the s channel piece, because, for $t = 4\mu^2$, u and s channel contributions to the final result are identical.

After substituting (7.5.1b) into (7.5.1a), the integral on $\mathrm{d}\cos\theta_t$ can be made with the help of the formula

$$Q_l(x) = \tfrac{1}{2}\int_{-1}^{+1} \mathrm{d}y\, \frac{P_l(y)}{x-y},$$

with Q_l the Legendre function of the second kind. This produces the Froissart–Gribov representation,

$$f_l(t) = \frac{1}{2k_t^2}\frac{1}{\pi}\int_{4\mu^2}^{\infty} \mathrm{d}s\ \operatorname{Im} F(s,t) Q_l\left(\frac{s}{2k_t^2}+1\right) + u \text{ channel}, \qquad k_t = \frac{\sqrt{t-4\mu^2}}{2}.$$

[30] Note, however, that the fulfillment of the dispersion relations is slightly less good than what (7.4.7), (7.4.10) seem to imply. If correlations are taken into account, the sum rules are stisfied only at the level of $0.9\,\sigma$ and $1.2\,\sigma$ respectively.

Taking now the limit $t \to 4\mu^2$ in both sides of (6.3.4) and using that

$$Q_l(z) \underset{z\to\infty}{\simeq} 2^{-l-1}\sqrt{\pi}\,\frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})} z^{-l-1}$$

we find, in general,

$$\frac{f_l(t)}{k_t^{2l}} \underset{t\to 4\mu^2}{=} \frac{\Gamma(l+1)}{\sqrt{\pi}\,\Gamma(l+3/2)} \int_{4\mu^2}^{\infty} \mathrm{d}s\, \frac{\mathrm{Im}\, F(s,t)}{(s+2k_t^2)^{l+1}} + \cdots . \tag{7.5.2}$$

We then use the formula, that can be easily verified for $l \geq 1$ from the effective range expression,

$$\frac{f_l(t)}{k_t^{2l}} \underset{t\to 4\mu^2}{\simeq} \frac{4\mu}{\pi} \left\{ a_l + k_t^2 b_l \right\}, \tag{7.5.3}$$

to find the integral representation for a_l, b_l,

$$\begin{aligned} a_l &= \frac{\sqrt{\pi}\,\Gamma(l+1)}{4\mu\Gamma(l+3/2)} \int_{4\mu^2}^{\infty} \mathrm{d}s\, \frac{\mathrm{Im}\, F(s,4\mu^2)}{s^{l+1}}, \\ b_l &= \frac{\sqrt{\pi}\,\Gamma(l+1)}{2\mu\Gamma(l+3/2)} \int_{4\mu^2}^{\infty} \mathrm{d}s\, \left\{ \frac{4\,\mathrm{Im}\, F'_{\cos\theta}(s,4\mu^2)}{(s-4\mu^2)s^{l+1}} - \frac{(l+1)\,\mathrm{Im}\, F(s,4\mu^2)}{s^{l+2}} \right\}. \end{aligned} \tag{7.5.4}$$

Here $F'_{\cos\theta}(s,4\mu^2) = (\partial/\partial\cos\theta)F(s,t)|_{t=4\mu^2}$, and an extra factor of 2 should be added to the l.h. side for identical particles (as occurs if the a_l, b_l refer to a state with well defined isospin). The method holds, as it is, for waves with $l = 1$ and higher. For the S wave, the corresponding integrals are divergent; one thus needs subtractions and the method becomes much less useful. We remark that the formulas (7.5.4) are valid, when $l =$ even, *only* for amplitudes F symmetric under $s \leftrightarrow u$ crossing, and, for $l =$ odd, for amplitudes F which are antisymmetric.

For actual calculations we will, as before, replace μ by the charged pion mass, M_π.

7.5.2. D waves

For the D wave scattering lengths, the ones that we will calculate now,

$$a_2 = \frac{4}{15M_\pi} \int_{4M_\pi^2}^{\infty} \mathrm{d}s\, \frac{\mathrm{Im}\, F(s,4M_\pi^2)}{s^3}. \tag{7.5.5}$$

For these D waves, we will consider the combinations

$$a_{00} = 2\left[\tfrac{1}{3}a_2^{(0)} + \tfrac{2}{3}a_2^{(2)}\right]; \quad a_{0+} = 2\left[\tfrac{1}{3}a_2^{(0)} - \tfrac{1}{3}a_2^{(2)}\right]. \tag{7.5.6}$$

They correspond, respectively, to the processes $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$. The factor of 2 is due to the identity of the particles; it is introduced so that the projected amplitudes are F_{0+}, F_{00} as given in (7.3.5) so, for example,

$$a_{0+} = \frac{4}{15M_\pi} \int_{4M_\pi^2}^{\infty} \mathrm{d}s\, \frac{\mathrm{Im}\, F_{0+}(s,4M_\pi^2)}{s^3}.$$

We will here illustrate the method with a detailed evaluation of a_{0+} and a_{00}; we start with the first. The contribution of the high energy ($s^{1/2} \geq 1.42$ GeV) is obtained integrating (7.5.4) in that region with the Regge formulas of Sect. 2.4. For the low energy pieces we use, for the D, F waves

and for the S, P waves above $s^{1/2} = 0.82$ GeV, the fits given in Chapter 6. For the S, P waves below 0.82 GeV, we use either the phase shifts of Colangelo, Gasser and Leutwyler (2001), that we denote by CGL, or ours here, that we denote by PY (since they correspond to the calculation made in Peláez and Ynduráin, 2003). We find, in units of $10^{-4}\ M_\pi^{-5}$

$$\begin{array}{cc} \text{CGL, Froissart} - \text{Gribov} & \text{PY, Froissart} - \text{Gribov} \\ 8.43 \pm 0.09 \quad [\text{CGL: S, P, } s^{1/2} \leq 0.82 \text{ GeV}] & 8.09 \pm 0.15 \quad [\text{PY S, P, }] \\ 1.84 \pm 0.05 \quad [\text{Rest, } s^{1/2} \leq 1.42 \text{ GeV}] & \\ 0.68 \pm 0.07 \quad [\text{Regge, } I_t = 0] & \\ -0.06 \pm 0.02 \quad [\text{Regge, } I_t = 2] & \\ 10.90 \pm 0.13 \quad [\text{CGL, F.–G.}] & 10.51 \pm 0.15 \quad [\text{PY, F.–G.}] \end{array} \tag{7.5.7}$$

The value found with this method with the CGL phase shifts disagrees by several standard deviations with the value given in the paper of these authors (Colangelo, Gasser and Leutwyler, 2001),

$$a_{0+} = (10.53 \pm 0.10) \times 10^{-4}\ M_\pi^{-5}\ (\text{CGL}).$$

We will discuss this mismatch later on.

This combination we have calculated is the one that may be evaluated with less ambiguity; the values of other low energy parameters depend substantially on the S wave scattering length. Thus, from the $\pi^0\pi^0$ scattering amplitude we calculate the combination

$$a_{00} = 2\left[\tfrac{1}{3}a_2^{(0)} + \tfrac{2}{3}a_2^{(2)}\right].$$

We find, again in units of $10^{-4}\ M_\pi^{-5}$,

$$\begin{array}{cc} \text{CGL, Froissart} - \text{Gribov} & \text{PY, F.} - \text{G.} \\ 11.73 \pm 0.32 \quad [\text{CGL S, P, } s^{1/2} \leq 0.82 \text{ GeV}] & 12.24 \pm 0.62 \quad [\text{PY S, P}] \\ 1.91 \pm 0.04 \quad [\text{Rest, } s^{1/2} \leq 1.42 \text{ GeV}] & \\ 0.68 \pm 0.07 \quad [\text{Regge, } I_t = 0] & \\ 0.12 \pm 0.04 \quad [\text{Regge, } I_t = 2] & \\ 14.44 \pm 0.33 & 14.95 \pm 0.65 \quad [\text{Total}]; \end{array} \tag{7.5.8}$$

The value given for this quantity in Colangelo, Gasser and Leutwyler (2001) is

$$a_{00} = (13.94 \pm 0.32) \times 10^{-4}\ M_\pi^{-5}.$$

The effective range parameters

$$b_{00} = 2\left[\tfrac{1}{3}b_2^{(0)} + \tfrac{2}{3}b_2^{(2)}\right]; \quad b_{0+} = 2\left[\tfrac{1}{3}b_2^{(0)} - \tfrac{1}{3}b_2^{(2)}\right]$$

may also be calculated; we only give the final results, in units of $10^{-4}\ M_\pi^{-7}$ now:

$$\begin{array}{rccc} b_{0+}: & [\text{CGL, direct}]; & [\text{CGL, F.–G.}]; & [\text{PY, F.–G.}]; \\ & -0.189 \pm 0.016 & -0.233 \pm 0.036 & -0.170 \pm 0.083; \\ b_{00}: & [\text{CGL, direct}]; & [\text{CGL, F.–G.}]; & [\text{PY, F.–G.}]. \\ & -6.72 \pm 0.22 & -6.61 \pm 0.23 & -6.85 \pm 0.47 \end{array} \tag{7.5.9}$$

Here the quantities labeled "CGL, direct" are obtained from the values given in Colangelo, Gasser and Leutwyler (2001) for scattering lengths and ranges.

7.5.3. P and F waves

We use now (7.5.4), with $\mathrm{Im}\, F \equiv \mathrm{Im}\, F^{(I_t=1)}$. For the P wave scattering length, the integral is slowly convergent; the integrand behaves like $\mathrm{Im}\, F^{(I_t=1)}/s^2 \sim s^{-1.48}$, and we do not have the factor $s - 4M_\pi^2$ in the denominator of the Olsson sum rule that favoured low energies. Because of this, the details of the energy region $1.42 \leq s^{1/2} \leq 1.80$ GeV are not negligible. We have represented $\mathrm{Im}\, F^{(I_t=1)}(s, 4M_\pi^2)$ there by a Regge formula, given by the rho trajectory. To this we could add the resonances $\rho(1450)$, $\rho(1700)$ and $\rho_3(1690)$, whose contribution is easily evaluated in the narrow width approximation,

$$\mathrm{Im}\, f \simeq \frac{2s^{1/2}}{\pi k}\pi M \Gamma_{2\pi}\delta(s - M^2),$$

with M, $\Gamma_{2\pi}$ the mass and two-pion width of the resonance. Their contribution is small. Alternatively, we may supplement the rho Regge piece with a smooth background; see for example Peláez and Ynduráin (2003); this is the method we will use. The contribution of this background is also small.

We find the results, in units of $10^{-3} \times M_\pi^{-3}$,

$$\begin{array}{cccc} \text{CGL, Froissart} - \text{Gribov} & & \text{PY, F.} - \text{G.} & \text{TY(St.+Sys.)} \\ 18.5 \pm 0.2 \text{ [CGL S, P, } s^{1/2} \leq 0.82 \text{ GeV]} & & 19.4 \pm 0.3 & \\ 9.1 \pm 0.3 \quad \text{[Rest, } s^{1/2} \leq 1.42 \text{ GeV]} & & & \\ 8.3 \pm 1.1 \quad \text{[Regge, } \rho\text{]} & & & \\ 1.0 \pm 0.3 \quad \text{[Regge, Bk]} & & & \\ 37.0 \pm 1.3 \quad \text{[Total]} & & 38.1 \pm 1.4 & 38.6 \pm 1.2. \end{array} \tag{7.5.10}$$

The number labeled "TY (St.+Sys.)" refers to what we obtained in Subsect. 7.2.1 from the fit to the pion form factor.

As we see, there is good agreement, within errors, among all determinations, and also with the value given by Colangelo, Gasser and Leutwyler (2001),

$$a_1 = (37.9 \pm 0.5) \times 10^{-3} \times M_\pi^{-3}, \text{ (CGL)}.$$

The value coming from the pion form factor, if we had not taken into account systematic errors, would have been

$$a_1 = (40.6 \pm 1.4) \times 10^{-3} \times M_\pi^{-3};$$

this is a bit too high, which is one of the reasons why we prefer the fit *including* systematic normalization errors.

For the effective range parameter, we find, in units of $10^{-3} \times M_\pi^{-5}$ now,

$$\begin{array}{cccc} \text{CGL, Froissart} - \text{Gribov} & & \text{PY, F.} - \text{G.} & \text{TY(St.+Sys.)} \\ -.92 \pm 0.05 \text{ [CGL S, P, } s^{1/2} \leq .82 \text{ GeV]} & & -0.57 \pm 0.10 & \\ 1.02 \pm 0.04 \quad \text{[Rest, } s^{1/2} \leq 1.42 \text{ GeV]} & & & \\ 4.82 \pm 0.86 \quad \text{[Regge, } \rho\text{]} & & & \\ 0.50 \pm 0.16 \quad \text{[Regge, Bk]} & & & \\ 5.50 \pm 0.82 \quad \text{[Total.]} & & 5.15 \pm 0.90 & 4.47 \pm 0.29. \end{array} \tag{7.5.11}$$

Here the Regge contribution is particularly important because the lower energy pieces cancel almost completely; we use, as we did in Eq. (7.5.10), the quadratic expression for the rho trajectory. The value following from the pion form factor, without taking into account systematic errors, would be $(4.18 \pm 0.43) \times 10^{-3} \times M_\pi^{-5}$.

The value obtained with the Froissart–Gribov representation and the phases of Colangelo, Gasser and Leutwyler (2001), or that given by these authors, $b_1 = (5.67 \pm 0.13) \times 10^{-3} \times M_\pi^{-5}$, is $4\,\sigma$ away from the value obtained from the fit to the pion form factor, labeled "TY (St. +Sys.)", or from the same without systematic errors, which gave $b_1 = (4.18 \pm 0.43) \times 10^{-3}\, M_\pi^{-5}$.

We conclude the present Subsection with the F wave scattering length. Here the high energy part is negligible; we give only those contributions that are sizable. We have, if using the phase shifts given here,

$$a_3 = (6.00 \pm 0.07) \times 10^{-5}\ M_\pi^{-5} \quad \text{(PY)}. \tag{7.5.12}$$

This is displaced by about $2\,\sigma$ from the value given by Colangelo, Gasser and Leutwyler (2001):

$$a_3 = (5.60 \pm 0.19) \times 10^{-5}\ M_\pi^{-5} \quad \text{(CGL)}.$$

7.5.4. G waves

The scattering lengths and effective range parameters may be calculated for the G waves; we only give the values of the first:

$$a_4^{(0)} = (8.0 \pm 0.2) \times 10^{-6}\, M_\pi^{-9}, \quad a_4^{(2)} = (4.5 \pm 0.2) \times 10^{-6}\, M_\pi^{-9}. \tag{7.5.13}$$

7.6. Summary and conclusions

In this Section we discuss two recent alternate sets of phase shifts due to the Bern group and to Descotes et al. (2002), and compare them between themselves and with what we have found.

7.6.1. The S, P waves of Colangelo, Gasser and Leutwyler

In a recent paper, Ananthanarayan et al. (2001) have made a detailed calculation of $\pi\pi$ scattering using experimental information and the Roy equations (but *not* full information on the pion form factor). This calculation has been refined by Colangelo, Gasser and Leutwyler (2001) who use Roy equations and chiral perturbation theory, including (estimated) terms of order p^6. In the last paper, that we will here denote by CGL, these authors pretend to obtain an extremely precise representation of low energy $\pi\pi$ parameters that, in some cases, reaches beyond the 1% level.

Unfortunately, and as made clear in particular in the articles of Peláez and Ynduráin (2003), Ynduráin (2003b) (and as we will also see briefly here) the precision claimed by CGL is excessively optimistic. First of all, the high energy scattering amplitude that CGL (and also Ananthanarayan et al., 2001) use is excessively different from what standard Regge theory (or its QCD version) implies.[31] Secondly, it is not clear that the accuracy of their chiral expansions is what one can

[31] In section B.4 of their paper, Ananthanarayan et al. (2001) explain that they take an asymptotic value for the total (Pomeron) cross section of 5 ± 3 mb. This is a factor three smaller than the value implied by factorization, as discussed in our Sect. 2.4 here. They also take a rho residue 50% higher than the

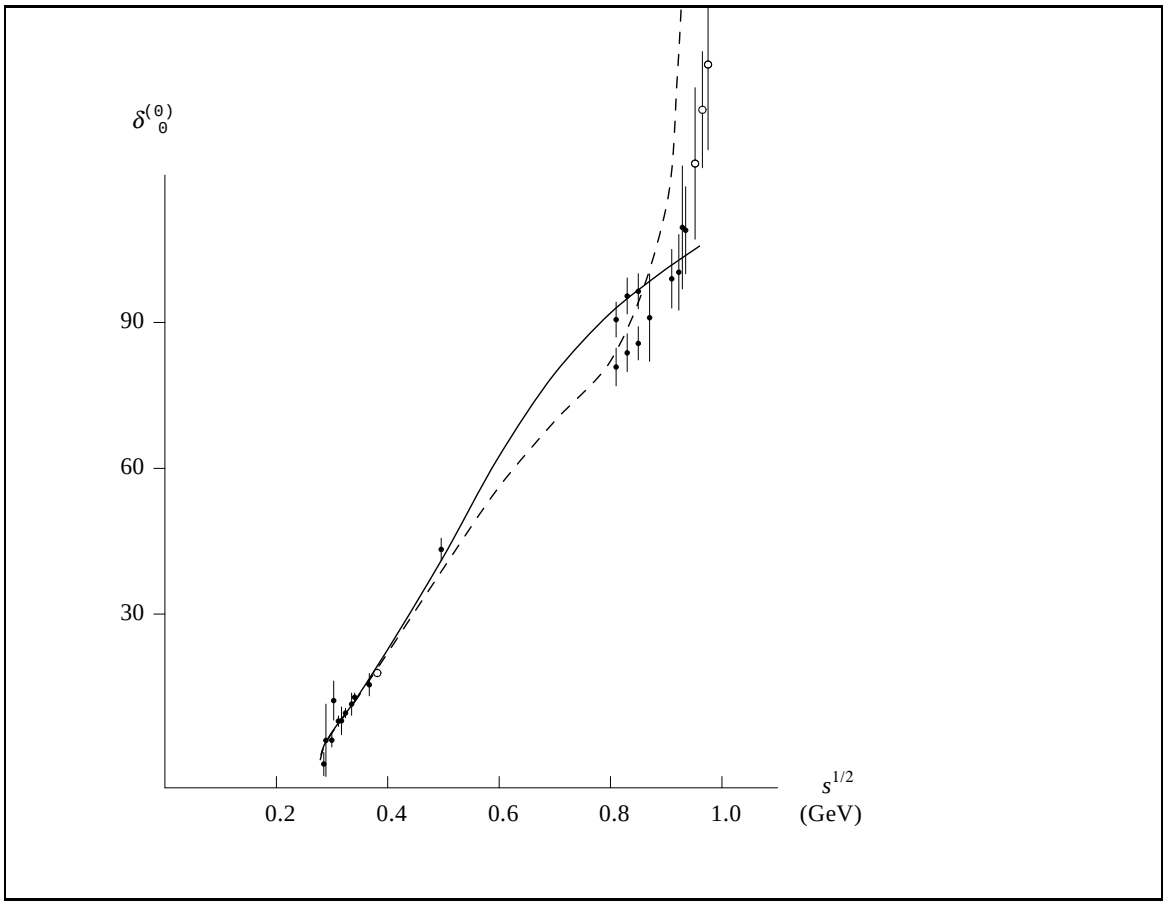

FIGURE 7.6.1. The $I = 0$, S-wave phase shifts corresponding to Eq. (6.4.11) (continuous line) and CGL, Eqs. (7.6.1) (broken line). Note that CGL only fit up to 0.8 GeV, but we have represented the upper tail because its lashing out is very suggestive; see text.

reasonably expect. Because of these reasons and, perhaps, also due to optimism of CGL when evaluating their errors, mismatches occur in the CGL phase shifts that reach the 2 to $4\,\sigma$ level. Some examples have already been seen; we come back to them below.

CGL only present in their paper details for the S, P waves up to an energy of 0.82 GeV, in

standard value and use incorrect formulas for the slopes of the Regge trajectories. The incompatibility of such Regge parameters with experimental data on $\pi\pi$ scattering makes all the results found by these authors very suspect.

the form of the following parametrizations, whose form they have taken from Schenk (1991):

$$\begin{aligned}\tan\delta_l^{(I)}(s) &= k^{2l}\sqrt{\frac{1-4M_\pi^2}{s}} \\ &\times \left\{A_l^I + B_l^I k^2/M_\pi^2 + C_l^I k^4/M_\pi^4 + D_l^I k^6/M_\pi^6\right\}\frac{4M_\pi^2 - s_{lI}}{s - s_{lI}},\end{aligned} \tag{7.6.1a}$$

$k = \sqrt{s/4 - M_\pi^2}$, and the values of the parameters, as given by CGL, Eq. (17.2), are

$$\begin{array}{cccc} A_0^0 = 0.220, & B_0^0 = 0.268, & C_0^0 = -0.0139, & D_0^0 = -1.39/10^3, \\ A_0^2 = -0.0444/10, & B_0^2 = -0.0857, & C_0^2 = -2.21/10^3, & D_0^2 = -1.29/10^4, \\ A_1 = 0.379/10, & B_1 = 0.140/10^4, & C_1 = -6.73/10^5, & D_1 = 1.63/10^8, \end{array} \tag{7.6.1b}$$

and

$$s_{00} = 36.77\, M_\pi^2, \quad s_{02} = -21.62\, M_\pi^2, \quad s_1 = 30.72\, M_\pi^2. \tag{7.6.1c}$$

In Appendix B we will criticise this type of parametrization, which is very inefficient.

The values of the low energy parameters will be given below. The errors to Eq. (7.6.1) may be found in an Appendix to the paper by Ananthanarayan et al. (2001).

As stated before, this CGL solution is slightly different from that given in Peláez and Ynduráin (2003), reproduced in the present notes: S0, S2 phase shifts are displaced with respect to the PY ones by about $1\,\sigma$ (Figs. 7.6.1, 6.4.1), and the errors are about a half of the errors we have found with our direct fits.

An important fact to remark is that Ananthanarayan et al. (2001) as well as CGL impose the (experimental) values of the phase shifts at the highest energy at which they cut off the Roy equations, $s^{1/2} = 0.8$ GeV. The values they take for the S2, P waves are reasonable but, for the S0 wave they require

$$\delta_0^{(0)}((0.8\text{ GeV})^2) = 82.3 \pm 3.4^\circ\,.$$

The error is a factor 2 or 3 times too small; cf. our discussion in Subsect. 6.4.2. The fit is thus a forced fit, something that appears very clearly in Fig. 7.6.1: the phase shoots up as soon as the energy is above 0.8 GeV, where these authors constrain it to go through the narrow corridor of a value of $82.3 \pm 3.4^\circ$.

7.6.2. The S wave scattering lengths of Descotes et al., and Kamiński et al.

The Roy equations (and of course experimental data) have also been used in a recent paper by Descotes et al. (2002) to find the low energy $\pi\pi$ scattering, especially the S0, S2 scattering lengths. The results these authors find are about two standard deviations different from those of Colangelo, Gasser and Leutwyler (2001), but more similar –both in central value and errors– to the ones given in the present notes: with $M_\pi = 1$, Descotes et al. give the values

$$a_0^{(0)} = 0.228 \pm 0.012, \quad a_0^{(2)} = -0.0382 \pm 0.0038.$$

The phase shifts of Descotes et al. (2002) are also compatible with the Olsson sum rule evaluated with standard Regge behaviour, as we showed in Subsect. 7.4.1.

A similar calculation, using also Roy equations, has been given by Kamiński, Leśniak and Loiseau (2003). These authors find

$$a_0^{(0)} = 0.224 \pm 0.013, \quad a_0^{(2)} = -0.0343 \pm 0.0036$$

and the effective range parameters

$$b_0^{(0)} = 0.252 \pm 0.011, \quad b_0^{(2)} = -0.075 \pm 0.015.$$

These results are compatible with those of Descotes et al. (2002), less so with our results here, and even less with the results of Colangelo, Gasser and Leutwyler (2001). We will discuss more about this calculation below.

7.6.3. Comparison of different calculations. Low energy parameters for $\pi\pi$ scattering

In this Subsection we summarize and discuss the results obtained in the two last chapters. In them, we have found simple, explicit formulas that satisfy the general requirements of analyticity and unitarity and which fit well the experimental data; then we have verified that our solutions are compatible with a few crossing and analyticity constraints: forward dispersion relations at threshold and the Froissart–Gribov representation.

It should be clear that we have not made an exhaustive analysis: nor was it intended. Thus, we have not tried to improve our parameters by fully imposing consistency requirements. For example, it is easily verified that a change in the parameters of the P and S0 waves to decrease (in the case of the first) the scattering length by of about $1/2\ \sigma$ would improve agreement of the Froissart–Gribov and direct determinations of a_1. However, for the improvements to be more than cosmetic, we should also consider dispersion relations in a wider range of s and t values or, equivalently, the Roy equations. This is the path followed by Ananthanarayan et al. (2001), Colangelo, Gasser and Leutwyler (2001), and Descotes et al. (2002), where the interested reader may find details. The results found in the first of these papers (which does not impose chiral perturbation theory) are compatible with ours, at the 1 to 1.5 σ level. Also the errors are similar, with theirs generally smaller. The price they pay, however, is that all their numbers are correlated, whereas ours are not: in this sense, our results are more robust. The method of the Roy equations and ours here are complementary.

	DFGS	ACGL	CGL	PY
$a_0^{(0)}$	0.228 ± 0.012	0.240 ± 0.060	0.220 ± 0.005	0.230 ± 0.010
$a_0^{(2)}$	-0.0382 ± 0.0038	-0.036 ± 0.013	-0.0444 ± 0.0010	-0.0422 ± 0.0022
$b_0^{(0)}$		0.276 ± 0.006	0.280 ± 0.001	0.268 ± 0.010
$b_0^{(2)}$		-0.076 ± 0.002	-0.080 ± 0.001	-0.071 ± 0.004

TABLE II

We will start the comparison with the S wave scattering lengths and effective rage parameters (Table II). Here we compare the results of Descotes et al. (2002), denoted by DFGS, Ananthanarayan et al. (2001) , denoted by ACGL, Colangelo, Gasser and Leutwyler (2001), denoted by

CGL, and the results in the present notes, denoted by PY. We use units with $M_\pi = 1$. As already remarked, the results of Descotes et al. (2002) for the S waves are in disagreement, at the level of $1\,\sigma$ per wave, with those of CGL, but agree well with the results presented in the present work (PY). The effective range parameters, $b_0^{(I)}$, are also compared between ACGL, CGL and PY.

We next turn to higher waves. Here we have, in the case of the analysis of CGL, two results: those given in the paper of Colangelo, Gasser and Leutwyler (2001), or those obtained using the Froissart–Gribov representation with the low energy S, P waves of CGL. Both types of results are shown in Table III.

	Nagel	PSGY	CGL	PY
a_1	38 ± 2	38.5 ± 0.6	37.0 ± 0.13 $[37.9 \pm 0.5]$ [a]	38.1 ± 1.4 $[38.6 \pm 1.2]$ [b] $\times\ 10^{-3}$
b_1		3.2 ± 0.3	5.50 ± 0.82 $[5.67 \pm 0.13]$ [a]	5.15 ± 0.90 $[4.47 \pm 0.27]$ [b] $\times\ 10^{-3}$
$a_2^{(0)}$	17 ± 3	18.5 ± 0.6	17.5 ± 0.3	$18.0 \pm 0.2\ \times 10^{-4}$
$a_2^{(2)}$	1.3 ± 3	1.9 ± 0.6	1.70 ± 0.13	$2.2 \pm 0.2\ \times 10^{-4}$
a_{0+}	10.5 ± 3	11.07 ± 0.52	10.90 ± 0.13 $[10.53 \pm 0.10]$ [a]	$10.51 \pm 0.15\ \times 10^{-4}$
a_{00}	13.1 ± 5	14.9 ± 0.8	14.44 ± 0.33 $[13.94 \pm 0.32]$ [a]	$14.95 \pm 0.32\ \times 10^{-4}$
b_{0+}		-0.12 ± 0.07	-0.233 ± 0.036 $[-0.189 \pm 0.016]$ [a]	$-0.170 \pm 0.083\ \times 10^{-4}$
b_{00}		-7.1 ± 0.8	-6.61 ± 0.23 $[-6.72 \pm 0.22]$ [a]	$-6.85 \pm 0.47\ \times 10^{-4}$
a_3	6 ± 2	6.2 ± 0.9	5.60 ± 0.19	$6.0 \pm 0.7\ \times 10^{-4}$
b_3		-5.0 ± 1.2	-4.02 ± 0.18	$\times 10^{-4}$
$a_4^{(0)}$		0.9 ± 0.1		$0.8 \pm 0.2\ \times\ 10^{-5}$
$a_4^{(2)}$		0.45 ± 0.12		$0.45 \pm 0.2\ \times\ 10^{-5}$

Units of M_π. Nagel: Nagel et al., (1979); PSGY: Palou, Sánchez-Gómez and Ynduráin (1975). (These two sets are included for historical interest). CGL: Colangelo, Gasser and Leutwyler (2001); PY: the results in Peláez and Ynduráin (2003), as given in the present review. The errors in PSGY only include the error due to variation of $a_0^{(0)}$ between 0.2 and 0.3 M_π^{-1}; full errors could be a factor 2 larger.
([a]) The numbers in braces are those given by CGL themselves; the numbers outside are from the Foissart–Gribov representation. ([b]) Numbers in braces are from the pion form factor (de Trocóniz and Ynduráin, 2002). Other numbers in "PY" are from the Froissart–Gribov projection.

TABLE III

We have also, not included in the Table III, the results of Kamiński, Leśniak and Loiseau (2003) for the P wave (the S waves were discussed already in Subsect. 6.3.2). These authors give the numbers

$$a_1 = (39.6 \pm 2.4) \times 10^{-3}\, M_\pi^{-3}, \quad b_1 = (2.63 \pm 0.67) \times 10^{-3}\, M_\pi^{-3}.$$

The central value for a_1 is too high, although it is compatible within its errors with other determinations (Table III). The value of b_1, however, is almost $3\,\sigma$ below the lowest one in Table III. The reason could be that Kamiński, Leśniak and Loiseau use some calculation techniques (effective, separable potentials and Padé approximants) which may bias their results.

We think the numbers in Table III speak for themselves, but there are a few remarks that can be made. First of all, we would also like to comment on the agreement of the old results of PSGY with the more modern determinations, both here and in CGL. This is noteworthy, and lends weight to the suitability of the Froissart–Gribov method to calculate low energy parameters for P, D and higher waves: compare for example PSGY with Nagel, in spite of the fact that the last is two year more modern (apparently, Nagel and collaborators were not aware of the PSGY evaluations). Secondly, we have not presented the results of Ananthanarayan et al. (2001). They are similar to those of CGL, but the errors are a factor ~ 2.5 times larger. Thirdly, we have the inconsistency of the results of CGL, both with themselves (different values for different determinations, as for a_{0+}, a_{00}) and with the TY value (for b_1), which reaches the $4\,\sigma$ level. This is discussed in detail in Peláez and Ynduráin (2003), Ynduráin (2003b) and, together with the independent indications found by Descotes et al., (2002), indicates that the CGL solution is displaced, for some quantities, by about one to two sigma, and their error estimates are probably a factor 1.5 or 2 too optimistic.

The reasons for this are, probably, a mixture of the following: first, use of an unrealistic set of Regge parameters and intermediate energy ($1.42\ \text{GeV} \leq s^{1/2} \leq 2\ \text{GeV}$) partial wave amplitudes. Secondly, the chiral perturbative expansions these authors use probably do not converge as fast as they assume. And thirdly, their experimental input is, at times, endowed with irrealistically small error estimates: see for example the discussion of this last in Ynduráin (2003b).

8. QCD, PCAC and chiral symmetry for pions and kaons

8.1. The QCD Lagrangian. Global symmetries; conserved currents

In the previous chapters we have discussed *general* properties of pion interactions. In this chapter[32] we remember that pions are bound states of quark-antiquark, and that we have a theory of strong interactions, QCD. We will then study properties of pion physics which follow, in one way or another, from the QCD Lagrangian,

$$\mathcal{L} = -\sum m_l \bar{q}_l q_l + \mathrm{i}\sum_{l=1}^{n} \bar{q}_l \not{D} q_l - \tfrac{1}{4}(D\times B)^2 + \text{gauge fixing} + \text{ghost terms}, \tag{8.1.1}$$

where q_l is the quark operator for flavour l, B is the gluon field operator, etc.; sum over omitted colour indices is generally understood. In (8.1.1) we assume that we have only *light* quarks (u, d and, eventually, s). The existence of heavy quarks has little influence in the physics of small momenta in which we are interested here.

In the present section we start with the *global* symmetries of the QCD Lagrangian. Since its form is unaltered by renormalization, we can neglect the distinction between bare and renormalized $\mathcal{L}$.

Clearly, $\mathcal{L}$ is invariant under Poincaré transformations, $x \to \Lambda x + a$. The currents corresponding to (homogeneous) Lorentz transformations Λ are not of great interest for us here. Space-time translations generate the *energy-momentum tensor*. Its form is fixed by Noether's theorem, which gives

$$\Theta^{\mu\nu} = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}\,\partial_\nu \Phi_i - g^{\mu\nu}\mathcal{L}, \tag{8.1.2a}$$

and the sum over i runs over all the fields in the QCD Lagrangian. These currents are conserved,

$$\partial_\mu \Theta^{\mu\nu} = 0,$$

and the corresponding "charges" are the components of the four-momentum

$$P^\mu = \int \mathrm{d}^3\mathbf{x}\,\Theta^{0\mu}(x).$$

The explicit expression for $\Theta^{\mu\nu}$ in QCD is

$$\begin{aligned}\Theta^{\mu\nu} =& \,\mathrm{i}\sum_q \bar{q}\gamma^\mu D^\nu q - \mathrm{i}g^{\mu\nu}\sum_q \bar{q}\not{D} q + g^{\mu\nu}\sum_q m_q \bar{q} q \\ & - g_{\alpha\beta}G^{\mu\alpha}G^{\nu\beta} + \tfrac{1}{4}g^{\mu\nu}G^2 + \text{gauge fixing} + \text{ghost terms}.\end{aligned} \tag{8.1.2b}$$

In the quantum version, we understand that products are replaced by Wick ordered products. Θ is not unique and, as a matter of fact, direct application of (8.1.2a) does not yield a gauge

[32] This chapter follows, to a large extent, the corresponding one in the text of the author, Ynduráin (1999), where we send for more details on QCD.

invariant tensor. To obtain the gauge invariant expression (8.1.2b) one may proceed by replacing derivatives by covariant derivatives. A more rigorous procedure would be to reformulate (8.1.2a) in a way consistent with gauge invariance by performing gauge transformations simultaneously to the spacetime translation. For an infinitesimal one, $x^\mu \to x^\mu + \epsilon^\mu$, we then define

$$B_a^\mu \to B_a^\mu + (\epsilon_\alpha \partial^\alpha B_a^\mu \equiv D^\mu \epsilon_\alpha B_a^\alpha + \epsilon_\alpha G_a^{\alpha\mu}).$$

The term $D^\mu \epsilon_\alpha B_a^\alpha$ may be absorbed by a gauge transformation, so we may write the transformation as $B_a^\mu \to B_a^\mu + \epsilon_\alpha G_a^{\alpha\mu}$. For a discussion of the arbitrariness in the definition of the energy–momentum tensor, see Callan, Coleman and Jackiw (1970) and Collins, Duncan and Joglekar (1977).

Next, we have the currents and charges associated with colour rotations. We leave it to the reader to write them explicitly; they are particular cases of colour gauge transformations (with constant parameters). We now pass over to a different set of currents *not* associated with interactions of quarks and gluons among themselves.

If all the quark masses vanished, we would have invariance of $\mathcal{L}$ under the transformations,

$$q_f \to \sum_{f'=1}^{n_f} W_{ff'} q_{f'}, \quad q_f \to \sum_{f'=1}^{n_f} W^5_{ff'} \gamma_5 q_{f'} \tag{8.1.3}$$

where f, f' are flavour indices, and W, W^5 unitary matrices. This implies that the currents

$$\begin{aligned} V^\mu_{qq'}(x) &= \bar q(x)\gamma^\mu q'(x), \\ A^\mu_{qq'}(x) &= \bar q(x)\gamma^\mu\gamma_5 q'(x) \end{aligned} \tag{8.1.4}$$

would be each separately conserved. When mass terms are taken into account, only the diagonal V^μ_{qq} are conserved; the others are what is called *quasi-conserved* currents, i.e., their divergences are proportional to masses. These divergences are easily calculated: since the transformations in (8.1.3) commute with the interaction part of $\mathcal{L}$, we may evaluate them with free fields, in which case use of the free Dirac equation $\mathrm{i}\partial\!\!\!/\, q = m_q q$ gives

$$\partial_\mu V^\mu_{qq'} = \mathrm{i}(m_q - m_{q'})\bar q q', \quad \partial_\mu A^\mu_{qq'} = \mathrm{i}(m_q + m_{q'})\bar q\gamma_5 q'. \tag{8.1.5}$$

In fact, there is a subtle point concerning the divergence of axial currents. Eq. (8.1.5) is correct as it stands for the nondiagonal currents, $q \neq q'$; for $q = q'$, however, one has instead

$$\partial_\mu A^\mu_{qq}(x) = \mathrm{i}(m_q + m_q)\bar q(x)\gamma_5 q(x) + \frac{T_F g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}(x) G_{\rho\sigma}(x), \tag{8.1.6}$$

with $T_F = 1/2$ a colour factor. This is the so-called Adler–Bell–Jackiw anomaly, that we will discuss later.

The *equal time commutation relations* (ETC) of the V, A with the fields are also easily calculated, for free fields. Using (8.1.4) and the ETC of quark fields, one finds,

$$\begin{aligned} \delta(x^0 - y^0)[V^0_{qq'}(x), q''(y)] &= -\,\delta(x-y)\delta_{qq''} q'(x), \\ \delta(x^0 - y^0)[A^0_{qq'}(x), q''(y)] &= -\,\delta(x-y)\delta_{qq''}\gamma_5 q'(x), \text{ etc.} \end{aligned} \tag{8.1.7}$$

The V, A commute with gluon and ghost fields. The equal time commutation relations of the V, A among themselves (again for free fields) are best described for three flavours, $f = 1, 2, 3 = u, d, s$

by introducing the Gell-Mann λ^a matrices in flavour space (for two quarks u, d, replace the λ^a by the τ^a of Pauli, and the f_{abc} by ϵ_{abc} in (8.1.9) below). So we let

$$V_a^\mu(x) = \sum_{ff'} \bar{q}_f(x)\lambda^a_{ff'}\gamma^\mu q_{f'}(x), \quad A_a^\mu(x) = \sum_{ff'} \bar{q}_f(x)\lambda^a_{ff'}\gamma^\mu\gamma_5 q_{f'}(x), \tag{8.1.8}$$

and we then obtain the commutation relations

$$\begin{aligned}
\delta(x^0-y^0)[V_a^0(x), V_b^\mu(y)] &= 2\mathrm{i}\delta(x-y)\sum f_{abc}V_c^\mu(x),\\
\delta(x^0-y^0)[V_a^0(x), A_b^\mu(y)] &= 2\mathrm{i}\delta(x-y)\sum f_{abc}A_c^\mu(x),\\
\delta(x^0-y^0)[A_a^0(x), A_b^\mu(y)] &= 2\mathrm{i}\delta(x-y)\sum f_{abc}V_c^\mu(x), \text{ etc.}
\end{aligned} \tag{8.1.9}$$

Equations (8.1.7) and (8.1.9) have been derived only for free fields. However, they involve short distances; therefore in QCD, and because of asymptotic freedom, they will hold as they stand even in the presence of interactions.

Equal time commutation relations of conserved or quasi-conserved currents with the Hamiltonian may also be easily obtained. If J^μ is conserved, then the corresponding charge

$$Q_J(t) = \int \mathrm{d}^3\mathbf{x}\, J^0(t,\mathbf{x}), \quad t = x_0,$$

commutes with $\mathcal{H}$:

$$[Q_J(t), \mathcal{H}(t,\mathbf{y})] = 0.$$

Here $\mathcal{H}$ is the *Hamiltonian density*, $\mathcal{H} = \Theta^{00}$. If J is quasi-conserved, let $\mathcal{H}_m$ be the mass term in $\mathcal{H}$,

$$\mathcal{H}_m = \sum_q m_q\bar{q}q.$$

Then,

$$[Q_J(t), \mathcal{H}_m(t,\mathbf{y})] = \mathrm{i}\partial_\mu J^\mu(t,\mathbf{y}). \tag{8.1.10}$$

Of course, Q_J still commutes with the rest of $\mathcal{H}$.

8.2 Mass terms and invariances: chiral invariance

In this section we will consider quarks with masses $m \ll \Lambda$ (with Λ the QCD mass parameter), to be referred to as *light quarks*.[33] Because the only dimensional parameter intrinsic to QCD is, we believe, Λ, we may expect that to some approximation we may neglect the masses of such quarks, which will yield only contributions of order m^2/Λ^2.

[33] It is, of course, unclear whether the meaningful parameter in this respect is Λ, connected to the strong interaction coupling by

$$\alpha_s(t) = \frac{12\pi}{(33-2n_f)\log t/\Lambda^2},$$

or Λ_0 defined by $\alpha_s(\Lambda_0) \approx 1$. From considerations of chiral dynamics (see later), it would appear that the scale for smallness of quark masses is $4\pi f_\pi \sim 1$ GeV, where f_π is the pion decay constant; but even if we accept this, it is not obvious at which scale m has to be computed. We will see that m_u, $m_d \sim 4$ to 10 MeV so there is little doubt that u, d quarks should be classed as "light" with any reasonable definition; but the situation is less definite for the s quark with $m_s \sim 180$ MeV.

To study this, we consider the QCD Lagrangian,

$$\mathcal{L} = -\sum_{l=1}^{n} m_l \bar{q}_l q_l + \mathrm{i}\sum_{l=1}^{n} \bar{q}_l \not{D} q_l - \tfrac{1}{4}(D \times B)^2 + \text{gauge fixing} + \text{ghost terms}. \tag{8.2.1}$$

The sum now runs only over *light* quarks; the presence of heavy quarks will have no practical effect in what follows and consequently we neglect them. We may then split the quark fields into left-handed and right-handed components:

$$q_l = q_{L,l} + q_{R,l}; \quad q_{L,l} \equiv q_{-,l} = \frac{1-\gamma_5}{2} q_l, \quad q_{R,l} \equiv q_{+,l} = \frac{1+\gamma_5}{2} q_l.$$

In terms of these, the quark part of the Lagrangian may be written as

$$\mathcal{L} = -\sum_{l=1}^{n} m_l \left(\bar{q}_{R,l} q_{L,l} + \bar{q}_{L,l} q_{R,l}\right) + \mathrm{i}\sum_{l=1}^{n} \left(\bar{q}_{L,l} \not{D} q_{L,l} + \bar{q}_{R,l} \not{D} q_{R,l}\right) + \cdots .$$

We then consider the set of transformations $W^{\pm}$ (left-handed times right-handed) given by the independent transformations of the $q_{R,l}$, $q_{L,l}$:

$$q_{R,l} \to \sum_{l'} W^{+}_{ll'} q_{R,l'}, \quad q_{L,l} \to \sum_{l'} W^{-}_{ll'} q_{L,l'}; \quad W^{\pm} \text{ unitary}. \tag{8.2.2}$$

Clearly, the only term in $\mathcal{L}$ that is not invariant under all the transformations (8.2.2) is the mass term,

$$\mathcal{M} = \sum_{l=1}^{n} m_l \bar{q}_l q_l = \sum_{l=1}^{n} m_l \left(\bar{q}_{R,l} q_{L,l} + \bar{q}_{L,l} q_{R,l}\right). \tag{8.2.3}$$

When written in this form, the mass term is invariant under the set of transformations $[U(1)]^n$,

$$q_l \to \mathrm{e}^{\mathrm{i}\theta_l} q_l, \tag{8.2.4}$$

but this would not have been the case if we had allowed for nondiagonal terms in the mass matrix. To resolve this question of which are the general invariance properties of a mass term, we will prove two theorems.[34]

THEOREM 1. *Any general mass matrix can be written in the form (8.2.3) by appropriate redefinition of the quark fields. Moreover, we may assume that $m \geq 0$. Thus, (8.2.3) is actually the most general mass term possible.*

For the proof we consider that the most general mass term compatible with hermiticity is

$$\mathcal{M}' = \sum_{ll'} \left\{\bar{q}_{L,l} M_{ll'} q_{R,l'} + \bar{q}_{R,l} M^{*}_{ll'} q_{L,l'}\right\}. \tag{8.2.5}$$

Let us temporarily denote matrices in flavour space by putting a tilde under them. If $\underset{\sim}{M}$ is the matrix with components $M_{ll'}$, then the well-known polar decomposition, valid for any matrix, allows us to write

$$\underset{\sim}{M} = \underset{\sim}{m}\underset{\sim}{U},$$

[34]The theorems are valid for *any* quark mass matrix, i.e., also including heavy flavours.

where $\underset{\sim}{m}$ is positive-semidefinite, so all its eigenvalues are ≥ 0, and $\underset{\sim}{U}$ is unitary. Eq. (8.2.5) may then be cast in the form

$$\mathcal{M}' = \sum_{ll'} \left\{ \bar{q}_{L,l} m_{ll'} q'_{R,l'} + \bar{q'}_{R,l} m_{ll'} q_{L,l'} \right\}, \quad q'_{R,l} = \sum_{l'} U_{ll'} q_{R,l'}, \tag{8.2.6}$$

and we have used the fact that $\underset{\sim}{m}$ is Hermitian. Define $q' = q_L + q'_R$; because $\bar{q}_R q_R = \bar{q}_L q_L = 0$, (8.2.6) becomes, in terms of q',

$$\mathcal{M}' = \sum \bar{q}'_l m_{ll'} q'_{l'}.$$

It then suffices to transform q' by the matrix that diagonalizes $\underset{\sim}{m}$ to obtain (8.2.3) with positive m_l. The term $\bar{q}\not{D}q$ in the Lagrangian is left invariant by all these transformations, so the theorem is proved.

THEOREM 2. *If all the m_l are nonzero and different, then the only invariance left is the $[U(1)]^n$ of (8.2.4).*

Let us consider the $\underset{\sim}{W}_{\pm}$ of (8.2.2), and assume that $\underset{\sim}{W}_{+} = \underset{\sim}{W}_{-} \equiv \underset{\sim}{W}$; to show that this must actually be the case is left as an exercise. The condition of invariance of $\mathcal{M}$ yields the relation

$$\underset{\sim}{W}^{\dagger} \underset{\sim}{m} \underset{\sim}{W} = \underset{\sim}{m}, \quad \text{i.e.,} \quad [\underset{\sim}{m}, \underset{\sim}{W}] = 0. \tag{8.2.7}$$

It is known that any $n \times n$ diagonal matrix can be written as $\sum_{k=0}^{n-1} c_k \underset{\sim}{m}^k$ if, as occurs in our case, all the eigenvalues of $\underset{\sim}{m}$ are different and nonzero. Because of (8.2.7), it then follows that $\underset{\sim}{W}$ commutes with all diagonal matrices, and hence it must itself be diagonal: because it is also unitary, it consists of diagonal phases, i.e., it may be written as a product of transformations (8.2.4), as was to be proved. We leave it to the reader to check that the conserved quantity corresponding to the $U(1)$ that acts on flavour q_f is the corresponding flavour number.

In the preceding theorems, we have not worried whether the masses m were bare, running or invariant masses. This is because, in the $\overline{\text{MS}}$ scheme, the mass matrix becomes renormalized as a whole:

$$\underset{\sim}{M} = Z_m^{-1} \underset{\sim}{M}_u,$$

where Z_m is a *number*. The proof of this last property is easy: all we have to do is to repeat the standard renormalization of the quark propagator, allowing for the matrix character of M, Z_m. We find, for the divergent part and in an arbitrary covariant gauge, with gauge parameter ξ,

$$\begin{aligned} \underset{\sim}{S}_R^{\xi}(p) = & \frac{\mathrm{i}}{\not{p} - \underset{\sim}{M}} + \frac{1}{\not{p} - \underset{\sim}{M}} \Bigg\{ - [\underset{\sim}{\Delta}_F (\not{p} - \underset{\sim}{M}) + (\not{p} - \underset{\sim}{M}) \underset{\sim}{\Delta}_F^{\dagger}] - \delta \underset{\sim}{M} \\ & - (1-\xi)(\not{p} - \underset{\sim}{M}) N_\epsilon C_F \frac{g^2}{16\pi^2} + 3 N_\epsilon C_F \frac{g^2}{16\pi^2} \underset{\sim}{M} \Bigg\} \frac{\mathrm{i}}{\not{p} - \underset{\sim}{M}}, \end{aligned}$$

and we have defined

$$\underset{\sim}{M} = \underset{\sim}{M}_u + \delta \underset{\sim}{M}, \quad \underset{\sim}{Z}_F = 1 - \underset{\sim}{\Delta}_F.$$

The renormalization conditions then yield

$$\begin{aligned} \underset{\sim}{\Delta}_F{}^{\dagger} + \underset{\sim}{\Delta}_F &= -(1-\xi) N_\epsilon C_F \frac{g^2}{16\pi^2} = \text{diagonal}, \\ [\underset{\sim}{\Delta}_F, \underset{\sim}{M}] &= 0, \quad [\underset{\sim}{M}, \delta \underset{\sim}{M}] = 0, \\ \delta \underset{\sim}{M} &= 3 N_\epsilon C_F \frac{g^2}{16\pi^2} \underset{\sim}{M}. \end{aligned}$$

Thus, the set of fermion fields and the mass matrix get renormalized as a whole:

$$\underset{\sim}{Z}_F^{-1} = 1 + (1-\xi)N_\epsilon C_F \frac{g^2}{16\pi^2}, \quad \underset{\sim}{Z}_m = 1 - 3N_\epsilon C_F \frac{g^2}{16\pi^2}, \tag{8.2.8a}$$

i.e.,

$$\underset{\sim}{Z}_F = Z_F\, 1, \quad \underset{\sim}{Z}_m = Z_m\, 1. \tag{8.2.8b}$$

We have proved this to lowest order, but the renormalization group equations guarantee the result to leading order in α_s.

This result can be understood in yet another way. The invariance of $\mathcal{L}$ under the transformations (8.2.4) implies that we may choose the counterterms to satisfy the same invariance, so the mass matrix will remain diagonal after renormalization. In fact, this proof shows that in mass independent renormalization schemes (such as the $\overline{\text{MS}}$), Eqs. (8.2.8b) actually hold to all orders.

The results we have derived show that, if all the m_i are different and nonvanishing,[35] the only global symmetries of the Lagrangian are those associated with flavour conservation, (8.2.4). As stated above, however, under certain conditions it may be a good approximation to neglect the m_l. In this case, all the transformations of Eq. (8.2.2) become symmetries of the Lagrangian. The measure of the accuracy of the symmetry is given by, for example, the divergences of the corresponding currents or, equivalently, the conservation of the charges. This has been discussed in Sect. 8.1, and we now present some extra details.

Let us parametrize the W as $\exp\{(\mathrm{i}/2)\sum \theta_a \lambda^a\}$, where the λ are the Gell-Mann matrices. (We consider the case $n = 3$; for $n = 2$, replace the λ by the τ of Pauli). We may denote by $U_\pm(\theta)$ the operators that implement (8.2.2):

$$U_\pm(\theta)\frac{1\pm\gamma_5}{2}q_l U_\pm^{-1}(\theta) = \sum_{l'}\left(\mathrm{e}^{(\mathrm{i}/2)\sum\theta_a\lambda^a}\right)_{ll'}\frac{1\pm\gamma_5}{2}q_{l'}. \tag{8.2.9}$$

For infinitesimal θ, we write

$$U_\pm(\theta) \simeq 1 - \frac{\mathrm{i}}{2}\sum \theta_a L_\pm^a, \quad (L_\pm^a)^\dagger = L_\pm^a,$$

so that (8.2.9) yields

$$[L_\pm^a, q_{\pm,l}(x)] = -\sum_{l'}\lambda_{ll'}^a q_{\pm,l'}(x), \quad q_{\pm,l} \equiv \frac{1\pm\gamma_5}{2}q_l. \tag{8.2.10}$$

Because the $U_\pm$ leave the interaction part of the Lagrangian invariant, and since QCD is a free field theory at zero distance, we may solve (8.2.10) using free-field commutation relations. The result is

$$L_\pm^a(t) =: \int \mathrm{d}^3\mathbf{x}\sum_{ll'}\bar{q}_{\pm,l}(x)\gamma^0\lambda_{ll'}^a q_{\pm,l'}(x) :, \quad t = x^0. \tag{8.2.11}$$

These will be recognized as the charges corresponding to the currents

$$J_\pm^{a\mu}(x) =: \sum_{ll'}\bar{q}_l(x)\lambda_{ll'}^a\gamma^\mu\frac{1\pm\gamma_5}{2}q_{l'}(x) : . \tag{8.2.12}$$

[35] As seems to be the case in nature. As we will see, one finds $m_d/m_u \sim 2$, $m_s/m_d \sim 20$, $m_u \sim 5$ MeV.

If the symmetry is exact, $\partial_\mu J_\pm^{a\mu} = 0$, and a standard calculation shows that the $L_\pm^a(t)$ are actually independent of t. Otherwise, we have to define *equal time* transformations and modify (8.2.9, 10) writing, for example,

$$[L_\pm^a(t), q_{\pm,l}(x)] = -\sum_{l'} \lambda_{ll'}^a q_{\pm,l'}(x), \quad t = x^0. \tag{8.2.13}$$

The set of transformations

$$U_\pm(\theta, t) = \exp\left\{-\frac{\mathrm{i}}{2}\sum L_\pm^a(t)\theta_a\right\},$$

builds up the group of *chiral transformations* generated by the currents (8.2.12). In our present case we find the chiral $SU_F^+(3) \times SU_F^-(3)$ group. Its generators may be rearranged in terms of the set of vector and axial currents $V_{ll'}^\mu(x)$, $A_{ll'}^\mu(x)$ introduced in Sect. 8.2. (Actually, not all diagonal elements are in $SU_F^+(3) \times SU_F^-(3)$, but they are in the group $U_F^+(3) \times U_F^-(3)$). An important subgroup of $SU_F^+(3) \times SU_F^-(3)$ is that generated by the vector currents, which is simply the flavour $SU(3)$ of Gell-Mann and Ne'eman.

The exactness of the symmetries is related to the time independence of the charges $L_\pm$, which in turn is linked to the divergence of the currents. These divergences are proportional to differences of masses, $m_l - m_{l'}$ for the vector, and sums $m_l + m_{l'}$ for the axial currents. Thus, we conjecture that $SU_F(3)$ will be good to the extent that $|m_l - m_{l'}|^2 \ll \Lambda^2$ and chiral $SU_F^+(3) \times SU_F^-(3)$ to the extent that $m_l \ll \Lambda$. In the real world, it appears that mass differences are of the same order as the masses themselves, so we expect chiral symmetries to be almost as good as flavour symmetries. This seems to be the case experimentally.

8.3 Wigner–Weyl and Nambu–Goldstone realizations of symmetries

The fact that flavour and chiral $SU(3)$ (or $SU(2)$) appear to be valid to similar orders of approximation does not mean that these symmetries are realized in the same manner. In fact, we will see that there are good theoretical and experimental reasons why they are very different.

Let us begin by introducing the charges with definite parity,

$$Q^a = L_+^a + L_-^a, \quad Q_5^a = L_+^a - L_-^a. \tag{8.3.1}$$

Their *equal time commutation relations* are

$$\begin{aligned}
[Q^a(t), Q^b(t)] &= 2\mathrm{i}\sum f^{abc} Q^c(t), \\
[Q^a(t), Q_5^b(t)] &= 2\mathrm{i}\sum f^{abc} Q_5^c(t), \\
[Q_5^a(t), Q_5^b(t)] &= 2\mathrm{i}\sum f^{abc} Q^c(t).
\end{aligned} \tag{8.3.2}$$

The set Q^a builds the group $SU_F(3)$. In the limit $m_l \to 0$, all Q, Q_5 are t-independent and

$$[Q^a, \mathcal{L}] = [Q_5^a, \mathcal{L}] = 0. \tag{8.3.3}$$

The difference between Q^a, Q_5^a, however, lies in the vacuum. In general, given a set of generators L^j of symmetry transformations of $\mathcal{L}$, we have two possibilities:

$$L^j|0\rangle = 0, \tag{8.3.4}$$

which is called a *Wigner–Weyl* symmetry, or

$$L^j|0\rangle \neq 0, \tag{8.3.5}$$

or *Nambu–Goldstone* symmetry. Obviously, we will in general have a mixture of the two symmetries, with some L^i, $i = 1, \ldots, r$, verifying (8.3.4) and the rest, L^k, $k = r+1, \ldots, n$, satisfying (8.3.5). Since the commutator of two operators that annihilate the vacuum also annihilates the vacuum, it follows that the subset of Wigner–Weyl symmetries forms a subgroup.

Two theorems are especially relevant with respect to these questions. The first, due to Coleman (1966), asserts that "the invariance of the vacuum is the invariance of the world", or, in more transparent terms, that the physical states (including bound states) are invariant under the transformations of a Wigner–Weyl group of symmetries. It follows that, if we assumed that chiral symmetry was all of it realized in the Wigner–Weyl mode, we could conclude that the masses of all mesons in a flavour multiplet would be degenerate, up to corrections of order m_q^2/M_h^2, with M_h the (average) hadron mass. This is true of the ω, ρ, K^*, ϕ, but if we include parity doublets this is no longer the case. Thus, for example, there is no scalar meson with a mass anywhere near that of the pion, and the axial vector meson masses are more than half a GeV larger than the masses of ω or ρ. Thus it is strongly suggested that $SU_F(3)$ is a Wigner–Weyl symmetry, but chiral $SU_F^+(3) \times SU_F^-(3)$ contains generators of the Goldstone–Nambu type. We assume, therefore,

$$Q^a(t)|0\rangle = 0, \quad Q_5^a(t)|0\rangle \neq 0. \tag{8.3.6}$$

The second relevant theorem is Goldstone's (1961). It states that, for each generator that fails to annihilate the vacuum, there must exist a massless boson with the quantum numbers of that generator. Therefore, we "understand" the smallness of the masses of the pion or kaon[36] because, in the limit m_u, m_d, $m_s \to 0$, we would also have $\mu \to 0$, $m_K \to 0$. Indeed, we will later show that

$$\mu^2 \sim m_u + m_d, \quad m_K^2 \sim m_{u,d} + m_s. \tag{8.3.7}$$

We will not prove either theorem here, but we note that (8.3.7) affords a quantitative criterion for the validity of chiral symmetries; they hold to corrections of order μ^2/m_ρ^2 for $SU(2)$ and of $m_K^2/m_{K^*}^2$ for $SU(3)$.

We also note that a Nambu–Goldstone realization (Nambu, 1960; Nambu and Jona–Lasinio, 1961a,b) is never possible in perturbation theory. Since the symmetry generators are Wick-ordered products of field operators, it is clear that to all orders of perturbation theory $Q_5^a(t)|0\rangle = 0$. This means that the physical vacuum is different from the vacuum of perturbation theory in the limit $m \to 0$. We emphasize this by writing $|0\rangle$ for the perturbation-theoretic vacuum and $|\text{vac}\rangle$ for the physical one when there is danger of confusion. So we rewrite (8.3.6) as

$$Q^a(t)|\text{vac}\rangle = 0, \quad Q_5^a(t)|\text{vac}\rangle \neq 0. \tag{8.3.8}$$

It is not difficult to see how this may come about in QCD. Let $a_P^\dagger(\mathbf{k})$ be the creation operator for a particle P with three-momentum $\mathbf{k}$. The states

$$a_P^\dagger(\mathbf{0}) \overbrace{\ldots}^{n} a_P^\dagger(\mathbf{0})|0\rangle = |n\rangle$$

[36]The particles with zero flavour quantum numbers present problems of their own (the so-called $U(1)$ problem) that will be discussed later.

are all degenerate in the limit $m_P \to 0$. Therefore, the physical vacuum will be, in this limit,

$$|\text{vac}\rangle = \sum_n c_n |n\rangle,$$

i.e., it will contain zero-frequency massless particles (Bogoliubov's model). In QCD we have the gluons which are massless, and so will the light quarks be, to a good approximation, in the chiral limit.

8.4. PCAC, π^+ decay, the pion propagator and light quark mass ratios

8.4.1. The weak axial current and π^+ decay

We are now in a position to obtain quantitative results on the masses of the light quarks, relating them to the masses of pions and kaons. To do so, consider the current

$$A^\mu_{ud}(x) = \bar{u}\gamma^\mu\gamma_5 d(x),$$

and its divergence

$$\partial_\mu A^\mu_{ud}(x) = \mathrm{i}(m_u + m_d)\bar{u}\gamma_5 d(x).$$

The latter has the quantum numbers of the π^+, so we can use it as a composite pion field operator. We thus write

$$\partial_\mu A^\mu_{ud}(x) = \sqrt{2} f_\pi \mu^2 \phi_\pi(x). \tag{8.4.1a}$$

The factors in (8.4.1a) are chosen for historical reasons (our convention is not universal, however). $\phi_\pi(x)$ is the pion field normalized to

$$\langle 0|\phi_\pi(x)|\pi(p)\rangle = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{-\mathrm{i}p\cdot x}, \tag{8.4.1b}$$

with $|\pi(p)\rangle$ the state of a pion with momentum p. The constant f_π may be obtained from experiment as follows. Consider the weak decay $\pi^+ \to \mu^+\nu$. With the effective Fermi Lagrangian for weak interactions (see, e.g., Marshak, Riazzudin and Ryan, 1969)

$$\mathcal{L}^{\text{Fermi}}_{\text{int}} = \frac{G_F}{\sqrt{2}} \bar{\mu}\gamma_\lambda(1-\gamma_5)\nu_\mu \bar{u}\gamma^\lambda(1-\gamma_5)d + \cdots,$$

we find the decay amplitude

$$F(\pi \to \mu\nu) = \frac{2\pi G_F}{\sqrt{2}} \bar{u}_{(\nu)}(p_2)\gamma_\lambda(1-\gamma_5)v_{(\mu)}(p_1,\sigma)\langle 0|A^\lambda_{ud}(0)|\pi(p)\rangle. \tag{8.4.2a}$$

Now, on invariance grounds,

$$\langle 0|A^\lambda_{ud}(0)|\pi(p)\rangle = \mathrm{i}p^\lambda C_\pi; \tag{8.4.2b}$$

contracting with p_μ we find $C_\pi = f_\pi\sqrt{2}/(2\pi)^{3/2}$ and hence

$$\mu^2 C_\pi = \langle 0|\partial_\lambda A^\lambda_{ud}(0)|\pi(p)\rangle = \sqrt{2} f_\pi \mu^2 \frac{1}{(2\pi)^{3/2}}. \tag{8.4.2c}$$

Therefore

$$\tau(\pi \to \mu\nu) = \frac{4\pi}{(1 - m_\mu^2/\mu^2)^2 G_F^2 f_\pi^2 \mu m_\mu^2},$$

and we obtain f_π from the decay rate. An accurate evaluation of f_π requires taking into account the Cabibbo rotation and electromagnetic radiative corrections; one gets $f_\pi \simeq 93$ MeV. A remarkable fact is that, if we repeat the analysis for kaons,

$$\partial_\mu A^\mu_{us}(x) = \sqrt{2} f_K m_K^2 \phi_K(x), \tag{8.4.3}$$

we find that, experimentally, $f_K \approx 110$ MeV: it agrees with f_π to 20%. Actually, this is to be expected because, in the limit $m_{u,d,s} \to 0$, there is no difference between pions and kaons, and we would find strict equality. That f_π, f_K are so similar in the real world is a good point in favour of $SU_F(3)$ chiral ideas.

The relations (8.4.1) and (8.4.3) are at times called PCAC[37] but this is not very meaningful, for these equations are really *identities*. One may use any pion field operator one wishes, in particular (8.4.1), provided that it has the right quantum numbers and its vacuum-one pion matrix element is not zero. The nontrivial part of PCAC will be described below.

8.4.2. The pion propagator; quark mass ratios

The next step is to consider the two-point function, or *correlator* (we drop the ud index from A_{ud})

$$\Pi^{\mu\nu}(q) = \mathrm{i} \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}q\cdot x} \langle \mathrm{T} A^\mu(x) A^\nu(0)^\dagger \rangle_{\mathrm{vac}},$$

and contract with q_μ, q_ν:

$$\begin{aligned}
q_\nu q_\mu \Pi^{\mu\nu}(q) &= -\, q_\nu \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}q\cdot x} \partial_\mu \langle \mathrm{T}\, A^\mu(x) A^\nu(0)^\dagger \rangle_{\mathrm{vac}} \\
&= -\, q_\nu \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}q\cdot x} \delta(x^0) \langle [A^0(x), A^\nu(0)^\dagger] \rangle_{\mathrm{vac}} \\
&\quad -\, q_\nu \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}q\cdot x} \langle \mathrm{T}\, \partial \cdot A(x) A^\nu(0)^\dagger \rangle_{\mathrm{vac}} \\
&= 2\mathrm{i} \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}q\cdot x} \delta(x^0) \langle [A^0(x), \partial \cdot A(0)^\dagger] \rangle_{\mathrm{vac}} \\
&\quad +\, \mathrm{i} \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}q\cdot x} \langle \mathrm{T}\, \partial \cdot A(x) \partial \cdot A(0)^\dagger \rangle_{\mathrm{vac}}.
\end{aligned}$$

Using Eqs. (8.4.1, 2) and evaluating the commutator, we find

$$\begin{aligned}
q_\nu q_\mu \Pi^{\mu\nu}(q) = {} & 2(m_u + m_d) \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}q\cdot x} \delta(x) \langle \bar{u}(x) u(x) + \bar{d}(x) d(x) \rangle_{\mathrm{vac}} \\
& + 2\mathrm{i} f_\pi^2 \mu^4 \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}q\cdot x} \langle \mathrm{T}\, \phi_\pi(x) \phi_\pi(0)^\dagger \rangle_{\mathrm{vac}},
\end{aligned}$$

[37] Partially conserved axial current. In fact, in the limit $\mu^2 \to 0$, the right hand side of (8.4.1a) vanishes.

or, in the limit $q \to 0$,

$$\begin{aligned} &2(m_u + m_d)\langle : \bar{u}(0)u(0) + \bar{d}(0)d(0) : \rangle_{\rm vac} \\ &= -2{\rm i} f_\pi^2 \mu^4 \int {\rm d}^4x\, {\rm e}^{{\rm i}q\cdot x} \langle {\rm T}\, \phi_\pi(x)\phi_\pi(0)^\dagger \rangle_{\rm vac}\big|_{q\to 0}, \end{aligned}$$

and we have reinstated explicitly the colons of normal ordering. The right hand side of this equality has contributions from the pion pole and from the continuum; by writing a dispersion relation (Cauchy representation) for $\mathit{\Pi}(t)$, defined by

$$\mathit{\Pi}(q^2) = {\rm i} \int {\rm d}^4x\, {\rm e}^{{\rm i}q\cdot x} \langle {\rm T}\, \phi_\pi(x)\phi_\pi(0)^\dagger \rangle_{\rm vac},$$

they can be expressed as[38]

$$\begin{aligned} {\rm i} \int {\rm d}^4x\, {\rm e}^{{\rm i}q\cdot x} \langle {\rm T}\, \phi_\pi(x)\phi_\pi(0)^\dagger \rangle_{\rm vac}\big|_{q\to 0} &= \left\{ \frac{1}{\mu^2 - q^2} + \frac{1}{\pi} \int {\rm d}t\, \frac{{\rm Im}\, \mathit{\Pi}(t)}{t - q^2} \right\}_{q\to 0} \\ &= \frac{1}{\mu^2} + \frac{1}{\pi} \int {\rm d}t \frac{{\rm Im}\, \mathit{\Pi}(t)}{t}. \end{aligned}$$

The order of the limits is essential; we first must take $q \to 0$ and the chiral limit afterwards. In the limit $\mu^2 \to 0$, the first term on the right hand side above *diverges*, and the second remains finite.[39] We then get

$$\begin{aligned} (m_u + m_d)\langle \bar{u}u + \bar{d}d \rangle &= -2 f_\pi^2 \mu^2 \left\{ 1 + O(\mu^2) \right\}, \\ \langle \bar{q}q \rangle &\equiv \langle : \bar{q}(0)q(0) : \rangle_{\rm vac}, \quad q = u, d, s, \ldots. \end{aligned} \tag{8.4.4}$$

This is a strong indication that $\langle \bar{q}q \rangle \neq 0$ because, in order to ensure that it vanishes, we would require $f_\pi = 0$ (or very large $O(\mu^4)$ corrections). We also note that we have not distinguished in e.g. (8.4.4), between bare (u) or renormalized (R) quark masses and operators; the distinction is not necessary because m_q and $\langle \bar{q}q \rangle$ acquire opposite renormalization, so that $m_u \langle \bar{q}q \rangle_u = m_R \langle \bar{q}q \rangle_R$.

We may repeat the derivation of (8.4.4) for kaons. We find, to leading order in m_K^2,

$$\begin{aligned} (m_s + m_u)\langle \bar{s}s + \bar{u}u \rangle &\simeq -\, 2 f_{K^+}^2 m_{K^+}^2, \\ (m_s + m_d)\langle \bar{s}s + \bar{d}d \rangle &\simeq -\, 2 f_{K^0}^2 m_{K^0}^2. \end{aligned} \tag{8.4.5}$$

We may assume $f_{K^+} = f_{K^0}$ since, in the limit $m_{u,d}^2 \ll \mathit{\Lambda}^2$ they should be strictly equal. For the same reason, one can take it that the VEVs $\langle \bar{q}q \rangle$ are equal for all light quarks. Under these circumstances, we may eliminate the VEVs and obtain

$$\frac{m_s + m_u}{m_d + m_u} \simeq \frac{f_K^2 m_{K^+}^2}{f_\pi^2 \mu^2}, \quad \frac{m_d - m_u}{m_d + m_u} \simeq \frac{f_K^2 (m_{K^0}^2 - m_{K^+}^2)}{f_\pi^2 \mu^2}.$$

[38] The equation below should have been written with subtractions, to compensate for the growth of $\mathit{\Pi}(q^2)$ for large q^2; but these do not alter the conclusions.

[39] Properly speaking, this is the PCAC limit, for in this limit the axial current is conserved.

A more careful evaluation requires consideration of electromagnetic contributions to the observed π, K masses (Bijnens, 1993; Donoghue, Holsten and Wyler, 1993) and higher order chiral corrections (Kaplan and Manohar, 1986; Bijnens, Prades and de Rafael, 1995).[40] In this way we find

$$\frac{m_s}{m_d} = 18 \pm 5, \quad \frac{m_d}{m_u} = 2.0 \pm 0.4. \tag{8.4.6}$$

If we couple this with the phenomenological estimate (coming from meson and baryon spectroscopy) $m_s - m_d \approx 100$ to 200 MeV, $m_d - m_u \approx 4$ MeV, we obtain the masses (in MeV)

$$\bar{m}_u(Q^2 \sim m_\rho^2) \approx 5, \quad \bar{m}_d(Q^2 \sim m_\rho^2) \approx 9, \quad \bar{m}_s(Q^2 \sim m_\rho^2) \approx 190, \tag{8.4.7}$$

where the symbol $\approx$ here means that a 50% error would not be very surprising.

This method for obtaining light quark masses is admittedly very rough; in the next section we will describe more sophisticated ones.

To conclude this section we make a few comments concerning light quark condensates, $\langle \bar{q}q \rangle$. The fact that these do not vanish implies spontaneous breaking of chiral symmetry because, under $q \to \gamma_5 q$, $\langle \bar{q}q \rangle \to -\langle \bar{q}q \rangle$. One may thus wonder whether chiral symmetry would not be restored in the limit $m_q \to 0$, which would imply

$$\langle \bar{q}q \rangle \underset{m_q \to 0}{\to} 0. \tag{8.4.8}$$

This possibility is discussed for example by Gasser and Leutwyler (1982). The equation (8.4.8) is highly unlikely. If it held, one would expect in particular the ratios,

$$\langle \bar{s}s \rangle : \langle \bar{d}d \rangle : \langle \bar{u}u \rangle \sim 190 : 9 : 5,$$

which runs contrary to all evidence, from hadron spectroscopy to SVZ sum rules which suggest

$$\langle \bar{s}s \rangle \sim \langle \bar{d}d \rangle \sim \langle \bar{u}u \rangle$$

to a few percent. Thus we obtain an extra indication that chiral symmetry is indeed spontaneously broken in QCD.

8.5. Bounds and estimates of light quark masses in terms of the pion and kaon masses

In this section we describe a method for obtaining bounds and estimates of light quark masses. The method was first used (to get rough estimates) by Vainshtein et al. (1978) and further refined by Becchi, Narison, de Rafael and Ynduráin (1981), Gasser and Leutwyler (1982), etc. One starts with the correlator,

$$\begin{aligned} \Psi_{ij}^5(q^2) &= \mathrm{i} \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}q\cdot x} \langle \mathrm{T} \partial \cdot A_{ij}(x) \partial \cdot A_{ij}(0)^\dagger \rangle_{\mathrm{vac}} \\ &= \mathrm{i}(m_i + m_j)^2 \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}q\cdot x} \langle \mathrm{T} J_{ij}^5(x) J_{ij}^5(0)^\dagger \rangle_{\mathrm{vac}}, \end{aligned} \tag{8.5.1}$$

[40] The method originates in the work of Glashow and Weinberg (1968a,b) and Gell-Mann, Oakes and Renner (1968). In QCD, see Weinberg (1978a), Domínguez (1978) and Zepeda (1978). Estimates of the quark masses essentially agreeing with (8.4.6, 7) below had been obtained even before QCD by e.g. Okubo (1969), but nobody knew what to do with them. The first evaluation in the context of QCD is due to Leutwyler (1974).

where $A^\mu_{ij} = \bar q_i \gamma^\mu \gamma_5 q_j$, $J^5_{ij} = \bar q_i \gamma_5 q_j$, $i,j = u,d,s$.

To all orders of perturbation theory, the function

$$F_{ij}(Q^2) = \frac{\partial^2}{\partial (q^2)^2} \Psi^5_{ij}(q^2), \quad Q^2 = -q^2,$$

vanishes as $Q^2 \to \infty$. Hence, we may write a dispersion relation of the form

$$F_{ij}(Q^2) = \frac{2}{\pi} \int_{m_P^2}^\infty \mathrm{d}t \, \frac{\operatorname{Im} \Psi^5_{ij}(t)}{(t+Q^2)^3}, \quad P = \pi,\, K. \tag{8.5.2}$$

For large values of Q^2, t we may calculate $F_{ij}(Q^2)$, $\operatorname{Im}\Psi^5_{ij}(t)$. The calculation has been improved along the years due to increasing precision of the QCD evaluations of these quantities.[41] Here, however, we will consider only leading effects and first order subleading corrections. We then have,

$$\begin{aligned} F_{ij}(Q^2) = & \frac{3}{8\pi^2} \frac{[\bar m_i(Q^2) + \bar m_j(Q^2)]^2}{Q^2} \\ & \times \left\{ 1 + \tfrac{11}{3} \frac{\alpha_s(Q^2)}{\pi} + \frac{m_i^2 + m_j^2 + (m_i - m_j)^2}{Q^2} + \frac{2\pi}{3} \frac{\langle \alpha_s G^2 \rangle}{Q^4} \right. \\ & \left. - \frac{16\pi^2}{3Q^4} \left[\left(m_j - \frac{m_i}{2} \right) \langle \bar q_i q_i \rangle + \left(m_i - \frac{m_j}{2} \right) \langle \bar q_j q_j \rangle \right] \right\} \end{aligned} \tag{8.5.3a}$$

and

$$\operatorname{Im} \Psi^5_{ij}(t) = \frac{3[\bar m_i(t) + \bar m_j(t)]^2}{8\pi} \left\{ \left[1 + \tfrac{17}{3} \frac{\alpha_s(t)}{\pi} \right] t - (m_i - m_j)^2 \right\}. \tag{8.5.3b}$$

The contributions containing the condensates are easily evaluated taking into account the nonperturbative parts of the quark and gluon propagators. The quantities $m_i \langle \bar q_j q_j \rangle$ may be reexpressed in terms of experimentally known quantities, $f_{K,\pi}$, $m_{K,\pi}$ as in (8.4.4, 5). For the case $ij = ud$, which is the one we will consider in more detail, their contribution is negligible, as are the terms of order m^2/Q^2 in Eqs. (8.5.3). We will henceforth neglect these quantities. Because one can write the imaginary part of the spectral function as

$$\operatorname{Im} \Psi^5_{ij}(t) = \tfrac{1}{2} \sum_\Gamma \left| \langle \mathrm{vac} | \partial^\mu A^{ij}_\mu(0) | \Gamma \rangle \right|^2 (2\pi)^4 \delta_4(q - p_\Gamma)$$

it follows that $\operatorname{Im}\Psi^5_{ij}(t) \geq 0$: it is this positivity that will allow us to derive quite general bounds. To obtain tight ones it is important to use the information contained in both Eqs. (8.5.3a,b); to this end, we define the function

$$\begin{aligned} \varphi_{ij}(Q^2) &= F_{ij}(Q^2) - \int_{Q^2}^\infty \mathrm{d}t \, \frac{1}{(t+Q^2)^3} \, \frac{2 \operatorname{Im} \Psi^5_{ij}(t)}{\pi} \\ &= \int_{m_P^2}^{Q^2} \mathrm{d}t \, \frac{1}{(t+Q^2)^3} \, \frac{2 \operatorname{Im} \Psi^5_{ij}(t)}{\pi}. \end{aligned} \tag{8.5.4}$$

[41] Broadhurst (1981) and Chetyrkin et al. (1995) for subleading mass corrections; Becchi, Narison, de Rafael and Ynduráin (1981), Generalis (1990), Sugurladze and Tkachov (1990), Chetyrkin, Gorishnii and Tkachov (1982), Groshny, Kataev, Larin and Sugurladze (1991) and Pascual and de Rafael (1982) for radiative corrections to various terms.

For sufficiently large Q^2 we may use (8.5.3) and integrate the imaginary part to obtain, for $ij = ud$,

$$\varphi_{ud}(Q^2) = \frac{3}{8\pi^2}\left\{\frac{[\bar{m}_u(Q^2)+\bar{m}_d(Q^2)]^2}{Q^2}\left[\tfrac{1}{4}+\left(\tfrac{5}{12}+2\log 2\right)\frac{\alpha_s}{\pi}\right] + \frac{1}{3Q^6}\left[8\pi^2 f_\pi^2\mu^2 + 2\pi\langle\alpha_s G^2\rangle\right]\right\}. \tag{8.5.5a}$$

For the $ij = us, ds$ cases, and neglecting $m_{u,d}/m_s$,

$$\varphi_{us,ds}(Q^2) = \frac{3}{8\pi^2}\left\{\frac{\bar{m}_s^2}{Q^2}\left[\tfrac{1}{4}+\left(\tfrac{5}{12}+2\log 2\right)\frac{\alpha_s}{\pi}\right] - \frac{2\bar{m}_s^4}{Q^4}\left[\tfrac{3}{4}+(6+4\log 2)\frac{\alpha_s}{\pi}\right] + \frac{1}{3Q^6}\left[8\pi^2 f_K^2 m_K^2 + 2\pi\langle\alpha_s G^2\rangle\right]\right\}. \tag{8.5.5b}$$

We can extract the pion (or kaon, as the case may be) pole explicitly from the low energy dispersive integral in (8.5.4) thus getting for e.g., φ_{ud}

$$\varphi_{ud}(Q^2) = \frac{4f_\pi^2\mu^4}{(\mu^2+Q^2)^3} + \int_{t_0}^{Q^2} \mathrm{d}t\, \frac{1}{(t+Q^2)^3}\,\frac{2\,\mathrm{Im}\,\Psi_{ij}^5(t)}{\pi}; \tag{8.5.6}$$

the continuum threshold t_0 is $3\mu^2$ for $ij = ud$ or $(m_K + 2\mu)^2$ for $ij = (u,d)s$. Because of the positivity of $\mathrm{Im}\,\Psi$ this immediately gives bounds on $m_i(Q^2) + m_j(Q^2)$ as soon as $Q^2 \geq Q_0^2$, where Q_0^2 is a momentum large enough for the QCD estimates (8.5.5) to be valid: thus, to leading order,

$$\bar{m}_u(Q_0^2) + \bar{m}_d(Q_0^2) \geq \left\{\frac{2^7\pi^2 f_\pi^2\mu^4}{3}\,\frac{Q_0^2}{(Q_0^2+\mu^2)^3}\right\}^{\frac{1}{2}}; \tag{8.5.7a}$$

for the combination us,

$$\bar{m}_s(Q_0^2) \geq \left\{\frac{2^7\pi^2 f_K^2 m_K^4}{3}\,\frac{Q_0^2}{(Q_0^2+m_K^2)^3}\right\}^{\frac{1}{2}}. \tag{8.5.7b}$$

The bound depends a lot on the value of Q_0^2. We find, for example, the bounds

$$\begin{aligned}\bar{m}_u(1\ \mathrm{GeV}^2) + \bar{m}_d(1\ \mathrm{GeV}^2) &\geq 13\ \mathrm{MeV}, \quad Q_0^2 = 1.75\ \mathrm{GeV}^2,\\ \bar{m}_u(1\ \mathrm{GeV}^2) + \bar{m}_d(1\ \mathrm{GeV}^2) &\geq 7\ \mathrm{MeV}, \quad Q_0^2 = 3.5\ \mathrm{GeV}^2\end{aligned} \tag{8.5.8a}$$

and

$$\begin{aligned}\bar{m}_s(1\ \mathrm{GeV}^2) &\geq 245\ \mathrm{MeV}, \quad Q_0^2 = 1.75\ \mathrm{GeV}^2,\\ \bar{m}_s(1\ \mathrm{GeV}^2) &\geq 150\ \mathrm{MeV}, \quad Q_0^2 = 3.5\ \mathrm{GeV}^2.\end{aligned} \tag{8.5.8b}$$

As is customary, we have translated the bounds (as we will also do for the estimates later on) to bounds on the running masses defined at 1 GeV. To do so, we have used the three loop expression for the running quark masses,

$$m(t) = \hat{m}\left(\tfrac{1}{2}\log t/\Lambda^2\right)^{-d_m}\left[1 - d_1\frac{\log\log t/\Lambda^2}{\log t/\Lambda^2} + d_2\frac{1}{\log t/\Lambda^2}\right];$$

$$d_m = \frac{4}{\beta_0},\ d_1 = 8\,\frac{51-\frac{19}{3}n_f}{\beta_0^3},\ d_2 = \frac{8}{\beta_0^3}\Big[\left(\tfrac{101}{12}-\tfrac{5}{18}n_f\right)\beta_0 - 51 + \tfrac{19}{3}n_f\Big],$$

and also the three loop running coupling constant,

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 L}\left\{1 - \frac{\beta_1 \log L}{\beta_0^2 L} + \frac{\beta_1^2 \log^2 L - \beta_1^2 \log L + \beta_2\beta_0 - \beta_1^2}{\beta_0^4 L^2}\right\};$$
$$L = \log \frac{Q^2}{\Lambda^2}.$$

Here

$$\beta_0 = 11 - \tfrac{2}{3}n_f, \quad \beta_1 = 102 - \tfrac{38}{3}n_f,$$
$$\beta_2 = \tfrac{2857}{2} - \tfrac{5033}{18}n_f + \tfrac{325}{54}n_f^2.$$

The value for the QCD parameter that we use is

$$\Lambda(3 \text{ loop}, n_f = 3) = 340 \pm 120 \text{ MeV};$$

see for example Ynduráin (1999) or the more recent Particle Data Tables values, that essentially agree with this.

The bounds can be stabilized somewhat by considering derivatives of F_{ij}^5, but (8.5.8) do not change much.

To get *estimates* for the masses, a model is necessary for the low energy piece of the dispersive integral (8.5.6). At very low energy, one can calculate $\mathrm{Im}\,\Psi^5$ using chiral perturbation theory (see for example Pagels and Zepeda, 1972; Gasser and Leutwyler, 1982); the contribution is minute. The important region is that where the quasi-two body channels are open, the $(\rho, \omega) - \pi$ channels for the ud case. This is expected to be dominated by the π' resonance, with a mass of 1.2 GeV. One can take the residue of the resonance as a free parameter, and fit the QCD expression (8.5.5). This is the procedure followed by Narison and de Rafael (1981), Hubschmid and Mallik (1981), Gasser and Leutwyler (1982), Kataev, Krasnikov and Pivovarov (1983), Domínguez and de Rafael (1987), Chetyrkin, Pirjol and Schilcher (1997), etc. The errors one finds in the literature are many times *overoptimistic* because they do not take into account the important matter of the value Q_0^2 at which the perturbative QCD evaluation is supposed to be valid (Ynduráin, 1998). Now, as is clear from Eq. (8.5.5), the radiative corrections feature a large coefficient, so it is difficult to estimate reliably a figure for Q_0^2. Both bounds (as shown above) and estimates will depend on this. As reasonably safe estimates we may quote the values

$$\begin{aligned} \bar{m}_u(Q^2 = 1 \text{ GeV}^2) &= 4.2 \pm 2 \text{ MeV}, \\ \bar{m}_d(Q^2 = 1 \text{ GeV}^2) &= 8.9 \pm 4.3 \text{ MeV}, \\ \bar{m}_s(Q^2 = 1 \text{ GeV}^2) &= 200 \pm 50 \text{ MeV}, \end{aligned} \tag{8.5.9a}$$

and we have, to reduce the errors a bit, taken also into account the chiral theory estimates of the mass ratios given in the previous section, Eq. (8.4.6).

For the s quark, independent estimates (Chen et al., 2001, Narison, 1995) following from

$$\tau \to \nu_\tau + \text{ strange particles}, \quad e^+e^- \to \text{ strange particles}$$

give slightly smaller numbers. Taking them into account we may write

$$\bar{m}_s(Q^2 = 1 \text{ GeV}^2) = 183 \pm 30 \text{ MeV}\,. \tag{8.5.9b}$$

8.6. The triangle anomaly; π^0 decay. The gluon anomaly. The $U(1)$ problem

8.6.3. The triangle anomaly and the π^0 decay

Historically, one of the first motivations for the colour degree of freedom came from the study of the decay $\pi^0 \to \gamma\gamma$, which we now consider in some detail.

The amplitude for the process $\pi^0 \to \gamma\gamma$ may be written, using the reduction formulas, as

$$\begin{aligned}\langle\gamma(k_1,\lambda_1),\gamma(k_2,\lambda_2)|S|\pi^0(q)\rangle &= \frac{-\mathrm{i}e^2}{(2\pi)^{9/2}}\,\epsilon^*_\mu(k_1,\lambda_1)\epsilon^*_\nu(k_2,\lambda_2)\\ \times\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{d}^4z\,\mathrm{e}^{\mathrm{i}(x_1\cdot k_1+x_2\cdot k_2-z\cdot q)}&(\Box_z+\mu^2)\langle\mathrm{T}J^\mu_{\mathrm{em}}(x_1)J^\nu_{\mathrm{em}}(x_2)\phi_{\pi^0}(z)\rangle_0,\end{aligned} \tag{8.6.1}$$

and we have used the relation $\Box A^\mu_{\mathrm{ph}}(x) = J^\mu_{\mathrm{em}}(x)$, with A^μ_{ph} the photon field. We leave it as an exercise for the reader to check this, as well as to verify that, in our particular case, one can replace

$$\Box_{x_1}\Box_{x_2}\mathrm{T}\{A^\mu_{\mathrm{ph}}(x_1)A^\nu_{\mathrm{ph}}(x_2)\phi_{\pi^0}(z)\} \to \mathrm{T}\{(\Box A^\mu_{\mathrm{ph}}(x_1))(\Box A^\nu_{\mathrm{ph}}(x_2))\phi_{\pi^0}(z)\},$$

i.e., that potential delta function terms that appear when the derivatives in the d'Alembertians act on the theta functions $\theta(x_1 - z), \ldots$ implicit in the T-product make no contribution. Separating off the delta of four-momentum conservation, we then find

$$F\left(\pi^0 \to \gamma(k_1,\lambda_1),\gamma(k_2,\lambda_2)\right) = \frac{e^2(q^2-\mu^2)}{\sqrt{2\pi}}\epsilon^*_\mu(k_1,\lambda_1)\epsilon^*_\nu(k_2,\lambda_2)F^{\mu\nu}(k_1,k_2), \tag{8.6.2a}$$

$q = k_1 + k_2$, where we have defined the VEV

$$F^{\mu\nu}(k_1,k_2) = \int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{e}^{\mathrm{i}(x\cdot k_1+y\cdot k_2)}\langle\mathrm{T}J^\mu(x)J^\nu(y)\phi(0)\rangle_0. \tag{8.6.2b}$$

We henceforth suppress the indices "em" and "π^0" in J and ϕ respectively.

We next use the equation (8.4.3), generalized to include the π^0:

$$\begin{aligned}\partial_\mu A^\mu_0(x) &= \sqrt{2}f_\pi\mu^2\phi(x), \quad \phi \equiv \phi_{\pi^0},\\ A^\mu_0(x) &= \frac{1}{\sqrt{2}}\left\{\bar{u}(x)\gamma^\mu\gamma_5 u(x) - \bar{d}(x)\gamma^\mu\gamma_5 d(x)\right\}.\end{aligned} \tag{8.6.3a}$$

It will prove convenient to use, instead of A_0, the current A_3, defined as

$$A^\mu_3(x) = \left\{\bar{u}(x)\gamma^\mu\gamma_5 u(x) - \bar{d}(x)\gamma^\mu\gamma_5 d(x)\right\}; \tag{8.6.3b}$$

with it, we write

$$\begin{aligned}F^{\mu\nu}(k_1,k_2) &= \frac{1}{f_\pi\mu^2}T^{\mu\nu}(k_1,k_2),\\ T^{\mu\nu}(k_1,k_2) &= \tfrac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{e}^{\mathrm{i}(x\cdot k_1+y\cdot k_2)}\langle\mathrm{T}J^\mu(x)J^\nu(y)\partial\cdot A_3(0)\rangle_0.\end{aligned} \tag{8.6.4}$$

Up to this point, everything has been exact. The next step involves using the PCAC hypothesis in the following form: we assume that $F(\pi \to \gamma\gamma)$ can be approximated by its leading term in the

limit $q \to 0$. On purely kinematic grounds, this is seen to imply that also $k_1, k_2 \to 0$. One may write

$$T^{\mu\nu}(k_1, k_2) = \epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta} \Phi + O(k^3). \tag{8.6.5}$$

The PCAC hypothesis means that we retain only the first term in Eq. (8.6.5). As will be seen presently, this will lead us to a contradiction, the resolution of which will involve introducing the so-called *axial*, or *triangle anomaly*, and will allow us actually to calculate $T^{\mu\nu}$ exactly to all orders of perturbation theory (in the PCAC approximation).

The first step is to consider the quantity

$$R^{\lambda\mu\nu}(k_1, k_2) = \mathrm{i} \int \mathrm{d}^4x \, \mathrm{d}^4y \, \mathrm{e}^{\mathrm{i}(x\cdot k_1 + y\cdot k_2)} \langle \mathrm{T} J^\mu(x) J^\nu(y) A_3^\lambda(0) \rangle_0. \tag{8.6.6}$$

On invariance grounds, we may write the general decomposition,

$$R^{\lambda\mu\nu}(k_1, k_2) = \epsilon^{\mu\nu\lambda\alpha} k_{1\alpha} \Phi_1 + \epsilon^{\mu\nu\lambda\alpha} k_{2\alpha} \Phi_2 + O(k^3), \tag{8.6.7}$$

where the $O(k^3)$ terms are of the form

$$\epsilon^{\mu\lambda\alpha\beta} k_{i\alpha} k_{j\beta} k_{l\lambda} \Phi_{ijl} + \text{ permutations of } i, j, l = 1, 2, 3,$$

and, for quarks with nonzero mass, the Φ are regular as $k_i \to 0$.

The conservation of the e.m. current, $\partial \cdot J = 0$, yields two equations:

$$k_{1\mu} R^{\mu\nu\lambda} = k_{2\nu} R^{\mu\nu\lambda} = 0. \tag{8.6.8}$$

The first implies

$$\Phi_2 = O(k^2); \tag{8.6.9a}$$

the second gives

$$\Phi_1 = O(k^2). \tag{8.6.9b}$$

Now we have, from (8.6.4) and (8.6.6),

$$q_\lambda R^{\lambda\mu\nu}(k_1, k_2) = T^{\mu\nu}(k_1, k_2), \quad \text{i.e., } \Phi = \Phi_2 - \Phi_1, \tag{8.6.10}$$

and hence we find the result of Veltman (1967) and Sutherland (1967),

$$\Phi = O(k^2). \tag{8.6.11}$$

Because the scale for k is μ, this means that Φ should be of order μ^2/M^2, where M is a typical hadronic mass. Thus, we expect that Φ would be vanishing in the chiral limit, and hence very small in the real world. Now, this is in disagreement with experiment, as the decay $\pi^0 \to 2\gamma$ is in no way suppressed; but worse still, (8.6.11) contradicts a direct calculation. In fact, we may use the equations of motion and write

$$\partial_\mu A_3^\mu(x) = 2\mathrm{i} \left\{ m_u \bar{u}(x) \gamma_5 u(x) - m_d \bar{d}(x) \gamma_5 d(x) \right\}. \tag{8.6.12}$$

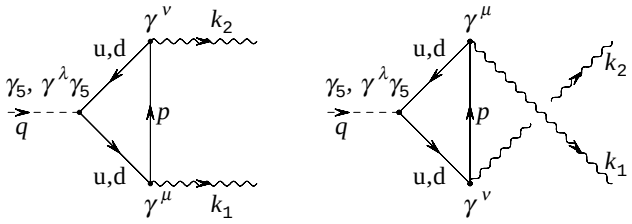

FIG. 8.6.1. Diagrams connected with the anomaly ($\pi^0 \to \gamma\gamma$ decay).

We will calculate first neglecting strong interactions; (8.6.11) should certainly be valid in this approximation. This involves the diagrams of Fig. 8.6.1 with a γ_5 vertex. The result, as first obtained by Steinberger (1949) is, in the limit $k_1, k_2 \to 0$, and defining $\delta_u = 1$, $\delta_d = -1$,

$$\begin{aligned} T^{\mu\nu}(k_1,k_2) &= 2N_c \sum_{f=u,d} \delta_f Q_f^2 m_f \\ &\quad \times \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\operatorname{Tr}\gamma_5(\not{p}+\not{k}_1+m_f)\gamma^\mu(\not{p}+m_f)\gamma^\nu(\not{p}-\not{k}_2+m_f)}{[(p+k_1)^2-m_f^2][(p-k_2)^2-m_f^2](p^2-m_f^2)} \\ &= -\frac{1}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}\left\{3(Q_u^2-Q_d^2)\right\}+O(k^4) \\ &= -\frac{1}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}+O(k^4). \end{aligned}$$

The factor $N_c = 3$ comes from the sum over the three colours of the quarks and the factor 2 from the two diagrams in Fig. 8.6.1 (which in fact contribute equally to the amplitude). We thus find that

$$\Phi = -\frac{1}{4\pi}, \tag{8.6.13}$$

which contradicts (8.6.11). This is the *triangle anomaly* (Bell and Jackiw, 1969; Adler, 1969).

What is wrong here? Clearly, we cannot maintain (8.6.12), which was obtained with free-field equations of motion, $\mathrm{i}\not{\partial} q = m_q q$; we must admit that in the presence of interactions with vector fields (the photon field in our case), Eq. (8.6.12) is no longer valid. To obtain agreement with (8.6.13) we have to write (Adler, 1969)

$$\begin{aligned} \partial_\mu A_3^\mu(x) = {} & 2\mathrm{i}\left\{m_u\bar{u}(x)\gamma_5 u(x) - m_d\bar{d}(x)\gamma_5 d(x)\right\} \\ & + N_c(Q_u^2-Q_d^2)\frac{e^2}{16\pi^2}F_{\mu\nu}(x)\widetilde{F}^{\mu\nu}(x), \end{aligned} \tag{8.6.14}$$

where the *dual* $\widetilde{F}$ has been defined as

$$\widetilde{F}^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \quad F^{\mu\nu} = \partial^\mu A^\nu_{\mathrm{ph}} - \partial^\nu A^\mu_{\mathrm{ph}}.$$

More generally, for fermion fields interacting with vector fields with strength h, we find

$$\partial_\mu \bar{f}\gamma^\mu\gamma_5 f = 2\mathrm{i}m_f\bar{f}\gamma_5 f + \frac{T_F h^2}{8\pi^2}H^{\mu\nu}\widetilde{H}_{\mu\nu}; \tag{8.6.15}$$

$H^{\mu\nu}$ is the vector field strength tensor.

Let us return to the decay $\pi^0 \to 2\gamma$. From (8.6.13) we calculate the amplitude, in the PCAC limit $\mu \sim 0$,

$$F(\pi^0 \to 2\gamma) = \frac{\alpha}{\pi} \frac{\epsilon^{\mu\nu\alpha\beta} k_{1\alpha} k_{2\beta} \epsilon^*_\mu(k_1, \lambda_1) \epsilon^*_\mu(k_2, \lambda_2)}{\sqrt{2\pi}}, \tag{8.6.16}$$

and the decay rate

$$\Gamma(\pi^0 \to 2\gamma) = \left(\frac{\alpha}{\pi}\right)^2 \frac{m_\pi^3}{64\pi f_\pi^2} = 7.25 \times 10^{-6} \text{ MeV},$$

to be compared with the experimental figure,

$$\Gamma_{\text{exp}}(\pi^0 \to 2\gamma) = 7.95 \times 10^{-6} \text{ MeV}.$$

Actually, the sign of the decay amplitude can also be measured (from the Primakoff effect) and it agrees with the theory. It is important to note that, if we had no colour, our result would have decreased by a factor $1/N_c^2$, i.e., it would have been off experiment by a full order of magnitude.

One may wonder what credibility to attach to this calculation: after all, it was made to zero order in α_s. In fact, the calculation is exact to all orders in QCD;[42] the only approximation is the PCAC one $\mu \approx 0$. To show this we will give an alternate derivation of the basic result, Eq. (8.6.13). We then return to (8.6.6). To zero order in α_s,

$$\begin{aligned} R^{\mu\nu\lambda} = & \sum \delta_f Q_f^2 \\ & \times \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\operatorname{Tr} \gamma^\lambda \gamma_5 (\not{p} + \not{k}_1 + m_f) \gamma^\mu (\not{p} + m_f) \gamma^\nu (\not{p} - \not{k}_2 + m_f)}{[(p+k_1)^2 - m_f^2][(p-k_2)^2 - m_f^2](p^2 - m_f^2)} \\ & + \text{crossed term} \end{aligned}$$

(Fig. 8.6.1 with the $\gamma^\lambda \gamma_5$ vertices). More generally, we regulate the integral by working in dimension D, and consider an arbitrary axial triangle with

$$R^{\mu\nu\lambda}_{ijl} = 2 \int \frac{\mathrm{d}^D p}{(2\pi)^D} \operatorname{Tr} \gamma^\lambda \gamma_5 \frac{1}{\not{p} + \not{k}_1 - m_i} \gamma^\mu \frac{1}{\not{p} - m_j} \gamma^\nu \frac{1}{\not{p} - \not{k}_2 - m_l}. \tag{8.6.17}$$

We would like to calculate $q_\lambda R^{\mu\nu\lambda}_{ijl}$. Writing identically

$$(\not{k}_1 + \not{k}_2)\gamma_5 = -(\not{p} - \not{k}_2 - m_l)\gamma_5 + (\not{p} + \not{k}_1 + m_i)\gamma_5 - (m_i + m_l)\gamma_5,$$

we have

$$\begin{aligned} q_\lambda R^{\mu\nu\lambda}_{ijl} = & -2(m_i + m_l) \\ & \times \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{\operatorname{Tr} \gamma_5 (\not{p} + \not{k}_1 + m_i) \gamma^\mu (\not{p} + m_j) \gamma^\nu (\not{p} - \not{k}_2 + m_l)}{[(p+k_1)^2 - m_i^2][(p-k_2)^2 - m_l^2](p^2 - m_j^2)} \\ & + a^{\mu\nu\lambda}_{ijl}, \end{aligned} \tag{8.6.18a}$$

[42] The proof is essentially contained in the original paper of Adler and Bardeen (1969). See also Wilson (1969), Crewther (1972) and Bardeen (1974).

$$a_{ijl}^{\mu\nu\lambda} = -2\int \mathrm{d}\hat{p}\ \mathrm{Tr}\left\{(\not{p} - \not{k}_2 - m_l)\gamma_5 - (\not{p} + \not{k}_1 + m_i)\gamma_5\right\} \times \frac{1}{\not{p} + \not{k}_1 - m_i}\gamma^\mu \frac{1}{\not{p} - m_j}\gamma^\nu \frac{1}{\not{p} - \not{k}_2 - m_l}. \tag{8.6.18b}$$

The first term on the right hand side of (8.6.18a) is what we would have obtained by naive use of the equations of motion, $\partial_\mu \bar{q}_i\gamma^\mu\gamma_5 q_l = \mathrm{i}(m_i + m_l)\bar{q}_i\gamma_5 q_l$; $a_{ijl}^{\mu\nu\lambda}$ is the anomaly. If we accepted the commutation relations $\{\gamma^\mu, \gamma_5\} = 0$ also for dimension $D \neq 4$, we could rewrite it as

$$a_{ijl}^{\mu\nu\lambda} = -2\int \mathrm{d}\hat{p}\left\{ \mathrm{Tr}\,\gamma_5 \frac{1}{\not{p} + \not{k}_1 - m_i}\gamma^\mu \frac{1}{\not{p} - m_l}\gamma^\nu + \mathrm{Tr}\,\gamma_5\gamma^\mu \frac{1}{\not{p} - m_j}\gamma^\mu \frac{1}{\not{p} - \not{k}_2 - m_l}\right\}. \tag{8.6.18c}$$

Then we could conclude that $a_{ijl}^{\mu\nu\lambda}$ vanishes because each of the terms in (8.6.18c) consists of an antisymmetric tensor that depends on a single vector (k_1 for the first term, k_2 for the second) and this is zero. It is thus clear that the nonvanishing of $a_{ijl}^{\mu\nu\lambda}$ is due to the fact that it is given by an ultraviolet divergent integral: if it was convergent, one could take $D \to 4$ and $a_{ijl}^{\mu\nu\lambda}$ would vanish. Incidentally, this shows that $a_{ijl}^{\mu\nu\lambda}$ is actually independent of the masses because $(\partial/\partial m)a_{ijl}^{\mu\nu\lambda}$ is convergent, and thus the former argument applies. We may therefore write $a_{ijl}^{\mu\nu\lambda} = a^{\mu\nu}$, where $a^{\mu\nu}$ is obtained by setting all masses to zero. A similar argument shows that $a^{\mu\nu}$ has to be of the form

$$a^{\mu\nu}(k_1, k_2) = a\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}, \quad a = \text{constant}, \tag{8.6.19a}$$

and thus we may obtain a as

$$a\epsilon^{\mu\nu\alpha\beta} = \frac{\partial^2}{\partial k_{1\alpha}\partial k_{2\beta}} a^{\mu\nu}(k_1, k_2)\Big|_{k_i=0}. \tag{8.6.19b}$$

If we could write the formula (8.6.18c) for a, we would immediately conclude from (8.6.19b) that $a = 0$, in contradiction with the Veltman–Sutherland theorem. But this is easily seen to be inconsistent: if we would have shifted variables in (8.6.18c), say $p \to p - \xi k_2$, we would have found a finite but nonzero value, actually ξ-dependent for a, $a = -\xi/2\pi^2$. This shows that the commutation relations[43] $\{\gamma^\mu, \gamma_5\} = 0$ cannot be accepted for $D \neq 0$, for they lead to an undefined value for the anomaly. If, however, we start from (8.6.18b) and refrain from commuting γ_5 and γ^μs,

$$a\epsilon^{\mu\nu\alpha\beta} = -2\int \mathrm{d}\hat{p}\,\mathrm{Tr}\,\gamma_5\left\{\frac{1}{\not{p}}\gamma^\alpha\frac{1}{\not{p}}\gamma^\mu\frac{1}{\not{p}}\gamma^\nu\frac{1}{\not{p}}\gamma^\beta - \frac{1}{\not{p}}\gamma^\mu\frac{1}{\not{p}}\gamma^\nu\frac{1}{\not{p}}\gamma^\beta\frac{1}{\not{p}}\gamma^\alpha\right\}.$$

[43] These commutation relations are actually self-contradictory. For example, using only the commutation relations of the γ_μ, $\mu = 0, \ldots, D-1$ for $D \neq 4$, we have

$$\mathrm{Tr}\,\gamma_5\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\alpha\gamma^\sigma = (6 - D)\,\mathrm{Tr}\,\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma,$$

while, if we allow γ_5 anticommutation, we can obtain

$$\mathrm{Tr}\,\gamma_5\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\alpha\gamma^\sigma = -\,\mathrm{Tr}\,\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\alpha\gamma^\sigma\gamma^\alpha = (D-2)\,\mathrm{Tr}\,\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma,$$

which differs from the former by a term $O(D-4)$. These problems, however, only arise for arrays with an odd number of γ_5 and at least four other gammas.

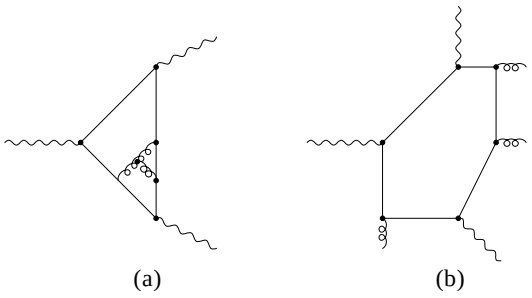

FIG. 8.6.2. (a) A nonanomalous diagram. (b) "Opened" diagram corresponding to (a).

Performing symmetric integration and using only the commutation rules for $D \neq 4$, we obtain an unambiguous result:

$$a\epsilon^{\mu\nu\alpha\beta} = \frac{8(D-1)(4-D)}{D(D+2)} \frac{\mathrm{i}}{16\pi^2} \frac{2}{4-D} \operatorname{Tr} \gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta + O(4-D)$$
$$\underset{D\to 4}{\to} -\frac{1}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}.$$

This is one of the peculiarities of the anomaly: a *finite* Feynman integral whose value depends on the regularization prescription. Fortunately, we may eschew the problem by using the Veltman–Sutherland theorem to conclude that, at any rate, there is a *unique* value of $a^{\mu\nu}$ compatible with gauge invariance for the e.m. current, viz.,

$$a^{\mu\nu}_{ijl} = a^{\mu\nu} = -\frac{1}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}. \tag{8.6.20}$$

We have explicitly checked that our regularization leads to precisely this value; to verify that it also respects gauge invariance is left as as simple exercise.

Before continuing, a few words on the Veltman–Sutherland theorem for zero quark masses are necessary. In this case, the first term on the right hand side of (8.6.18a) is absent: it would appear that we could not maintain our result for the anomaly, Eq. (8.6.20), because this would imply

$$q_\lambda R^{\mu\nu\lambda}_{ijl} = -\frac{1}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta} \neq 0,$$

thus contradicting the Veltman–Sutherland conclusion, $q_\lambda R^{\mu\nu\lambda}_{ijl} = 0$. This is not so. The relation $q_\lambda R^{\mu\nu\lambda}_{ijl} = a^{\mu\nu}$ and the value of $a^{\mu\nu}$ *are* correct. What occurs is that for vanishing masses the functions Φ_i in (8.6.7) possess singularities of the type $1/k_1 \cdot k_2$, singularities coming from the denominators in, for example, Eq. (8.6.17) when $m_i = 0$. Therefore, the Veltman–Sutherland theorem is *not* applicable. This is yet another peculiarity of the anomalous triangle: we have the relation

$$\lim_{m\to 0} q_\lambda R^{\mu\nu\lambda}_{ijl} = 0$$

but, if we begin with $m = 0$,

$$q_\lambda R^{\mu\nu\lambda}_{m\equiv 0} = a^{\mu\nu} \neq 0.$$

Let us return to our original discussion, in particular for $m \neq 0$. The present method shows how one can prove that the result does not get renormalized. The Veltman–Sutherland theorem is exact; so we have actually shown that it is sufficient to prove that (8.6.20) is not altered by higher orders in α_s. Now, consider a typical higher order contribution (Fig. 8.6.2a). It may be written as an integral over the gluon momenta and an integral over the quark momenta. But for the latter, the triangle has become an hexagon (Fig. 8.6.2b) for which the quark integral is convergent and here the limit $D \to 4$ may be taken: it vanishes identically. In addition, the above arguments have shown that the anomaly is in fact related to the large momentum behaviour of the theory and thus we expect that the exactness of (8.6.13) will not be spoiled by nonperturbative effects. We will not make the proof more precise, but refer to the literature.[44]

8.6.2. The $U(1)$ problem and the gluon anomaly

In the previous section, we discussed the triangle anomaly in connection with the decay $\pi^0 \to \gamma\gamma$. As remarked there, the anomaly is not restricted to photons; in particular, we have a gluon anomaly. Although this lies outside the scope of the present review, we will say a few words on the subject. Defining the current

$$A_0^\mu = \sum_{f=1}^{n} \bar{q}_f \gamma^\mu \gamma_5 q_f, \tag{8.6.23}$$

we find that it has an anomaly

$$\partial^\mu A_0^\mu = \mathrm{i} \sum_{f=1}^{n} 2 m_f \bar{q}_f \gamma_5 q_f + \frac{n g^2}{16\pi^2} \widetilde{G} G, \tag{8.6.24}$$

where

$$\widetilde{G}_a^{\mu\nu} \equiv \tfrac{1}{2} \epsilon^{\mu\nu\alpha\beta} G_{a\alpha\beta}, \quad \widetilde{G} G = \sum_a \widetilde{G}_a^{\mu\nu} G_{a\mu\nu}.$$

The current (8.6.23) is the so-called $U(1)$ current (pure flavour singlet) and is atypical in more respects than one. In particular, it is associated with the $U(1)$ problem, to which we now turn.

Assume that we have n light quarks; we only consider these and will neglect (as irrelevant to the problem at hand) the existence of heavy flavours. We may take $n = 2$ (u, d) and then we speak of "the $U(1)$ problem of $SU(2)$" or $n = 3$ (u, d, s), which is the $SU(3)$ $U(1)$ problem. Consider now the $n^2 - 1$ matrices in flavour space $\lambda_1, \ldots, \lambda_{n^2-1}$; for $SU(3)$ they coincide with the Gell-Mann matrices, and for $SU(2)$ with the Pauli matrices. Define further $\lambda_0 \equiv 1$. Any $n \times n$ Hermitian matrix may be written as a linear combination of the n^2 matrices λ_α, $\alpha = 0, 1, \ldots, n^2 - 1$. Because of this completeness, it is sufficient to consider the currents

$$A_\alpha^\mu = \sum_{f f'} \bar{q}_f \gamma^\mu \gamma_5 \lambda_{f f'}^\alpha q_{f'}; \quad \alpha = 0, 1, \ldots, n^2 - 1.$$

[44] For a detailed discussion, see the reviews of Adler (1971) and Ellis (1976). The triangle graph is the only one that has *primitive* anomalies; it does, however, induce secondary anomalies in square and pentagon graphs. The triangle with three axial currents has an anomaly closely related to the one we have discussed, cf. the text of Taylor (1976). An elegant discussion of currents with anomalies for arbitrary interaction may be found in Wess and Zumino (1971). The derivation of the anomaly in the context of the path integral formulation of field theory, where it is connected with the *divergence* of the measure, may be found in Fujikawa (1980, 1984, 1985).

Of course, only A_0 has an anomaly.

Now let $N_1(x), \ldots, N_k(x)$ denote local operators (simple or composite) and consider the quantity

$$\langle \mathrm{vac}|\mathrm{T}A^\mu_\alpha(x)\prod_j N_j(x_j)|\mathrm{vac}\rangle. \tag{8.6.25}$$

For $\alpha = a \neq 0$, the Goldstone theorem implies that the masses of the pseudoscalar particles P_a with the quantum numbers of the A_a vanish in the chiral limit; introducing a common parameter ϵ for all the quark masses by letting $m_f = \epsilon r_f$, $f = 1, \ldots, n$, where the r_f remain fixed in the chiral limit, we have

$$m^2_{P_a} \approx \epsilon. \tag{8.6.26}$$

Therefore, in this limit, the quantity (8.6.25) develops a pole at $q^2 = 0$, for $\alpha = a \neq 0$. To be precise, what this means is that in the chiral limit (zero quark masses),

$$\lim_{q\to 0}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}q\cdot x}\langle \mathrm{vac}|\mathrm{T}A^\mu_\alpha(x)\prod_j N_j(x_j)|\mathrm{vac}\rangle \approx (\text{const.}) \times\, q^\mu \frac{1}{q^2}.$$

If we neglect anomalies, the derivation of (8.6.26) can be repeated for the case $\alpha = 0$ and we would thus find that the $U(1)$ (flavour singlet) particle P_0 would also have vanishing mass in the chiral limit (Glashow, 1968). This statement was made more precise by Weinberg (1975) who proved the bound $m_{P_0} \leq \sqrt{n}\times(\text{average } m_{P_a})$. Now, this is a catastrophe since, for the $SU(2)$ case, $m_\eta \gg \sqrt{2}\,\mu$ and, for $SU(3)$, the mass of the η' particle also violates the bound. This is the $U(1)$ problem. In addition, Brandt and Preparata (1970) proved that under these conditions the decay $\eta \to 3\pi$ is forbidden, which is also in contradiction with experiment. We are thus led to *assume* that (8.6.25) remains regular as $\epsilon \to 0$ for $\alpha = 0$. If we could *prove* that this is so, we would have solved the $U(1)$ problem. We will not discuss this matter any more here, sending to the standard references.[45]

[45] Adler (1969); Bardeen (1974); Crewther (1979b), etc.

9. Chiral perturbation theory

9.1 Chiral Lagrangians

9.1.1. The σ Model

In this and the following sections we will describe a method that has been devised to explore *systematically* the consequences of the chiral symmetries of QCD, in the limit of small momenta and neglecting the light quark masses (or to leading order in these). The method consists in writing Lagrangians consistent with chiral symmetry for pion field operators. These Lagrangians are not unique but, on the mass shell and for momenta p^2 much smaller than Λ^2, all produce the *same* results (Coleman, Wess and Zumino, 1969; Weinberg, 1968a). The Lagrangians are not renormalizable, but this is not important as they are to be used only at tree level (actually, it turns out to be possible to go beyond tree level, at the cost of introducing a number of phenomenological constants, as we will discuss later). One can then use these Lagrangians to calculate low energy quantities involving pions, if the symmetry we consider is chiral $SU(2)$, reproducing the results obtained in a more artisanal way with the help of current algebra and soft pion (PCAC) techniques. This general formulation of chiral dynamics was first proposed by Weinberg (1979) and later developed in much greater detail by Gasser and Leutwyler (1984, 1985a,b). We will begin in this section with a few examples, to proceed in next section to contact with PCAC and present a first application; the general formulation of chiral perturbation theory will be left for Sect. 9.3.

The starting point to formulate the effective chiral Lagrangian theories is to write the chiral transformation properties of pions,[46] whose field we denote by $\vec{\varphi}$, with the vector representing an isospin index, and a fictitious, scalar particle that we will denote by σ. This is the so-called *sigma model* for spontaneous symmetry breaking, devised by Gell-Mann and Lévy (1960). For infinitesimal chiral (i.e., parity changing) transformations we write,

$$\begin{aligned} \sigma \to & \sigma + \delta\sigma, \quad \delta\sigma = -\vec{\alpha}\vec{\varphi} \\ \vec{\varphi} \to & \vec{\varphi} + \delta\vec{\varphi}, \quad \delta\vec{\varphi} = \vec{\alpha}\sigma. \end{aligned} \tag{9.1.1a}$$

The $\vec{\alpha}$ are the parameters of the chiral transformations in $SU(2)\times SU(2)$, which would correspond, in a quark formulation, to the transformations involving γ_5.

For ordinary isospin transformations, with parameters $\vec{\theta}$, we have

$$\delta\sigma = 0, \qquad \delta\vec{\varphi} = \vec{\theta}\times\vec{\varphi}. \tag{9.1.1b}$$

Because we suppose invariance under the full $SU(2)\times SU(2)$ transformations it follows that σ and $\vec{\varphi}$ fields should have the same mass, that (in a first approximation) we take to be zero.

We now assume that the interaction is such that the field σ acquires a vacuum expectation value, $\langle\sigma\rangle = k \neq 0$; this will provide a large (i.e., of order Λ) mass for the sigma field, which will

[46] We will consider here explicitly only chiral $SU(2)$; the extension to chiral $SU(3)$, that is to say, to processes involving also kaons and the η, is straightforward.

then disappear from the low energy effective theory.[47] To formulate the last, we want to redefine fields which do no more mix under chiral transformations. It happens that this is not possible if using *linear* transformations; but can be achieved if nonlinear ones are allowed (*nonlinear sigma models*). A simple choice is to set $\sigma' = \sigma - k$ (so the VEV of σ' vanishes) and then define

$$\begin{aligned} R =& \sqrt{(\sigma'+k)^2 + \vec{\varphi}^2} - k, \\ \vec{\pi} =& \frac{k}{\sqrt{(\sigma'+k)^2 + \vec{\varphi}^2}}\,\vec{\varphi}. \end{aligned} \tag{9.1.2}$$

For small energies we can expand the new fields in terms of the old (in effect, this is an expansion in powers of k^{-1}),

$$R \simeq \sigma' + \cdots, \qquad \vec{\pi} \simeq \vec{\varphi} + \cdots$$

so that the R, $\vec{\pi}$ coincide, at leading order, with the old fields. However, the new fields do not mix under chiral transformations: we get

$$\delta R = 0, \qquad \delta\vec{\pi} = \vec{\alpha}\,\frac{k\sigma}{\sqrt{\sigma^2+\vec{\varphi}^2}} = \vec{\alpha}\sqrt{k^2 - \vec{\pi}^2}. \tag{9.1.3a}$$

Under ordinary isospin we still have,

$$\delta R = 0, \qquad \delta\vec{\pi} = \vec{\theta}\times\vec{\pi}. \tag{9.1.3b}$$

Because of these properties we can write a Lagrangian, invariant under chiral transformations, using only the field $\vec{\pi}$: we have succeeded in decoupling the sigma field. The Lagrangian is not unique; a choice, suggested by Coleman, Wess and Zumino (1969), is to take

$$\mathcal{L} = \tfrac{1}{2}\frac{1}{(1+a^2\vec{\Pi}^2)^2}\left(\partial\vec{\Pi}\right)^2, \quad (\partial\vec{\Pi})^2 \equiv (\partial_\mu\vec{\Pi})(\partial^\mu\vec{\Pi}); \qquad a = 1/2k, \tag{9.1.4a}$$

with $\vec{\Pi}$ a reparametrization of $\vec{\pi}$:

$$\vec{\Pi} = \frac{2\vec{\pi}}{1+\sqrt{1-\vec{\pi}^2/k^2}};$$

it transforms chirally as

$$\delta\vec{\Pi} = \frac{1}{a}\left[\vec{\alpha}\left(1 - a^2\vec{\Pi}^2\right) + 2a^2\vec{\Pi}(\vec{\alpha}-\vec{\Pi})\right].$$

We may expand $\mathcal{L}$ in powers of a getting

$$\mathcal{L} = \tfrac{1}{2}\left(\partial\vec{\Pi}\right)^2 - \frac{a^2}{2}\vec{\Pi}^2\left(\partial\vec{\Pi}\right)^2 + \frac{a^4}{2}\vec{\Pi}^4\left(\partial\vec{\Pi}\right)^2 + \cdots \tag{9.1.5}$$

[47] Alternatively, we could interpret it as the enhancement experimentally observed in the isospin zero S-wave in pion-pion scattering at an energy around 750 MeV. The key point, of course, is that at low energies only the pions give appreciable contributions; those from other particles are suppressed by powers p^2/M_σ^2.

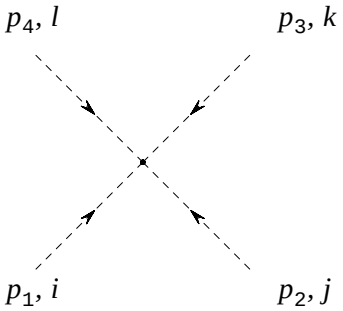

FIGURE 9.1.1
The four pion graph.

To show the usefulness of the effective Lagrangian formulation, we calculate $\pi\pi$ scattering to lowest order in a. Denote by i, j, k, l to the isospin indices, varying from 1 to 3. The Feynman rule corresponding to (9.1.5) to lowest order in a is, for a four-pion vertex with momenta p_1, p_2, p_3, p_4, all incoming (Fig. 9.1.1),

$$\begin{aligned} \mathrm{i}a^2 g_{\mu\nu} \Big[& \delta_{ij}\delta_{kl} \left(p_3^\mu p_4^\nu + p_1^\mu p_2^\nu\right) \\ & + \delta_{ik}\delta_{jl} \left(p_2^\mu p_4^\nu + p_1^\mu p_3^\nu\right) \\ & + \delta_{il}\delta_{jk} \left(p_2^\mu p_3^\nu + p_1^\mu p_4^\nu\right) \Big]. \end{aligned} \tag{9.1.6}$$

In terms of the Mandelstam variables

$$s = (p_1 + p_2)^2, \quad t = (p_2 + p_4)^2, \quad u = (p_2 + p_3)^2,$$

we can write the scattering amplitude that follows from (9.1.6) to lowest order as

$$F(i + j \to k + l) = \frac{a^2}{(2\pi)^2} \left\{ \delta_{ij}\delta_{kl} s + \delta_{ik}\delta_{jl} t + \delta_{il}\delta_{jk} u \right\}. \tag{9.1.7}$$

We will later identify a with $1/f_\pi$, the inverse of the pion decay constant, so (9.1.7) gives the low energy (s, t, $u \ll \Lambda^2$) pion-pion collision amplitude. The simplicity of this evaluation contrasts with that based on "old fashioned" PCAC, current algebra and soft pion techniques (Weinberg, 1966).

9.1.2. Exponential formulation

A more elegant, but equivalent formulation uses a matrix representation of the pion field. Letting $\vec{\tau}$ be the Pauli matrices, for isospin space, we construct the 2×2 matrix

$$\pi = \vec{\tau}\vec{\varphi} \tag{9.1.8a}$$

with $\vec{\varphi}$ the pion field. We then exponentiate π and set the matrix

$$\Sigma = \exp 2\mathrm{i}\pi/F. \tag{9.1.8b}$$

The chiral $SU(2) \times SU(2)$ transformations are defined in terms of the unitary matrices W_L, W_R:

$$\Sigma \to \Sigma' \equiv W_L \Sigma W_R^\dagger. \tag{9.1.8c}$$

The symmetry breaking condition is implemented by assuming a nonzero VEV for Σ:

$$\langle \Sigma \rangle = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}.$$

The advantage of the present method is that we only work with the pion field from the beginning.

It is convenient to parametrize the $W_{R,L}$ as

$$\begin{aligned} W_L =& \mathrm{e}^{\vec{\alpha}\vec{\tau}} \mathrm{e}^{\vec{\theta}\vec{\tau}}, \\ W_R =& \mathrm{e}^{-\vec{\alpha}\vec{\tau}} \mathrm{e}^{\vec{\theta}\vec{\tau}}. \end{aligned} \tag{9.1.9}$$

For ordinary isospin transformations we simply set $\vec{\alpha} = 0$ so that W_L and W_R coincide and (9.1.8) is equivalent to $\pi' = W(\vec{\theta})\pi W^{-1}(\vec{\theta})$. Then, for the pion field itself we have $\vec{\pi}' = R(\vec{\theta})\vec{\pi}$ with $R(\vec{\theta})$ the three-dimensional rotation corresponding to the $SU(2)$ matrix $W(\vec{\theta})$ given by the relation

$$W(\vec{\theta})\tau_i W^{-1}(\vec{\theta}) = \sum_j R^{-1}_{ij}(\vec{\theta})\tau_j.$$

Under an infinitesimal chiral transformation, (9.1.8) gives, after expanding,

$$\vec{\pi}' = \vec{\pi} + F\vec{\alpha} + \cdots.$$

Next we construct a Lagrangian invariant under (9.1.8). The one which contains *less* derivatives is

$$\mathcal{L} = \frac{F^2}{4} \operatorname{Tr}\left(\partial_\mu \Sigma^+\right) \partial^\mu \Sigma, \tag{9.1.10}$$

and the overall constant is chosen so that, after expanding, the kinetic term is $\frac{1}{2}(\partial_\mu \vec{\pi})\partial^\mu \vec{\pi}$. This shows clearly the arbitrariness of the method: we can add extra terms with higher derivatives to (9.1.10). However, they will, on dimensional grounds, contribute to higher orders in the momenta. But it is important to realize that the effective Lagrangian methods are only valid to give the first orders in the expansion in powers of the momenta, p^2/Λ^2. The theory says nothing *a priori* about higher corrections, which involve more and more arbitrary parameters.

In this formalism we can introduce in a natural manner leading order symmetry breaking by considering that it is due to a quark mass matrix,

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}.$$

This is not invariant under chiral (or even ordinary isospin) transformations. We may couple M and Σ; the lowest dimensionality scalar that can be formed is the function

$$v^3 \operatorname{Tr}(\Sigma^+ M + M\Sigma).$$

v is a constant with dimensions of mass, that we will identify later. Expanding in powers of π, we find that the first nonzero term is the quadratic one,

$$-\frac{4v^3}{F^2} \operatorname{Tr} M\pi^2 = -\frac{4v^3}{F^2}(m_u + m_d)\vec{\pi}^2, \tag{9.1.11}$$

and we have used that $(\vec{\lambda}\vec{\tau})^2 = \vec{\lambda}^2$ for any $\vec{\lambda}$. Eq. (9.1.11) provides the lowest order mass term for the pions; it has the nice feature that it reproduces (as it should) the result we had obtained with the help of PCAC and current algebra in (8.2.4). This allows us to realize that v^3 is proportional to the quark condensate. Applications of this to calculate some hadronic corrections to low energy weak interactions may be found in the book of Georgi (1984).

An alternate to the exponential formulation presented here will be given in Sect. 9.3.

9.2. Connection with PCAC, and a first application

Before starting to calculate with the chiral Lagrangians described in the previous section we have to interpret the constant (F or a) that appears there. For this we have to introduce the *axial current* in the present formalism, which we choose to do in the original Coleman–Wess–Zumino version; the derivation in the exponential version, somewhat messier, may be found in the text of Georgi (1984). To do so we use a method which is a variant of Noether's method, due to Adler (for more details on it, see Georgi, 1984 or Adler, 1971). Let us consider a general Lagrangian $\mathcal{L}(\phi)$ depending on the field ϕ, and make an infinitesimal transformation on the fields, characterized by the infinitesimal parameters ϵ_i:

$$\delta\phi = \sum_i \epsilon_i \xi_i(\phi).$$

The corresponding variation of the Lagrangian is then

$$\delta\mathcal{L} = K_i(\phi)\epsilon_i + L_i^\mu(\phi)\partial_\mu\epsilon_i + M_i^{\mu\nu}(\phi)\partial^\mu\partial^\nu\epsilon_i + \text{higher derivatives.}$$

(sum over repeated indices understood). The variation of the action can then be written, after integrating by parts, as

$$\delta\mathcal{A} = \int \mathrm{d}^4x \, \{K_i + \partial_\mu J_i^\mu\}\, \epsilon_i$$

and we have defined the current J by

$$J_i^\mu = -L_i^\mu + \partial_\nu M_i^{\mu\nu} + \cdots.$$

For a symmetry of the system, the change must leave the action unchanged, hence $\partial_\mu J_i^\mu = -K_i$. Moreover, choosing ϵ constant, $\mathcal{L}$ will be invariant only if $K_i = 0$. In this case, J_i^μ is obtained simply as the coefficient of $\partial_\mu\epsilon_i$ in the variation of $\mathcal{L}$. It is interesting to note that, if $\mathcal{L}$ only contains first order derivatives of the field ϕ, then all the terms M, etc. above vanish so J_i^μ coincides with $-L_i^\mu$.

This can be immediately applied to the Lagrangian (9.1.4a). Working to lowest order in Π, we find immediately the axial current to be

$$\vec{A}_\mu = -\frac{1}{a}\partial_\mu\vec{\Pi} + \text{higher orders} = -\frac{1}{a}\partial_\mu\vec{\varphi} + \text{higher orders.}$$

Taking derivatives of both sides and using the equations of motion this gives

$$\partial^\mu\vec{A}_\mu = \frac{1}{a}\mu^2\vec{\varphi}.$$

On comparing with the definitions in Sect. 7.3, we can identify

$$\frac{1}{a} = f_\pi,$$

f_π the pion decay constant, $f_\pi \simeq 93$ MeV. (The factor $\sqrt{2}$ in the definitions of Sect. 7.3 has disappeared because the physical pion states are related to the ones used now, $\vec{\pi}$, by $\pi^\pm = \mp 2^{-1/2}(\pi_1 \pm \mathrm{i}\pi_2)$).

With this identification we get the pion-pion scattering amplitude, given in Eq. (9.1.7), as

$$F(i+j \to k+l) = \frac{1}{4\pi^2 f_\pi^2}\left\{\delta_{ij}\delta_{kl}s + \delta_{ik}\delta_{jl}t + \delta_{il}\delta_{jk}u\right\}. \tag{9.2.1}$$

From this one can evaluate the low energy parameters for $\pi\pi$ scattering. For example, the isospin 1, P wave scattering length is calculated as follows. First, we identify the physical pion states in terms of the i, j, $\ldots = 1$, 2, 3 ones as

$$|\pi^0\rangle = |3\rangle, \qquad |\pi^\pm\rangle = \mp 2^{-1/2}\left\{|1\rangle \pm \mathrm{i}|2\rangle\right\};$$

the isospin 1 state will appear in particular in the combination $|\pi^0\pi^+\rangle$ as

$$|\pi^0\pi^+\rangle = 2^{-1/2}|I=1\rangle + 2^{-1/2}|I=2\rangle.$$

Moreover, we have the partial wave expansion, for states with well defined isospin I,

$$F^{(I)} = 2\sum_l (2l+1)P_l(\cos\theta) f_l^{(I)}; \qquad f_l^{(I)} = \frac{2s^{1/2}}{\pi k}\sin\delta_l^{(I)} \mathrm{e}^{\mathrm{i}\delta_l^{(I)}}, \tag{9.2.2}$$

with $\delta_l^{(I)}$ the phase shifts.[48]

At small energy we write the *partial wave amplitudes*, $f_l^{(I)}$, in terms of the *scattering lengths*, $a_l^{(I)}$:

$$f_l^{(I)}(s) \underset{s\to 4M_\pi^2}{\simeq} \frac{4M_\pi k^{2l}}{\pi} a_l^{(I)}.$$

k is the center of mass momentum; for massless pions, we can take $k^2 = s/4$. Following the custom in modern chiral perturbative calculations, we express our numbers in terms of $M_\pi = m_{\pi^+}$. With all this we find, for the P wave

$$a_1 = \frac{1}{24\pi f_\pi^2 M_\pi} \simeq 0.031\, M_\pi^{-3}. \tag{9.2.3}$$

Experimentally, and from the analysis of Sect. 6.8, we know that

$$a_1(\text{exp.}) = (0.0386 \pm 0.0012)\, M_\pi^{-3}.$$

The agreement between theory and experiment improves if including pion mass corrections, and higher order chiral perturbative theory terms (to be discussed later).

The S-wave phase shifts are similarly calculated, and we get,

$$a_0^{(0)} = \frac{7M_\pi}{32\pi f_\pi^2} \simeq 0.157\ M_\pi^{-1};$$
$$a_0^{(2)} = -\frac{M_\pi}{16\pi f_\pi^2} \simeq -0.045\ M_\pi^{-1}.$$

The agreement of these with experiment is less good than before. Including corrections, the predicted value for $a_0^{(0)}$ (for example) could go up to $0.22\, M_\pi^{-1}$, while experiment gives values in the range $0.215\, M_\pi^{-1}$ to $0.240\, M_\pi^{-1}$, as we saw in Chapter 6. Corrections will be discussed in more detail later on.

[48] Recall that the factor 2 in the partial wave expansion is due to the identity of the particles, in states with well-defined isospin.

9.3 Chiral perturbation theory: general formulation

There is a large number of further applications of chiral perturbation theory (at times also denoted by the name of χPT), to leading order, which the interested reader may find in the text of Georgi (1984). But one may ask if it is possible to go beyond. In fact, an enormous amount of work has been devoted to the matter in recent years, particularly following the basic papers of Gasser and Leutwyler (1984, 1985a,b).[49] In the present section we will indeed describe the general formalism of chiral perturbation theory, following, precisely, the excellent exposé of these authors. We will restrict ourselves to chiral isospin; the extension to chiral $SU(3)$ may be found in Gasser and Leutwyler (1985a).

The idea is the following: we will first extend the chiral symmetry in QCD to a gauge symmetry. Then we will construct the more general Lagrangians involving pions (for chiral $SU(2)$), first to leading order and then to higher orders, consistent with the PCAC definition $\partial \cdot \vec{A} = f_\pi m u^2 \vec{\phi}_\pi$ and verifying the gauge chiral symmetry. Because these Lagrangians share the symmetry with the QCD one, it will follow that the theory based on pions will satisfy identical Ward identities and commutation relations as QCD; therefore they will show the same low energy properties.

9.3.1. Gauge extension of chiral invariance

As stated, we start by extending the $SU(2) \times SU(2)$ symmetry to a gauge symmetry. We do so by introducing *sources* in the QCD Lagrangian. We denote by $\mathcal{L}_{\rm QCD0}$ to the QCD Lagrangian for massless u, d quarks,

$$\mathcal{L}_{\rm QCD0} = \sum_{\alpha=u,d} \bar{q}_\alpha \mathrm{i} \not{D} q_\alpha - \tfrac{1}{4} G^2. \tag{9.3.1a}$$

Then we consider $\mathcal{L}(v_\mu, a_\mu, s, p)$ where v_μ, a_μ, s, p are, respectively, vector, axial, scalar and pseudoscalar sources, and we define

$$\begin{aligned} \mathcal{L}(v_\mu, a_\mu, s, p) = & \mathcal{L}_{\rm QCD0} \\ & + \sum_{\alpha,\beta} \bar{q}_\alpha \gamma_\mu \left(v^\mu_{\alpha\beta} + a^\mu_{\alpha\beta} \gamma_5 \right) q_\beta + \sum_{\alpha,\beta} \bar{q}_\alpha \left(-s_{\alpha\beta} + \mathrm{i} p_{\alpha\beta} \gamma_5 \right) q_\beta. \end{aligned} \tag{9.3.1b}$$

We include the mass matrix in $s_{\alpha\beta}$ so that

$$s_{\alpha\beta} = m_\alpha \delta_{\alpha\beta} + \widetilde{s}_{\alpha\beta}. \tag{9.3.1c}$$

α, β are flavour indices that run over the values u, d, in our case.

The Lagrangian (9.3.1b) is invariant under independent local gauge transformations of the left and right components of the q, *provided* we at the same time transform the sources:

$$\begin{aligned} & q \to q' = \left\{ \tfrac{1}{2}(1+\gamma_5) W_R(x) + \tfrac{1}{2}(1-\gamma_5) W_L(x) \right\} q; \\ & v^\mu \pm a^\mu \to v'^\mu \pm a'^\mu = W_{R,L} \left(v^\mu \pm a^\mu \right) W^\dagger_{R,L} + \mathrm{i} W_{R,L} \partial^\mu W^\dagger_{R,L}, \\ & s + \mathrm{i} p \to s' + \mathrm{i} p' = W_R (s + \mathrm{i} p) W^\dagger_L. \end{aligned} \tag{9.3.2}$$

[49] We will not be able to give an amount of information comparable to that presented in these papers; we urge the reader to consult them for a more detailed treatment and further applications. The subject has had an enormous growth in the last years; a recent review, with references, is that by Scherer (2002). An introductory one is the text by Dobado et al. (1997).

Here the $W_{R,L}$ are independent $SU(2)$ matrices. The symmetry may be extended to a $U(2)\times U(2)$ symmetry; however, the current associated with the diagonal piece presents an anomaly, as we know. We will not study this piece here, but refer to Gasser and Leutwyler (1985a). To avoid it we will restrict the v^μ, a^μ to be traceless. This is automatic if we parametrize them in terms of the three-vectors v_i^μ, a_i^μ writing

$$v^\mu = \tfrac{1}{2}\sum_i v_i^\mu \tau_i, \quad a^\mu = \tfrac{1}{2}\sum_i a_i^\mu \tau_i \tag{9.3.3}$$

and the τ_i are the Pauli matrices in flavour space. The s, p may likewise be parametrized in terms of the (*Euclidean*) four dimensional vectors s_A, p_A with

$$s = \sum_A s_A \tau_A, \quad p = \sum_A p_A \tau_A; \qquad \tau_0 \equiv 1. \tag{9.3.4}$$

At low energy the only degrees of freedom are those associated with the pions; moreover, we have to take also into account that, in QCD, the scalar densities have a nonzero expectation value in the ground state (the physical vacuum). We will use the quantity B defined as

$$B = -\frac{\langle \bar{q}q\rangle}{f^2}. \tag{9.3.5}$$

We write f for the pion decay constant in the chiral limit ($m_{u,d} \to 0$). In Subsect. 9.3.3 we will see the connection with the physical decay constant, whose value we take to be $f_\pi \simeq 93$ MeV . In the chiral limit, B is independent of which q (u or d) we take. Comparing with (9.2.4) we have

$$B = \mu^2/(m_u + m_d).$$

9.3.2. Effective Lagrangians in the chiral limit

We will start by working in the chiral limit, $m_{u,d} = 0$. At low energies an effective Lagrangian should only include pion fields and, apart from the nonzero value of the condensate, should respect chiral gauge invariance.[50] To construct this Lagrangian we proceed as for the nonlinear σ-model of Sect. 9.1. We define a chiral four-dimensional vector φ_A, $A = 0, 1, 2, 3$ such that $\vec{\varphi} = \vec{\pi}$ (the pion field) and $\varphi_0 = \sigma$ (the σ field). We get rid of the last by imposing the invariant constraint

$$\sum_A \varphi_A \varphi_A = f^2. \tag{9.3.6a}$$

We could include this into the Lagrangian, using a multiplier or, more simply, by admitting that φ_0 is not an independent field, but one has

$$\varphi_0 = \sqrt{f^2 - \vec{\varphi}^2}. \tag{9.3.6b}$$

[50] This is, of course, a limitation of the chiral dynamics approach; it must fail at distances where the *composite* character of the pions becomes relevant; thus certainly at energies of the order of the ρ mass, as this particle is a quark-antiquark bound state, and decays into two pions.

The transformation properties of φ under $SU(2) \times SU(2)$ imply the following values for the chiral covariant derivative, that we denote by ∇^μ:

$$\begin{aligned} \nabla^\mu \varphi_0 &= \partial^\mu \varphi_0 + \vec{a}^\mu(x)\vec{\varphi}, \\ \nabla_\mu \vec{\varphi} &= \partial^\mu \vec{\varphi} + \vec{v}^\mu(x) \times \vec{\varphi} - \vec{a}^\mu(x)\varphi_0. \end{aligned} \tag{9.3.7}$$

We then construct the more general Lagrangians which are compatible with Eq. (9.3.2), and involve only φ_A. We start at lowest order in the momenta, $O(p^2)$. If we only allow two powers of the momenta at tree level, then only two derivatives can occur and the more general form of this first order Lagrangian is, simply,

$$\mathcal{L}_{\text{ch.1}} = \tfrac{1}{2} \sum_A (\nabla_\mu \varphi_A) \nabla^\mu \varphi_A. \tag{9.3.8}$$

The index "ch." reminds us that this is valid in the chiral limit, and the factor $1/2$ is included so that the kinetic energy term agrees with that for three real, (pseudo-)scalar fields. One can evaluate the axial current from (9.3.8) and identify f with the value of the pion decay constant, f_π, in the chiral limit. (In this case the identification of the axial current is simpler than before, as it is the current coupled to the axial source, $\vec{a}^\mu$).

In particular, to lowest order and replacing φ_0 in terms of $\vec{\varphi}$, this gives

$$\mathcal{L}_{\text{ch.1}} \simeq \tfrac{1}{2}(\partial_\mu \vec{\varphi})\partial^\mu \vec{\varphi} + \frac{1}{2f^2} (\vec{\varphi}\partial_\mu \vec{\varphi}) (\vec{\varphi}\partial^\mu \vec{\varphi}) + \text{ source terms } + \text{ higher orders.} \tag{9.3.9}$$

To order p^2, this is equivalent to (9.1.5).

Let us next consider $O(p^4)$. Simple power counting shows that the loop corrections generated by (9.3.8) are of relative order p^2 for each new loop; hence, one loop corrections induced by $\mathcal{L}_{\text{ch.1}}$ will be of order p^4. These corrections (which are necessary in order to respect unitarity of the effective theory) are, generally speaking, divergent. However, if we use a regularization that respects gauge invariance (such as dimensional regularization, in the absence of anomalies) these divergences will multiply chiral gauge invariant polynomials of degree p^4. They can thus be absorbed into suitable counterterms.

This leads us to construct all possible terms of order p^4 which will build the second order effective Lagrangian, $\mathcal{L}_{\text{ch.2}}$. After use of the equations of motion it can be seen (Gasser and Leutwyler, 1984) that its most general form will be (sum over repeated indices A, B, C understood)

$$\begin{aligned} \mathcal{L}_{\text{ch.2}} = \frac{1}{f^4} \Big\{ & l_1 \left(\nabla^\mu \varphi_A \nabla_\mu \varphi_A\right)^2 + l_2 \left(\nabla^\mu \varphi_A \nabla^\mu \varphi_A\right) \left(\nabla_\mu \varphi_B \nabla_\mu \varphi_B\right) \\ & + l_5 \varphi_A F^{\mu\nu}_{AB} F_{BC,\mu\nu} + l_6 \nabla_\mu \varphi_A F^{\mu\nu}_{AB} \nabla_\nu \varphi_B \\ & + h_2 \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} \Big\}. \end{aligned} \tag{9.3.10a}$$

Here F is defined by

$$\left(\nabla^\mu \nabla^\nu - \nabla^\nu \nabla^\mu\right) \varphi_A = F^{\mu\nu}_{AB} \varphi_B \tag{9.3.10b}$$

and the reason for the numbering of the constants $l_1, \ldots, h_2$ (that agrees with the definitions of Gasser and Leutwyler, 1984) will be seen below.

The constants $l_1, \ldots, h_2$ will be divergent: their divergence is to be adjusted so that it cancels the one loop divergences generated by $\mathcal{L}_{\text{ch.1}}$. The theory will, therefore, *predict* the coefficients of terms of type $p^4 \log p^2/\nu^2$, with ν a renormalization scale (and, when we take into account leading symmetry breaking by the pion mass, also terms in μ^4 and $p^2\mu^2$ multiplied by either $\log p^2/\nu^2$

or $\log \mu^2/\nu^2$). However, the finite parts of the constants $l_1, \ldots, h_2$ are *not* given by the theory. In fact, what one does is to *fix* these constants by requiring agreement of the predictions using $\mathcal{L}_{\text{ch.1}}$, $\mathcal{L}_{\text{ch.2}}$ with experiment. Chiral dynamics does *not* allow an evaluation from first principles of corrections of order p^4. What it does is to *correlate* these corrections to all processes in terms of a finite number of constants, the $l_1, \ldots, h_2$.

In principle one can extend this procedure to higher orders and, indeed, the $O(p^6)$ corrections have been considered in the literature,[51] but we will not discuss this in any detail here. Not only the number of constants to be fitted to experiment grows out of hand, but it is practically impossible to separate the $O(p^4)$ and $O(p^6)$ pieces of the $l_1, \ldots$, as we will see in some examples later. More interesting is to take into account the corrections due to the nonzero masses of the u, d quarks (or, equivalently, of the pions) to which we now turn.

9.3.3. Finite pion mass corrections

Because the mass of the pion will appear in pion propagator denominators, $1/(p^2-\mu^2)$, a consistent way to treat the finiteness of the pion mass requires that we consider p^2 and μ^2 to be of the same order of magnitude, and calculate to all orders in their ratio; otherwise we would be replacing

$$\frac{1}{p^2-\mu^2} \quad \text{by} \quad \frac{-1}{\mu^2}\left\{1+\frac{p^2}{\mu^2}+\frac{p^4}{\mu^4}+\cdots\right\},$$

not a very accurate procedure.

To leading order we have to find the lowest order terms that can be added to $\mathcal{L}_{\text{ch.1}}$ and which contain s_0; we recall that s_0 included the quark masses. There is only one such term that also preserves parity, Constant $\times$ $(s_0\varphi_0 + \vec{p}\vec{\varphi})$. The constant may be identified requiring that the new term reproduce the equality (9.2.4) for the pion propagator. We then have the full $\mathcal{L}_1$, correct to $O(p^2)$, $O(\mu^2)$,

$$\mathcal{L}_1 = \mathcal{L}_{\text{ch.1}} + 2Bf\,(s_0\varphi_0 + \vec{p}\vec{\varphi})\,, \tag{9.3.11a}$$

which corresponds to the pion mass

$$\mu^2 = (m_u + m_d)B. \tag{9.3.11b}$$

To next order,

$$\mathcal{L}_2 = \mathcal{L}_{\text{ch.2}} + \frac{1}{f^4}\Big\{l_3(\xi_A\varphi_A)^2 + l_4\nabla^\mu\xi_A\nabla_\mu\varphi_A + l_7(\eta_A\varphi_A)^2 + h_1\xi_A\xi_A + h_3\eta_A\eta_A\Big\}. \tag{9.3.12a}$$

We have defined

$$\xi_0 = 2Bs_0, \quad \vec{\xi} = 2Bp; \qquad \eta_0 = 2Bp_0, \quad \vec{\eta} = -2B\vec{s} \tag{9.3.12b}$$

and $\mathcal{L}_{\text{ch.1}}$, $\mathcal{L}_{\text{ch.2}}$ are as given in (9.3.8), (9.3.10).

For reference, we note the correspondence between our definitions and those of Gasser and Leutwyler (1984):

$$U_A = \frac{1}{f}\,\varphi_A, \quad \chi_A = \frac{1}{f}\,\xi_A, \quad \widetilde{\chi}_A = \frac{1}{f}\,\eta_A. \tag{9.3.13}$$

[51] Akhoury and Alfakih (1991); Fearing and Scherer (1996); Knecht et al. (1995, 1996); Bijnens et al. (1996); Bijnens, Colangelo and Eder (2000).

9.3.4. Renormalized effective theory

Renormalization for the one loop graphs generated by $\mathcal{L}_1$ proceeds in the usual manner. The divergences, as stated in the previous subsection, can be canceled by divergent pieces in the l_i, h_j. One finds (Gasser and Leutwyler, 1984, where the c_i are denoted by γ_i and the d_j by δ_j)

$$\begin{aligned} l_i &= l_i^{\text{loop.}}(\nu) = \frac{c_i}{32\pi^2}\left\{\frac{2}{D-4} + \log\nu^2 - (\log 4\pi - \gamma_{\text{E}} + 1)\right\}, \\ h_j &= h_j^{\text{loop.}}(\nu) = \frac{d_j}{32\pi^2}\left\{\frac{2}{D-4} + \log\nu^2 - (\log 4\pi - \gamma_{\text{E}} + 1)\right\}; \end{aligned} \tag{9.3.14a}$$

ν is the renormalization point and

$$\begin{aligned} &c_1 = \tfrac{1}{3}, \quad c_2 = \tfrac{2}{3}, \quad c_3 = -\tfrac{1}{2}, \quad c_4 = 2, \quad c_5 = -\tfrac{1}{6}, \quad c_6 = \tfrac{1}{3}, \quad c_7 = 0; \\ &d_1 = 2, \quad d_2 = \tfrac{1}{12}, \quad d_3 = 0. \end{aligned} \tag{9.3.14b}$$

The renormalized constants $l_i^{\text{ren.}}$ may be obtained by comparing with experimental quantities.[52] They depend on the renormalization point, ν. Alternatively, one may replace them by the quantities $\bar{l}_i$, defined as (proportional to) the $l_i^{\text{ren.}}(\nu)$ with $\nu = \mu_{\text{ch.}}$. (Here we denote by $\mu_{\text{ch.}}$ to the pion mass in the leading order in chiral symmetry breaking, that is to say, using (9.3.11b) but evaluating $B = -\langle\bar{q}q\rangle/f$ in the chiral limit). Then, we have

$$l_i^{\text{ren.}}(\nu) = \frac{c_i}{32\pi^2}\left\{\bar{l}_i + \log\frac{\mu_{\text{ch.}}}{\nu^2}\right\}. \tag{9.3.14c}$$

We remark that this implies that the $\bar{l}_i$ are divergent in the chiral limit, as we are renormalizing at $\nu = \mu_{\text{ch.}}$ which vanishes in this limit:

$$\bar{l}_i \underset{m_{u,d}\to 0}{\simeq} -\log\mu_{\text{ch.}}.$$

We can now compare the results of calculations made with $\mathcal{L}_1$ and $\mathcal{L}_2$ with experimental quantities, and obtain the $\bar{l}_i$. As an example we consider $\pi\pi$ scattering. If we use the full $\mathcal{L}_1$ and $\mathcal{L}_2$ we obtain, after a straightforward but tedious calculation (Gasser and Leutwyler, 1984)

$$F(i+j\to k+l) = \frac{1}{4\pi^2}\left\{\delta_{ij}\delta_{kl}A(s,t,u) + \delta_{ik}\delta_{jl}A(t,s,u) + \delta_{il}\delta_{jk}A(u,t,s)\right\} \tag{9.3.15a}$$

where now

$$A(s,t,u) = \frac{s-\mu_{\text{ch.}}^2}{f^2} + B(s,t,u) + C(s,t,u). \tag{9.3.15b}$$

Here B, C are, respectively, the logarithmic and polynomial fourth order corrections:

$$\begin{aligned} B(s,t,u) = \frac{1}{96\pi^2 f_\pi^4}\Big\{ & 3(s^2-\mu^2)I(s) \\ & + \left[t(t-u) - 2\mu^2 t + 4\mu^2 u - 2\mu^4\right] I(t) \\ & + \left[u(u-t) - 2\mu^2 u + 4\mu^2 t - 2\mu^4\right] I(u)\Big\}; \\ I(s) = \beta\log\frac{\beta-1}{\beta+1} + 2, \quad & \beta = \sqrt{1-4\mu_{\text{ch.}}^2/s}; \end{aligned} \tag{9.3.16a}$$

[52] The h_j depend on the renormalization scheme and, in fact, do not intervene in any physical observable. This is discussed in Gasser and Leutwyler, 1984.

$$C(s,t,u) = \frac{1}{96\pi^2 f_\pi^4}\Bigg\{ 2\left(\bar{l}_1 - \tfrac{4}{3}\right)(s-2\mu^2)^2 + \left(\bar{l}_2 - \tfrac{5}{6}\right)\left[s^2 + (t-u)^2\right] - 12\mu^2 s + 15\mu^4 \Bigg\}. \tag{9.3.16b}$$

The expression for B in the chiral limit ($\mu = 0$) has been known for a long time (Lehmann, 1972). To leading order B, $C \to 0$, $\mu \to 0$, and (9.3.15) of course reproduces (9.2.1).

The extension to $SU(3)$ (i.e., including kaons and η) may be found, for chiral perturbation theory, in Gasser and Leutwyler (1985a); for $\pi\pi$ scattering in Bernard, Kaiser and Meissner (1991) for some cases and, in general, in the paper of Gómez-Nicola and Peláez (2002).

9.3.5. The parameters of chiral perturbation theory

To one loop, the $\pi\pi$ scattering amplitude depends on the two unknown constants $\bar{l}_1$, $\bar{l}_2$ (besides, of course, f_π and M_π). A technical point to be cleared is that, in the amplitude A in Eq. (9.3.15b), we have the quantities f and $\mu_{\rm ch.}$ which we have to relate to the physical ones. The details may be found again in the paper of Gasser and Leutwyler (1984); we have, writing as usual[53] our results in terms of M_π

$$M_\pi^2 = \mu_{\rm ch.}^2\left\{1 - \frac{M_\pi^2}{32\pi^2 f_\pi^2}\bar{l}_3\right\}, \quad f_\pi = f\left\{1 + \frac{M_\pi^2}{16\pi^2 f_\pi^2}\bar{l}_4\right\}. \tag{9.3.17}$$

Thus, F in (9.3.15) depends also indirectly on the constants $\bar{l}_3$, $\bar{l}_4$. We can, however, obtain directly $\bar{l}_1$, $\bar{l}_2$ by selecting an observable that depends only on second order effects. Such observables are the D waves at low energy:

$$f_2^{(I)} \simeq \frac{(s-4M_\pi^2)^2}{4\pi} M_\pi a_2^{(I)},$$

and $I = 0$, 2 is the isospin index. Because this vanishes (for $M_\pi = 0$) as $s^2 = p^4$, the contributions to them start at second order and we find

$$\begin{aligned} a_2^{(0)} &= \frac{M_\pi^{-1}}{1440\pi^3 f_\pi^4}\left\{\bar{l}_1 + 4\bar{l}_2 - \tfrac{53}{8}\right\}, \\ a_2^{(2)} &= \frac{M_\pi^{-1}}{1440\pi^3 f_\pi^4}\left\{\bar{l}_1 + \bar{l}_2 - \tfrac{103}{40}\right\}. \end{aligned} \tag{9.3.18}$$

We can also improve our previous determination of the scattering lengths; for example, for the S and P waves, including pion mass and $O(p^4)$ terms gives

$$\begin{aligned} a_0^{(0)} &= \frac{7M_\pi}{32\pi f_\pi^2}\left\{1 + \frac{5M_\pi^2}{84\pi^2 f_\pi^2}\left[\bar{l}_1 + 2\bar{l}_2 - \tfrac{3}{8}\bar{l}_3 + \tfrac{21}{10}\bar{l}_4 + \tfrac{21}{8}\right]\right\}, \\ a_0^{(2)} &= -\frac{M_\pi}{16\pi f_\pi^2}\left\{1 - \frac{M_\pi^2}{12\pi^2 f_\pi^2}\left[\bar{l}_1 + 2\bar{l}_2 + \tfrac{3}{8}\right] + \frac{M_\pi^2}{32\pi^2 f_\pi^2}\left[\bar{l}_3 + 4\bar{l}_4\right]\right\}; \end{aligned} \tag{9.3.19a}$$

[53] In the limit in which we forget isospin breaking interactions, one would have $\mu = M_\pi$. Gasser and Leutwyler, and, following them, most modern authors, take M_π as the starting point from which to perturb with isospin breaking interactions, instead of using –as would appear more natural– μ as the starting point. As stated several times, we follow this custom when giving numerical results, for ease of comparison with other calculations.

$$a_1 = \frac{M_\pi^{-1}}{24\pi f_\pi^2}\left\{1 - \frac{M_\pi^2}{12\pi^2 f_\pi^2}\left[\bar{l}_1 - \bar{l}_2 + \tfrac{65}{48}\right] + \frac{M_\pi^2}{8\pi^2 f_\pi^2}\bar{l}_4\right\}. \tag{9.3.19b}$$

Note that we here use the definition (cf. (3.1.7, (7.5.3))

$$\frac{\pi}{4M_\pi k^{2l}}\,\mathrm{Re}\, f_l^{(I)}(s) = a_l^{(I)} + b_l^{(I)}k^2 + \cdots .$$

This may be compared with the standard effective range expansion:

$$k^{2l+1}\cot\delta_l^I(s) \underset{k\to 0}{\simeq} \frac{1}{a_l^{(I)}} + \tfrac{1}{2}r_0k^2 + O(k^4).$$

The connection between the corresponding parameters $a_l^I|_{\text{G.\&L.}}$, $b_l^I|_{\text{G.\&L.}}$ of Gasser and Leutwyler (1984) and our $a_l^{(I)}$, $b_l^{(I)}$, and also with r_0 and with the parameters $b_l^I|_{\text{P.S–G.Y.}}$ of Palou, Sánchez-Gómez and Ynduráin (1975) is

$$a_l^I|_{\text{G.\&L.}} = M_\pi a_l^{(I)}, \quad 4b_l^I|_{\text{P.S–G.Y.}} = b_l^I|_{\text{G.\&L.}} = b_l^{(I)} = a_l^{(I)}\frac{1 - M_\pi^2 a_l^{(I)} r_0}{2M_\pi}.$$

For the parameters $b_l^{(I)}$ one loop chiral perturbation theory gives

$$\begin{aligned} b_0^{(0)} &= \frac{M_\pi^{-1}}{4\pi f_\pi^2}\left\{1 + \frac{M_\pi^2}{12\pi^2 f_\pi^2}\left[2\bar{l}_1 + 3\bar{l}_2 - \tfrac{13}{16}\right] + \frac{M_\pi^2}{8\pi^2 f_\pi^2}\bar{l}_4\right\}, \\ b_0^{(2)} &= -\frac{M_\pi^{-1}}{8\pi f_\pi^2}\left\{1 - \frac{M_\pi^2}{12\pi^2 f_\pi^2}\left[\bar{l}_1 + 3\bar{l}_2 - \tfrac{5}{16}\right] + \frac{M_\pi^2}{8\pi^2 f_\pi^2}\bar{l}_4\right\}; \end{aligned} \tag{9.3.20a}$$

$$b_1 = \frac{M_\pi^{-1}}{288\pi^3 f_\pi^4}\left\{-\bar{l}_1 + \bar{l}_2 + \tfrac{97}{120}\right\}. \tag{9.3.20b}$$

The values of the a_0, b_0 given above imply that the S waves with isospin I have a zero each, for $s = z_I^2$ in the range $0 < s < 4M_\pi^2$, located at

$$\begin{aligned} z_0^2 =& 4M_\pi^2 - \frac{7M_\pi^2}{2}\left\{1 + \frac{5M_\pi^2}{84\pi^2 f_\pi^2}\left[\bar{l}_1 + 2\bar{l}_2 - \tfrac{3}{8}\bar{l}_4 + \tfrac{21}{8}\right]\right. \\ &\left. - \frac{M_\pi^2}{12\pi^2 f_\pi^2}\left[2\bar{l}_1 + 3\bar{l}_2 - \tfrac{13}{16}\right] + \frac{M_\pi^2}{8\pi^2 f_\pi^2}\bar{l}_4\right\}; \\ z_2^2 =& 4M_\pi^2 - 2M_\pi^2\left\{1 + \frac{M_\pi^2}{32\pi^2 f_\pi^2}\bar{l}_3 + \frac{M_\pi^2}{12\pi^2 f_\pi^2}\left[\bar{l}_2 + \tfrac{1}{16}\right]\right\}. \end{aligned} \tag{9.3.21}$$

These zeros are often called *Adler zeros*, after the work of Adler (1965) on zeros of scattering amplitudes implied by PCAC. It should be clear, however, that while the location of z_2 is probably well described by (9.3.21), there is no reason why the same should be the case for z_0. Indeed, to get this last, we have used the expansion of $f_l^{(I)}$ for $s = z_0^2 \simeq \frac{1}{2}M_\pi^2$ were, due to the vicinity of the left hand cut of $f_0^{(0)}(s)$, starting at $s = 0$, we would expect it to give a poor approximation. Actually, while fits to data do confirm z_2 (Subsect. 6.4.1), the situation is less clear for z_0.

9.4. Comparison of chiral perturbation theory to one loop with experiment

9.4.1. One loop coupling constants, and $\pi\pi$ scattering and the electromagnetic form factor of the pion

Using the experimental values of the quantities evaluated in Chapters 6,7, and others as well, we can find the constants $\bar{l}_i$. In fact, there are many more observables than constants; for example, $a_2^{(2)}$, $a_2^{(0)}$ and b_1 depend only on the two $\bar{l}_1$, $\bar{l}_2$. So the agreement of various determinations among themselves is a nontrivial check of second order chiral perturbation theory.[54]

We collect here some recently obtained values of the constants $\bar{l}_i$; the reader interested in the details of the calculations should consult the original papers. We have, from Bijnens, Colangelo and Talavera (1998) and Colangelo, Gasser and Leutwyler (2001),

$$\begin{aligned} &\bar{l}_1 = -0.4 \pm 0.6, \quad \bar{l}_2 = 6.0 \pm 1.3, \quad \bar{l}_3 = 2.9 \pm 2.4, \quad \bar{l}_4 = 4.4 \pm 0.2 \quad \text{(CGL)}; \\ &\bar{l}_5 = 13.0 \pm 1.0, \quad \bar{l}_6 = 16 \pm 1 \qquad \text{(BCT)}. \end{aligned} \tag{9.4.1}$$

The value of $\bar{l}_7$ is not known with any accuracy; an estimate for it is $\bar{l}_7 \sim 5 \times 10^{-3}$ (Gasser and Leutwyler, 1984).

Actually, the determinations in (9.4.1) include estimates of two loop effects. One should however remember that, as discussed in Sect. 7.6, the calculations of Colangelo, Gasser and Leutwyler (2001) of $\pi\pi$ scattering –on which their estimates of several of the $\bar{l}_i$ are based– are biased by up to 2 standard deviations, and their errors are underestimated by a factor between 1.5 and 2. Something similar happens for $\bar{l}_4$; see Subsect. 9.4.2 below. This should affect the numbers in (9.4.1).

Some of these constants can be calculated independently with greater accuracy (but only at the one loop level) using the results reported in Sect. 7.6 (cf. Table III) here. So, from the combination $a_{0+} = \frac{1}{3}[a_2^{(0)} - a_2^{(2)}]$, that we evaluated very precisely, there follows the value

$$\bar{l}_2 = 5.97 \pm 0.07. \tag{9.4.2a}$$

Likewise, the constant $\bar{l}_1$ can be deduced from the value of $a_{00} = \frac{1}{3}[a_2^{(0)} + 2a_2^{(2)}]$ that follows from the Froissart–Gribov representation for $\pi^0\pi^0$, and the value of $\bar{l}_2$. We find

$$\bar{l}_1 = -1.47 \pm 0.24, \tag{9.4.2b}$$

somewhat larger in magnitude than the value given in (9.4.1).

Use of the quadratic charge radius of the pion as input (see Eq. (9.4.5) below) allows us to get also an accurate evaluation of $\bar{l}_6$:

$$\bar{l}_6 = 16.35 \pm 0.14, \tag{9.4.2d}$$

which, in turn, implies a slightly more precise value for $\bar{l}_5$:

$$\bar{l}_5 = 13.7 \pm 0.7, \tag{9.4.2e}$$

[54] The tests are less impressive than what they may look at first sight. In fact, chiral perturbation theory is just a (very convenient) way to summarize properties that hold in any local field theory: analyticity, crossing and unitarity, plus the dynamical properties embodied in the constants f_π, μ and the $\bar{l}_i$. Thus for example, by comparing the r.h. sides of the Olsson sum rule (7.4.8) and the Froissart-Gribov representation for a_1, (7.5.4), we discover that, in any local field theory we must have the equality $2a_0^{(0)} - 5a_0^{(2)} = 18\mu^2 a_1 + O(\mu^4)$.

Our improvements, however, may be challenged because of the possible contributions of two loop corrections, that we have not taken into account, and of electromagnetic corrections, that may be important. For the second, see next section for a discussion of a few examples. As for higher order ch.p..t corrections, we will say a few words at the end of Subsect. 9.4.3.

In what respects S and P waves scattering lengths and effective range parameters, substituting the $\bar{l}_i$ into (9.3.19) we find the one-loop chiral perturbation theory predictions: in units of M_π,

$$\begin{aligned} a_0^{(0)} &= (0.207 \pm 0.003), \quad a_0^{(2)} = (-0.043 \pm 0.002), \quad a_1 = (38.0 \pm 0.4) \times 10^{-3}; \\ b_0^{(0)} &= (0.255 \pm 0.003), \quad b_0^{(0)} = (-0.076 \pm 0.001), \quad b_1 = (4.69 \pm 0.14) \times 10^{-3}. \end{aligned} \tag{9.4.3}$$

In the calculation we have used the values of the $\bar{l}_i$ in (9.4.2) and (9.4.12) below in preference to those of (9.4.1), when possible.

The predictions for the $a_0^{(2)}$, $b_0^{(I)}$, b_1 are in agreement with the experimental values we found before (Table II, Subsect. 7.6.2); the value of $a_0^{(0)}$ in (9.4.3) indicates that the higher order correction (two loop or otherwise) for this quantity must be relatively large. The value of the P wave scattering length in (9.4.3) is also compatible with the result of the direct fit, $a_1 = (38.6 \pm 1.2) \times 10^{-3}\, M_\pi^{-3}$. It is also compatible with the results of other authors:

$$a_1 = \begin{cases} (37.9 \pm 0.5) \times 10^{-3}\ M_\pi^{-3} & \text{(Colangelo, Gasser and Leutwyler, 2001)} \\ (37.2 \pm 2) \times 10^{-3}\ M_\pi^{-3} & \text{(Ananthanarayan et al., 2001)} \\ (38.0 \pm 2) \times 10^{-3}\ M_\pi^{-3} & \text{(Amorós, Bijnens and Talavera, 2000).} \end{cases}$$

We also note, as consistency tests, that the value of $\bar{l}_4$ that follows from a_1, via Eq. (9.3.19b), is compatible, within the rather large error, with (9.4.1a), as one gets $\bar{l}_4 = 5.5 \pm 2.0$, and that b_1 would yield a number for $\bar{l}_2 - \bar{l}_1$ compatible (also within errors) with what we found before. In what respects to b_1, however, a better, nontrivial consistency test is obtained by eliminating $\bar{l}_1$, $\bar{l}_2$ between (9.3.19a), (9.3.20b). We find the relation

$$b_1 = \tfrac{5}{2}\,[3a_{0+} - a_{00}] + (\tfrac{97}{120} + \tfrac{1}{8})\frac{1}{288\pi^3 f_\pi^4 M_\pi}, \tag{9.4.6a}$$

in which some of the larger higher order corrections cancel and, moreover, the r.h.s. is dominated by a_{0+}, which is known accurately. This gives, using the PY values for the a_{00}, a_{0+} described in Table III (Subsect. 7.6.3),

$$b_1 = (4.68 \pm 0.20) \times 10^{-3}\, M_\pi^{-5}, \tag{9.4.6b}$$

in excellent agreement with the value deduced from the electromagnetic form factor of the pion (see again Table III), $b_1 = (4.47 \pm 0.27) \times 10^{-3}\, M_\pi^{-5}$.

The very precise calculation of the pion form factor possible with the Omnès–Muskhelishvili techniques also allows a direct determination of a second order (two loop) parameter. According to Gasser and Meissner (1991), Colangelo, Finkelmeir and Urech (1996), and Fearing and Scherer (1966), one has

$$c_\pi = \frac{1}{16\pi^2 f_\pi^2}\left\{\frac{1}{60 M_\pi^2} + \frac{1}{16\pi^2 f_\pi^2}\bar{f}_2\right\}. \tag{9.4.5a}$$

With the value $c_\pi = 3.60 \pm 0.03$ GeV^{-4} given in de Trocóniz and Ynduráin (2002), this implies

$$\bar{f}_2 = 5.520 \pm 0.056. \tag{9.4.5b}$$

Note, however, that this result is purely formal; indeed, the nominally leading term $(1/60M_\pi^2)$ is in fact *smaller* than the nominally second order one, $\bar{f}_2/(16\pi^2 f_\pi^2)$. This shows clearly the limitations of chiral perturbation theory.

Another example is the charge radius of the pion, for which one has, to second order (Gasser and Meissner, 1991, and Colangelo, Finkelmeir and Urech, 1996),

$$\langle r_\pi^2\rangle = \frac{1}{16\pi^2 f_\pi^2}\left[\bar{l}_6 - 1 + \frac{M_\pi^2}{16\pi^2 f_\pi^2}\bar{f}_1\right]. \tag{9.4.6}$$

Here the two loop term is smaller than the leading one, for reasonable values of $\bar{f}_1$, but perhaps not totally negligible, given the accuracy of the experimental result for $\langle r_\pi^2\rangle$. The value of $\bar{l}_6$ given above was obtained neglecting $\bar{f}_1$; a value of this quantity of the order of $\bar{f}_2$ would alter $\bar{l}_6$ by 1%, a variation slightly larger than the nominal error in (9.4.2d).

It is also possible to give a prediction, based on chiral perturbation theory and the Froissart–Gribov representation, for scattering lengths for large l. We will give some details for the amplitude for isospin 1 in the t channel,

$$F^{(I_t=1)} = \tfrac{1}{3}F^{(I_s=0)} + \tfrac{1}{2}F^{(I_s=1)} - \tfrac{5}{6}F^{(I_s=2)}.$$

The corresponding scattering lengths are given by Eq. (7.5.4),

$$2a_l^{(1)} = \frac{\sqrt{\pi}\,\Gamma(l+1)}{4M_\pi\Gamma(l+3/2)}\int_{4M_\pi^2}^{\infty}\mathrm{d}s\,\frac{\operatorname{Im}F^{(I_t=1)}(s,4M_\pi^2)}{s^{l+1}}$$

and the factor 2 in the l.h. side is due to the identity of the pions. As $l\to\infty$, only the behaviour of $\operatorname{Im}F^{(I_t=1)}(s,4M_\pi^2)$ near threshold matters; hence we can replace

$$\begin{aligned}\operatorname{Im}F^{(I_t=1)}(s,4M_\pi^2) &\simeq \tfrac{1}{3}\operatorname{Im}F^{(I_s=0)} - \tfrac{5}{6}\operatorname{Im}F^{(I_s=2)}\\ &\simeq 2\times\frac{2s^{1/2}}{\pi k}\left\{\tfrac{1}{3}\sin^2\delta_0^{(0)}(s) - \tfrac{5}{6}\sin^2\delta_0^{(2)}(s)\right\}\\ &\simeq 2\times\frac{2s^{1/2}}{\pi k}\left\{\tfrac{1}{3}\left[a_0^{(0)}\right]^2 - \tfrac{5}{6}\left[a_0^{(2)}\right]^2\right\}.\end{aligned} \tag{9.4.7}$$

The factor 2 in the r.h. side also comes from the identity of the pions. Replacing the $a_0^{(I)}$ by their values at leading order in chiral perturbation theory, Eq. (9.3.19), we find that we can approximate

$$\operatorname{Im}F^{(I_t=1)}(s,4M_\pi^2) \simeq 2\frac{13M_\pi^2 s^{1/2}k}{512\pi^3 f_\pi^4}.$$

Substituting into the Froissart–Gribov representation, and performing the integral we get the result

$$a_l^{(1)} \underset{l\to\infty}{\simeq} \frac{13M_\pi^{3-2l}}{2\times 4^{l+5}\pi^2 f_\pi^4}\frac{\Gamma(l+1)\Gamma(l-1)}{\Gamma(l+3/2)\Gamma(l+1/2)} \underset{l\to\infty}{\simeq} \frac{13M_\pi^{3-2l}}{2\times 4^{l+5}\pi^2 l^2 f_\pi^4} \tag{9.4.8a}$$

and, in the last step, we have used the asymptotic properties of the gamma function and replaced μ by M_π. The same method gives a prediction for even waves; for example, for $I=0$ we have,

$$a_l^{(0)} \underset{l\to\infty}{\simeq} \frac{23M_\pi^{3-2l}}{2\times 4^{l+5}\pi^2 l^2 f_\pi^4}. \tag{9.4.8b}$$

Gasser and Leutwyler (1983) have produced a formula for all scattering lengths with $I = 1$ to leading order in chiral perturbation theory that is exact (and not only valid for $l \to \infty$), based on a direct calculation. They give the expression, valid for $l \geq 3$,

$$a_l^{(1)}(\text{G.}-\text{L.}) = \frac{M_\pi^{3-2l}}{512\pi^3 f_\pi^4} \frac{l!(l-3)!}{[(2l+1)!!]^2} \left(13l^2 + 5l - 22\right). \tag{9.4.9}$$

If, in (9.4.9), we replace $(2l+1)!! = \Gamma(2l+2)/2^l\Gamma(l+1)$ and use the duplication formula of the gamma function we find

$$\begin{aligned} a_l^{(1)}(\text{G.}-\text{L.}) = &\frac{13M_\pi^{3-2l}}{2\times 4^{l+5}\pi^2 f_\pi^4} \frac{\Gamma(l+1)\Gamma(l-1)}{\Gamma(l+3/2)\Gamma(l+1/2)} \frac{l^2}{(l-2)(l+1/2)} \\ &\times \left(1 + \frac{5}{13l} - \frac{22}{13l^2}\right) \end{aligned}$$

which, as $l \to \infty$, agrees with (9.4.7a).

The calculation using leading order chiral perturbation theory yields the figure $a_3 = 1.8 \times 10^{-5}\, M_\pi^{-7}$. From the Froissart–Gribov representation we found in Sect. 7.6 results ranging between 5.4 and 6.7, in units of $10^{-5}\, M_\pi^{-7}$. A large disagreement (a factor of 3 to 4) is thus found between the leading chiral perturbation result and the results based on experiment. It is not clear to the present author which is the reason for this disagreement; in fact, as far as I know, the Gasser–Leutwyler result in Eq. (9.4.9) has not been checked by an independent calculation.

For $l = 4$, our expression (9.4.7b) gives $a_4^{(0)} = 1.4 \times 10^{-5}\, M_\pi^{-9}$ while the "experimental" value (from the Froissart–Gribov representation) is $(0.8 \pm 0.2) \times 10^{-5}\, M_\pi^{-9}$. The disagreement for a_3, and the difference between the two values for $a_4^{(0)}$, show that, in some cases, the corrections due to subleading effects in chiral perturbation theory may be very large: for the quantity a_3, two to three times the nominally leading term. This is not surprising; as is clear in our derivation, the value we obtain depends on the *square* of the S wave scattering lengths, for which one loop corrections are at least of 25%.

9.4.2. The scalar form factor of the pion

For the scalar form factor of the pion, defined in (7.2.5), chiral perturbation theory gives[55]

$$\begin{aligned} F_S(0) &= M_\pi^2 \left\{1 - \frac{M_\pi^2}{32\pi^2 f_\pi^2}(\bar{l}_3 - 1)\right\}, \\ \langle r^2_{\text{S},\pi}\rangle &= \frac{3}{8\pi^2 f_\pi^2}\left\{\bar{l}_4 - \tfrac{13}{12}\right\}. \end{aligned} \tag{9.4.9}$$

It is also possible to give a formula for the quadratic scalar radius in terms of observable quantities

[55] Two loop evaluations of the scalar radii may be found in Gasser and Meißner (1991) and Frink, Kubis and Meißner (2002).

(Gasser and Leutwyler, 1985b):

$$\begin{aligned}\langle r^2_{\mathrm{S},\pi}\rangle &= \frac{6}{m_K^2 - M_\pi^2}\left(\frac{f_K}{f_\pi} - 1\right) + \delta_3;\\ \delta_3 &= -\frac{1}{64\pi^2 f_\pi^2}\,\frac{1}{m_K^2 - M_\pi^2}\left\{6(2m_K^2 - M_\pi^2)\log\frac{m_K^2}{M_\pi^2} + 9m_\eta^2\log\frac{m_\eta^2}{M_\pi^2}\right.\\ &\quad\left. - 2(m_K^2 - M_\pi^2)\left(10 + \tfrac{1}{3}\frac{M_\pi^2}{m_\eta^2}\right)\right\}\end{aligned} \tag{9.4.10a}$$

and $m_\eta = 547$ MeV. The same authors give also a formula for the mixed kaon-pion radius, defined by

$$\langle\pi(p)|(m_s - m_u)\bar{u}s(0)|K(p')\rangle \underset{t\to 0}{\simeq} f_{K\pi}(0)\Big\{1 + \tfrac{1}{6}\langle r^2_{\mathrm{S},K\pi}\rangle\, t\Big\}:$$

$$\begin{aligned}\langle r^2_{\mathrm{S},K\pi}\rangle &= \frac{6}{m_K^2 - M_\pi^2}\left(\frac{f_K}{f_\pi} - 1\right) + \delta_2;\\ \delta_2 &= -\frac{1}{192\pi^2 f_\pi^2}\left\{15h_2(M_\pi^2/m_K^2) + \frac{19m_K^2 + 3m_\eta^2}{m_K^2 + m_\eta^2}h_2(m_\eta^2/m_K^2) - 18\right\},\\ h_2(x) &= \tfrac{3}{2}\left(\frac{1+x}{1-x}\right)^2 + \frac{3x(1+x)}{(1-x)^3}\log x.\end{aligned} \tag{9.4.10b}$$

From these formulas follow the values, respectively,

$$\langle r^2_{\mathrm{S},\pi}\rangle_{\mathrm{GL}} = 0.55 \pm 0.15\ \mathrm{fm}^2. \tag{9.4.11a}$$

$$\langle r^2_{\mathrm{S},K\pi}\rangle_{\mathrm{GL}} = 0.20 \pm 0.05\ \mathrm{fm}^2, \tag{9.4.11b}$$

The value for the first that we obtained from experiment in Eq. (7.2.17), $\langle r^2_{\mathrm{S},\pi}\rangle = 0.75 \pm 0.07\,\mathrm{fm}^2$, and the one for the second following from K_{l3} decays,[56] $\langle r^2_{\mathrm{S},K\pi}\rangle = 0.31 \pm 0.06\,\mathrm{fm}^2$, are therefore about $2\,\sigma$ above the chiral theory prediction to one loop, Eqs. (9.4.11). This indicates that two loop corrections may be important for the relations (9.4.10). If we neglect them, however, we get a very precise value for the constant $\bar{l}_4$:

$$\bar{l}_4 = 5.4 \pm 0.5, \tag{9.4.12}$$

a number substantially larger than that given in (9.4.1) which, unfortunately, is based partially on the evaluation of Donoghue, Gasser and Leutwyler (1990) which, as discussed in Ynduráin (2003a) is not quite reliable within its estimated errors.

[56] See e.g. Ynduráin (2003) for details.

9.4.3. Summary of ch.p.t. predictions for $\pi\pi$ scattering

We will now present a summary of the two previous subsections. We will use the improved values of the $\bar{l}_i$ parameters we have obtained, *to one loop*:

$$\begin{aligned} &\bar{l}_1 = -1.47 \pm 0.24, \quad \bar{l}_2 = 5.97 \pm 0.07, \quad \bar{l}_3 = 2.9 \pm 2.4, \\ &\bar{l}_4 = 5.4 \pm 0.5, \quad \bar{l}_5 = 13.7 \pm 0.7, \quad \bar{l}_6 = 16.35 \pm 0.14. \end{aligned} \tag{9.4.13a}$$

(Actually, only the first four $\bar{l}_i$ enter for $\pi\pi$ scattering). With this, we may evaluate the low energy $\pi\pi$ parameters, and we find the values of Table IV:

Quantity	Exp. value (PY)	1 loop ch.p.t.
$a_0^{(0)}$	0.230 ± 0.010	0.207 ± 0.003 [0.157]
$a_0^{(2)}$	-0.0422 ± 0.0022	-0.043 ± 0.002 [-0.045]
$b_0^{(0)}$	0.268 ± 0.010	0.255 ± 0.003 [0.179]
$b_0^{(2)}$	-0.071 ± 0.004	-0.076 ± 0.001 [-0.089]
a_1	$(38.3 \pm 0.8) \times 10^{-3}$	$(38.0 \pm 0.04) \times 10^{-3}$ [33.6]
b_1	$(4.56 \pm 0.26) \times 10^{-3}$	$(4.69 \pm 0.14) \times 10^{-3}$
$a_2^{(0)}$	$(18.0 \pm 0.2) \times 10^{-4}$	input
$a_2^{(2)}$	$(2.2 \pm 0.2) \times 10^{-4}$	input
$\langle r^2_{S,\pi} \rangle$	0.77 ± 0.07 fm^2	input

Comparison of evaluations of low energy parameters from experiment, and from one loop chiral perturbation theory. The experimental numbers for $\pi\pi$ scattering are taken from the Peláez–Ynduráin $\pi\pi$ amplitudes in Tables II, III; $\langle r^2_{S,\pi} \rangle$ is as given in Eq. (7.2.17). The values of a_1, b_1 have been (very slightly) improved by compounding the independent determinations from the pion form factor, and from the Froissart–Gribov representation. In brackets: the quantities to leading order.

TABLE IV

Except for $a_0^{(0)}$, and a little for $b_0^{(0)}$, the agreement is perfect, so no there is no need of two loop corrections at the present level of accuracy for the remaining low energy parameters. For both $a_0^{(0)}$ and $b_0^{(0)}$ already the one loop corrections are quite large, so we expect also large two loop corrections, just by renormalization group arguments (Colangelo, 1995). In fact, a detailed estimate for $a_0^{(0)}$ (Bijnens et al., 1996) gives

$$\delta_{2\text{ loop}}\, a_0^{(0)} = 0.017 \pm 0.002$$

which brings the ch.p.t. value of $a_0^{(0)}$ to

$$a_0^{(0)} = 0.224 \pm 0.003 \quad \text{[incl. two loop]},$$

well inside the experimental error bars.

9.5. Weak and electromagnetic interactions. The accuracy of chiral perturbative calculations

Weak and electromagnetic interactions, at tree level, can be introduced by making the standard minimal replacement in the covariant derivatives; for e.g., electromagnetic interactions, $\nabla_\mu \to \nabla_\mu - eA_\mu$. In this way one can calculate chiral dynamics values of quantities like the pion electromagnetic form factor, or strong interactions corrections to weak decays. Another matter are virtual electromagnetic corrections. These break chiral invariance, and can be large. For example, the $\pi^+ - \pi^0$ mass difference is of order $(m_d - m_u)^2$ in chiral perturbation theory; the corresponding calculation yields a very small number,

$$m^2_{\pi^+} - m^2_{\pi^0} = (m_d - m_u)^2 \frac{2B^2 \bar{l}_7}{f^2_\pi}, \quad B = -\langle \bar{q}q \rangle / f^2$$

which would give $m_{\pi^+} - m_{\pi^0} \sim 0.2$ MeV . However, the experimental value is $m_{\pi^+} - m_{\pi^0} = 4.6\,\mathrm{MeV}$. In this case one can use current algebra techniques to estimate the electromagnetic contribution, which is indeed of the right order of magnitude (Das, Mathur and Okubo, 1967), but in general this is not possible; we expect (generally unknown) electromagnetic corrections of something up to this order of magnitude, $\sim 3.4\%$, to chiral perturbation theory calculations.

A case in which the electromagnetic corrections to the constants $\bar{l}_i$ is known is that of $\bar{l}_6$. The value reported in (9.4.2d) above is actually an average of those obtained from the charge radii of the pion with $\pi^0\pi^+$ and $\pi^+\pi^-$. If we use only the last (associated with the ρ^0), hence the parameters of (6.3.5c), we find instead

$$\bar{l}_6 = 16.07 \pm 0.18. \tag{9.5.1}$$

The difference between the two, a 1.5%, is the minimum extra error due to electromagnetic corrections that we should append to all the determinations of chiral perturbation theory parameters.

A place where isospin violation corrections are potentially large are the scattering lengths. If we repeat the fits of de Trocóniz and Ynduráin (2002) without imposing the constraint $a_1 = (38 \pm 3) \times 10^{-3}\, M_\pi^{-3}$, and fit separately $\pi^+\pi^-$ and τ decay data we find the numbers,

$$a_1(\pi^+\pi^-) = (37 \pm 3) \times 10^{-3}\, M_\pi^{-3},$$
$$a_1(\pi^+\pi^0) = (43 \pm 3) \times 10^{-3}\, M_\pi^{-3}.$$

The two values overlap, but only barely; a difference of the order of 3×10^{-3} (in units of M_π) cannot be excluded.

Another question is the scale of higher corrections in chiral perturbation theory. For the *logarithmic* corrections we know that this scale is $1/(4\pi f_\pi)^2$, so for energies of the order of M_π we expect corrections $O(M^2_\pi/(4\pi f_\pi)^2) \simeq 1.4\%$. However, this estimate forgets the *constant* contributions to the $\bar{l}_i$. There is no reason why they should be suppressed by powers of $1/(4\pi f_\pi)^2$; all we can expect is a suppression of order $O(M^2_\pi/\Lambda^2_0)$, with Λ^2_0 proportional to the QCD parameter $\Lambda \sim 400$ MeV (for 2 or 3 flavours). In some cases the coefficients of these terms will be small; in other they may be large. This last situation occurs for example for the S0 wave in $\pi\pi$ scattering, where the correction necessary to get agreement between the leading value obtained from chiral dynamics, $a^{(0)}_0 = 0.16\, M_\pi^{-1}$, with the experimental values which vary between $a^{(0)}_0 \simeq 0.24\, M_\pi^{-1}$ and $a^{(0)}_0 \simeq 0.21\, M_\pi^{-1}$ is at least a third of the leading one.[57]

[57] This possibility is particularly relevant in view of the doubts expressed by other researches on some aspects of chiral perturbation theory; see, for example, Fuchs, Sazdjian and Stern (1991); Knecht et al. (1996).

In this context, we would like to emphasize that the situation is even worse for the quantities a_3, c_π, for which the leading order calculations miss the experimental values by a factor ~ 3.

In some cases the size of the corrections may be gleaned from external arguments. For example, for the isospin zero S-wave in $\pi\pi$ scattering, chiral dynamics implies that its imaginary part should be suppressed with respect to the real part, at energy squared s, by powers s/Λ_0^2. However, already at $s^{1/2} = 500$ MeV, i.e., only 200 MeV above threshold, real and imaginary part are of the same order of magnitude; so, we would expect poor convergence in this case, as indeed happens.

Appendices

Appendix A: Summary of low energy, $s^{1/2} \leq 1.42$ GeV partial waves

A.1. The S wave with isospin zero below 960 MeV

We impose the Adler zero at $s = \frac{1}{2}{M_\pi}^2$ (no attempt is made to vary this), and a resonance with mass M_σ, a free parameter. We write

$$\cot\delta_0^{(0)}(s) = \frac{s^{1/2}}{2k}\,\frac{{M_\pi}^2}{s-\frac{1}{2}{M_\pi}^2}\,\frac{M_\sigma^2-s}{M_\sigma^2}\,\psi(s), \tag{A.1}$$

and

$$\psi(s) = \sum_n B_n\, w(s)^n; \quad w(s) = \frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}, \quad s_0 = 4m_K^2; \tag{A.2}$$

we have taken $m_K = 0.496\,\text{GeV}$. We will fit the phases that follow from K_{l4} decays, with the P wave as given below. We include in the fit the value $\delta_0^{(0)}(M_K^2) = 43.3 \pm 2.3^\circ$, as discussed in the text. Finally, in the region[58] where $s^{1/2}$ is between 0.81 GeV and 0.98 GeV, we also include some phases of Protopopescu et al. (1973), and of the s-wave solutions of Estabrooks and Martin (1974), as given in Subsect. 6.4.2.

Solution B2. If we take two parameters in (A.4a) we find what we call *solution 2B*,

$$\begin{aligned}
\cot\delta_0^{(0)}(s) &= \frac{s^{1/2}}{2k}\,\frac{M_\pi^2}{s-\frac{1}{2}M_\pi^2}\,\frac{M_\sigma^2-s}{M_\sigma^2}\,\left\{B_0 + B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}\right\};\\
B_0 &= 21.04, \quad B_1 = 6.62, \quad M_\sigma = 782 \pm 24\ \text{MeV}; \quad \frac{\chi^2}{\text{d.o.f.}} = \frac{15.7}{19-3};\\
a_0^{(0)} &= (0.230 \pm 0.010)\times M_\pi^{-1}; \quad \delta_0^{(0)}(M_K) = 41.0^\circ\ \pm 2.1^\circ\,;
\end{aligned} \tag{A.3}$$

this fit we take to be valid for $s^{1/2} \leq 0.96$ GeV. Uncorrelated errors are obtained if replacing the B_i by the parameters x, y with

$$B_0 = y - x; \quad B_1 = 6.62 - 2.59x; \quad y = 21.04 \pm 0.75, \quad x = 0 \pm 2.4. \tag{A.4a}$$

The corresponding phase shift is shown in Fig. 6.4.3 in the main text.

[58] This is the energy region in which most *experimental* phase shifts agree one with the other, within errors.

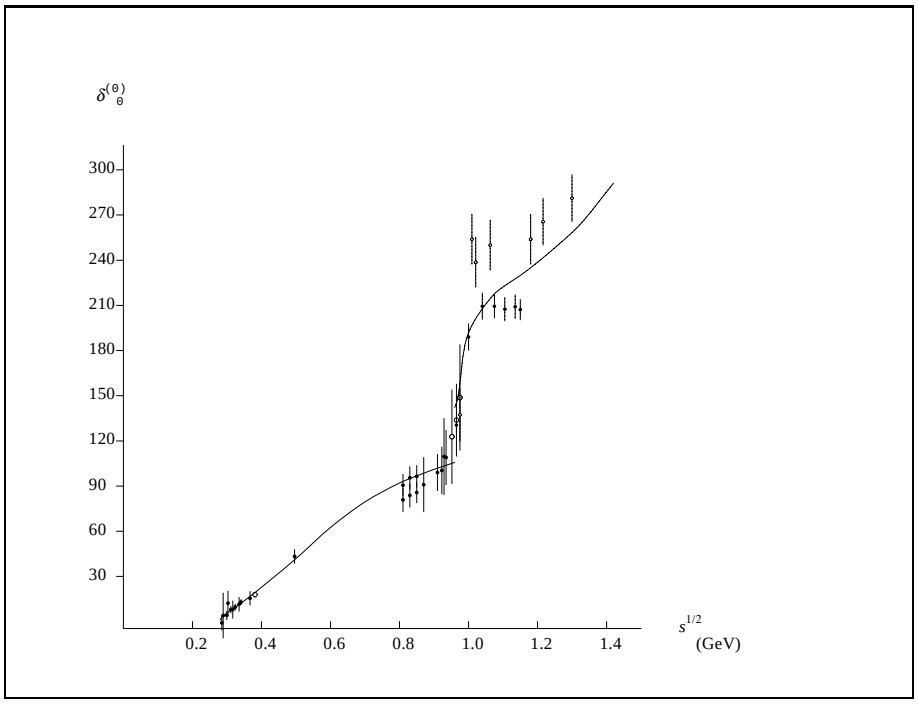

FIGURE A.1. The S0 phase shifts and inelasticities corresponding to Eqs. (A.4), (A.6). Also shown are the experimental points included in the fits.

Solution B3. With three parameters a new minimum (*solution 3B*) appears:

$$\begin{aligned}\cot\delta_0^{(0)}(s) &= \frac{s^{1/2}}{2k}\,\frac{M_\pi^2}{s-\frac{1}{2}M_\pi^2}\,\frac{M_\sigma^2-s}{M_\sigma^2}\\ &\quad\times\left\{B_0+B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}+B_2\left[\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}\right]^2\right\};\\ s_0^{1/2} &= 2M_K;\quad \chi^2/\text{d.o.f.} = 11.1/(19-4).\\ M_\sigma &= 806\pm 21,\ B_0 = 21.91\pm 0.62,\ B_1 = 20.29\pm 1.55,\ B_2 = 22.53\pm 3.48;\\ a_0^{(0)} &= (0.226\pm 0.015)\ M_\pi^{-1}.\end{aligned} \tag{A.5}$$

The central values in (A.5) are something between (A.4) and the solution of Colangelo, Gasser and Leutwyler (2001), which it comprises.

A.2. The $I = 0$ S wave between 960 MeV and 1420 MeV

We here present a semi-phenomenological fit to $\delta_0^{(0)}$ and $\eta_0^{(0)}$, as discussed in the main text. We

write

$$\cot\delta_0^{(0)}(s) = c_0\,\frac{(s-M_\sigma^2)(M_f^2-s)}{M_f^2 s^{1/2}}\,\frac{|k_2|}{k_2^2},\quad k_2=\frac{\sqrt{s-4m_K^2}}{2} \tag{A.6a}$$

and

$$\eta_0^{(0)} = 1-\left(c_1\frac{k_2}{s^{1/2}}+c_2\frac{k_2^2}{s}\right)\,\frac{M'^2-s}{s}. \tag{A.6b}$$

In the first, c_0 and M_σ are free parameters and we fix $M_f = 1320$ MeV. In (A.6b), the free parameters are c_1, c_2 and we adjust M' to the inelasticity of Hyams et al. on the $f_0(1370)$. For the selection of data points, see the main text, subsect 6.4.3. We find,

$$\begin{aligned} &c_0 = 1.36\pm 0.05,\quad M_\sigma = 802\pm 11\ \text{MeV},\ M' = 1500\ \text{MeV};\quad \frac{\chi^2}{\text{d.o.f.}} = \frac{36.2}{14-2};\\ &c_1 = 6.7\pm 0.17,\quad c_2 = -17.6\pm 0.8;\quad \chi^2/\text{d.o.f.} = 7.7/(8-2). \end{aligned} \tag{A.6c}$$

The errors for c_0, M_σ correspond to *three* standard deviations, since we have a χ^2 /d.o.f. $\simeq 3$. The value of M_σ coincides, *grosso modo*, with what we found below $\bar{K}K$ threshold.

The qualitative features of the fits to the S0 wave in the whole range $s^{1/2}\leq 1.42$ GeV may be seen in Fig. A.1.

A.3. Parametrization of the S wave for $I = 2$

As discussed in the main text, we consider three sets of experimental data. The first corresponds to solution A in the paper by Hoogland et al. (1977), who use the reaction $\pi^+p\to\pi^+\pi^+n$; the set from the work of Losty et al. (1974), who analyze instead $\pi^-p\to\pi^-\pi^-\Delta$; and the set of Cohen et al. (1973), obtained fron pion-deuteron scattering (which, however, were *not* included in the fit). We will not consider the so-called solution B in the paper of Hoogland et al.

For isospin 2, there is no low energy resonance, but $f_0^{(2)}(s)$ presents the feature that a zero is expected (and, indeed, confirmed by the fits) in the region $0<s<4M_\pi^2$. This zero of $f_0^{(2)}(s)$ is related to the so-called Adler zeros and, to lowest order in chiral perturbation theory, occurs at $s=2z_2^2$ with $z_2=M_\pi$. In view of this, we extract the zero and write

$$\cot\delta_0^{(2)}(s) = \frac{s^{1/2}}{2k}\,\frac{{M_\pi}^2}{s-2z_2^2}\left\{B_0+B_1 w(s)\right\}.$$

The inelasticity of $\pi\pi$ scattering in this channel, say $\pi^{\pm,0}\pi^\pm$, is very small until one crosses the $\rho^{\pm,0}\rho^\pm$ threshold. In our calculations we will take $s_0^{1/2}=1.45$ GeV for the effective opening of the inelastic channels.

We can improve on the quality of the results by requiring, simultaneously with the fit to the data, fulfillment of the Olsson sum rule, within the errors produced by the remaining waves. If moreover we fix $z_2=M_\pi$, and include all experimental data of Losty et al. and Hoogland et al., solution A, (up to $s^{1/2}=1350$ MeV) in the fit we find

$$\begin{aligned} \cot\delta_0^{(2)}(s) &= \frac{s^{1/2}}{2k}\,\frac{M_\pi^2}{s-2z_2^2}\left\{B_0+B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}\right\};\\ s_0^{1/2} &= 1.45\ \text{GeV};\quad \chi^2/\text{d.o.f.} = 17.2/(19-2).\\ B_0 &= -118\pm 2.5,\quad B_1 = -105\pm 2.5,\quad z_2 = 139.57\ \text{MeV}\ \text{[fixed]}. \end{aligned} \tag{A.7a}$$

Then one has $a_0^{(2)} = -0.0422 \pm 0.0022$.

The inelasticity may be obtained fitting the data of Cohen et al. (1973) and Losty et al. (1974). One finds

$$\eta_0^{(2)}(s) = 1 - c\left(1 - M_{\rm eff}^2/s\right)^{3/2}; \quad M_{\rm eff} = 0.96 \ \text{GeV}, \quad c = 0.28 \pm 0.12, \tag{A.7b}$$

which is valid for $s^{1/2} \geq 0.96$ GeV. The plot of $\delta_0^{(2)}$ may be seen in Fig. 7.6.2 in the main text. We will take (A.7) to be valid up to 1.42 GeV.

A.4. The P wave below 1 GeV

We will consider first the P wave for $\pi\pi$ scattering for energies below the region were the inelasticity reaches the 2% level; say, below $s_0 = 1.1$ GeV^2. We will neglect for the moment isospin invariance violations due to e.m. interactions or the mass difference of the u, d quarks.

The best values for our parameters are actually obtained from fits to the pion form factor. If we take systematic normalization errors into account, but neglect isospin violation, we find

$$\begin{gathered}\cot\delta_1(s) = \frac{s^{1/2}}{2k^3}\,(M_\rho^2 - s)\,\left\{B_0 + B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}\right\};\\ s_0^{1/2} = 1.05 \ \text{GeV}; \quad \chi^2/\text{d.o.f.} = 227/(209-3).\\ M_\rho = 773.5 \pm 0.85 \ \text{MeV}, \quad B_0 = 1.071 \pm 0.007, \quad B_1 = 0.18 \pm 0.05.\\ [s^{1/2} \leq 1.0 \ \text{GeV}]\end{gathered} \tag{A.8a}$$

and

$$\begin{aligned}a_1 &= (38.6 \pm 1.2) \times 10^{-3}\ M_\pi^{-3}, \quad b_1 = (4.47 \pm 0.29) \times 10^{-3}\ M_\pi^{-5},\\ \Gamma_\rho &= 145.5 \pm 1.1 \ \text{MeV}\,.\end{aligned}$$

Although the values of the experimental $\pi\pi$ phase shifts were *not* included in the fit, the phase shifts that (A.8) implies are en very good agreement with them, as shown in Fig. 6.3.2.

Eqs. (A.8) above were evaluated with an average of information on $F_\pi(t)$ from the two channels that contain the $I = 1$ P wave, that is, $\pi^+\pi^-$ (dominated by the ρ^0) and $\pi^0\pi^+$, dominated by the ρ^+. Experimentally the first are obtained from processes $e^+e^- \to \pi^+\pi^-$, $\pi e \to \pi e$; the second from the decays $\tau \to \nu_\tau\pi^0\pi^+$. The values of the parameters for a pure ρ^0 ($\pi^+\pi^-$) are slightly different. Including systematic errors in the analysis we would find

$$\begin{aligned}B_0 &= 1.065 \pm 0.007, \quad B_1 = 0.17 \pm 0.05, \quad M_{\rho^0} = 773.1 \pm 0.6,\\ \Gamma_{\rho^0} &= 147.4 \pm 1.0 \ \text{MeV};\end{aligned} \tag{A.8b}$$

a_1, b_1 do not change appreciably.

A.5. The P wave for $1\,\text{GeV} \leq s^{1/2} \leq 1.42\,\text{GeV}$

For the imaginary part of the P wave between 1 GeV and 1.42 GeV we use an empirical formula, obtained adding a resonance (with mass 1.45 GeV) to a nonresonant background:

$$\text{Im}\,\hat{f}_1(s) = \frac{1}{1 + [\lambda + 1.1k_2/s^{1/2}]^2} + \text{BR}\frac{M_{\rho'}^2\Gamma^2\left[k/k(M_{\rho'}^2)\right]^6}{(s - M_{\rho'}^2)^2 + M_{\rho'}^2\Gamma^2\left[k/k(M_{\rho'}^2)\right]^6}; \tag{A.9}$$

$[1.0 \leq s^{1/2} \leq 1.42 \ \text{GeV}] \quad M_{\rho'} = 1.45 \ \text{GeV}, \quad \Gamma = 0.31 \ \text{GeV}, \quad \lambda = 2.6 \pm 0.2.$

The parameters of the $\rho(1450)$ are poorly known. We have taken BR $= 0.25 \pm 0.05$, but the error could well be twice as large.

A.6. Parametrization of the D wave for $I = 0$

The D wave with isospin 0 in $\pi\pi$ scattering presents a resonance below 1.42 GeV: the $f_2(1270)$, that we will denote simply by f_2. Experimentally, $\Gamma_{f_2} = 185 \pm 4$ GeV. The f_2 couples mostly to $\pi\pi$; to a 15% accuracy we may neglect inelasticity up to $s_0^{1/2} = 1.43$ GeV.

We will first fit data on $\delta_2^{(0)}$ altogether neglecting inelasticity, which we will then add by hand. The data are scanty, and of poor quality. The phase shifts of Protopopescu et al. (1973) cover only the range $810 \leq s^{1/2} \leq 1150$ MeV, and are incompatible with those of the Cern-Munich experiment, that we take as given in the s-channel solution of Estabrooks and Martin (1974).[59] We impose in the fit the scattering length, as obtained from the Froissart–Gribov representation, and the experimental width of the f_2:

$$a_2^{(0)} = (18.1 \pm 0.4) \times 10^{-4} \times M_\pi^{-5}, \quad \Gamma_{f_2} = 185 \pm 4 \text{ GeV}\,.$$

We write

$$\cot \delta_2^{(0)}(s) = \frac{s^{1/2}}{2k^5}\,(M_{f_2}^2 - s)\,{M_\pi}^2\,\psi(s), \quad \psi(s) = B_0 + B_1 w(s) \tag{A.10a}$$

and

$$w(s) = \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}}, \quad s_0 = 1430 \text{ MeV}; \quad M_{f_2} = 1275.4 \text{ MeV}\,.$$

We find,

$$\frac{\chi^2}{\text{d.o.f.}} = 74/(21-2), \quad B_0 = 22.4 \pm 0.1, \quad B_1 = 23.3 \pm 3.0;$$

The very poor χ^2/d.o.f. is obviously due to the strong bias of the data of Protopopescu et al. (1973), clearly seen in Fig. 6.4.2 in the main text. Above values of the B_i would give $\Gamma_{f_2} = 196 \pm 6$ MeV. We will then take this solution up to $\bar{K}K$ threshold; on it, we join the solution to a new one, for which we impose the f_2 width; we get

$$B_0 = 22.5 \pm 0.1, \quad B_1 = 28.5 \pm 3.2.$$

Therefore, we have

$$B_0 = \begin{cases} 22.4 \pm 0.1, & s < 4M_K^2, \\ 22.5 \pm 0.1, & s > 4M_K^2 \end{cases}; \qquad B_1 = \begin{cases} 23.3 \pm 3.0, & s < 4M_K^2, \\ 28.5 \pm 3.2, & s > 4M_K^2. \end{cases} \tag{A.10b}$$

We take into account the inelasticity by writing

$$\eta_2^{(0)}(s) = \begin{cases} 1, & s < 4M_K^2; \\ 1 - 2 \times \epsilon_f \dfrac{k_2(s)}{k_2(M_{f_2}^2)}, & \epsilon_f = 0.131 \pm 0.015, \quad s > 4M_K^2. \end{cases} \tag{A.10c}$$

[59] We take only the values of the phase shifts of these authors at the energies 0.63, 0.71, 0.75, 0.79, 0.83, 0.87 and 0.91 GeV. Since they do not give errors for their numbers, we arbitrarily take a common error of 10%.

$k_2 = \sqrt{s/4 - M_K^2}$. We have fixed the coefficient ϵ_f fitting the inelasticities of Protopopescu et al., and the experimental inelasticity of the f_2; the overall χ^2/d.o.f. of this fit is ~ 1.8. This parametrization is different from the corresponding formula used in Peláez and Ynduráin (2003) and, probably, more exact, although the influence of the change in the various sum rules is negligible. For example, the Olsson integral has only increased by 0.002 by using the formulas given here.

The fit returns the values

$$\begin{aligned} &a_2^{(0)} = (18.4 \pm 7.6) \times 10^{-4}\, M_\pi^{-5}, \quad b_2^{(0)} = (-7.9^{+4.1}_{-11.0}) \times 10^{-4}\, M_\pi^{-7}, \\ &\Gamma_{f_2} = 185 \pm 5 \text{ MeV}. \end{aligned} \tag{A.10d}$$

A.7. Parametrization of the D wave for $I = 2$

For isospin equal 2, there are no resonances in the D wave. If we want a parametrization that applies down to threshold, we must incorporate the zero of the corresponding phase shift. So we write

$$\cot \delta_2^{(2)}(s) = \frac{s^{1/2}}{2k^5}\, [B_0 + B_1 w(s)]\, \frac{{M_\pi}^4 s}{4({M_\pi}^2 + \Delta^2) - s} \tag{A.11a}$$

with Δ a free parameter and

$$w(s) = \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}}, \quad s_0 = 1450 \text{ MeV}.$$

Moreover, we impose the value for the scattering length that follows from the Froissart–Gribov representation.

Solution B2. With two b_is we get a mediocre fit, χ^2 /d.o.f. $= 72/(22-3)$, for $s^{1/2}$ below 1 GeV, and the values of the parameters are

$$B_0 = (2.30 \pm 0.17) \times 10^3,\ B_1 = -267 \pm 750,\ \Delta = 103 \pm 11 \text{ MeV}; \quad s^{1/2} \le 1.1 \text{ GeV}. \tag{A.11b}$$

Actually, (A.11b) is valid up to $s^{1/2} \sim 1.2$ GeV. Above 1 GeV we simply write a polynomial fit:

$$\begin{aligned} &\delta_2^{(2)}(s) = (-0.051 \pm 0.004) + a\left(\frac{s}{1\,\text{GeV}^2} - 1\right) + b\left(\frac{s}{1\,\text{GeV}^2} - 1\right)^2; \\ &a = -0.081 \pm 0.033, \quad b = 0.042 \pm 0.005; \quad s \ge 1.0 \text{ GeV}. \end{aligned} \tag{A.11c}$$

The incompatibilities between the three sets of experimental data (obvious from a look at Fig. 6.4.1 in the main text), probably related to those for the S2 wave, preclude a better fit. We can include inelasticity,

$$\begin{aligned} &\eta_2^{(2)}(s) = 1 - c\,(1 - M_{\text{eff}}^2/s)^{3/2}, \quad M_{\text{eff}} = 0.96 \text{ GeV}, \quad c = 0.12 \pm 0.12; \\ &s^{1/2} \ge 0.96 \text{ GeV}. \end{aligned} \tag{A.11d}$$

The fit returns a good value for the scattering length, and also for the effective range parameter, $b_2^{(2)}$:

$$a_2^{(2)} = (2.20 \pm 0.16) \times 10^{-4}\, {M_\pi}^{-5}; \quad b_2^{(2)} = (-5.75 \pm 1.26) \times 10^{-4}\, {M_\pi}^{-7}, \tag{A.11e}$$

to be compared with what we found using the Froissart–Gribov representation (Sect. 7.5), which gives very precise results,

$$a_2^{(2)} = (2.22 \pm 0.33) \times 10^{-4}\, {M_\pi}^{-5}; \quad b_2^{(2)} = (-3.34 \pm 0.24) \times 10^{-4}\, {M_\pi}^{-7}.$$

Solution B4. With *four* parameters B_i, and including also the data of Cohen et al. (1973) one gets,

$$\begin{aligned} &B_0 = (1.94 \pm 0.14) \times 10^3, \quad B_1 = (10.15 \pm 1.3) \times 10^3, \quad B_2 = (18.68 \pm 2.4) \times 10^3, \\ &B_3 = (-31.04 \pm 5.5) \times 10^3; \qquad \Delta = 218 \pm 22 \text{ MeV}. \end{aligned} \tag{A.12a}$$

The errors here correspond to $3\,\sigma$. One has χ^2 /d.o.f. $= 57/(25-5)$ and the fit returns the values of the low energy parameters

$$a_2^{(2)} = (2.04 \pm 0.5) \times 10^{-4}\, M_\pi^{-5}, \quad b_2^{(2)} = (1.6 \pm 0.3) \times 10^{-4}\, M_\pi^{-7}. \tag{A.12b}$$

The large values of the parameters B_i, and the incompatibility of the three data sets, makes one suspect that the corresponding minimum is spureous, but it represents resonably well the data.

A.8. The F wave

For the imaginary part of the F wave below $s^{1/2} = 1.42$ GeV we write a background plus the tail of a Breit–Wigner formula for a resonance. The background is obtained fitting the low energy phase shifts of Protopopescu et al. (1973), plus the scattering length as given by the Froissart–Gribov representation. The resonance is the ρ_3 with its properties taken from the Particle Data Tables:

$$\begin{aligned} \text{Im}\, \hat{f}_3(s) &= \frac{1}{1+\cot^2\delta_3} + \left(\frac{k(s)}{k(M_{\rho_3}^2)}\right)^{14} \text{BR} \frac{M_{\rho_3}^2 \Gamma^2}{(s-M_{\rho_3}^2)^2 + M_{\rho_3}^2 \Gamma^2 (k(s)/k(M_{f_4}^2))^{14}}; \\ \cot\delta_3(s) &= \frac{s^{1/2}}{2k^7}\, M_\pi^6 \left\{ B_0 + B_1 \frac{\sqrt{s} - \sqrt{s_0 - s}}{\sqrt{s} + \sqrt{s_0 - s}} \right\}; \quad s_0^{1/2} = 1.5 \text{ GeV} \\ M_{\rho_3} &= 1.69 \text{ GeV}, \quad \Gamma = 0.161 \text{ GeV}, \quad \text{BR} = 0.24; \\ B_0 &= (1.07 \pm 0.03) \times 10^5, \quad B_1 = (1.35 \pm 0.03) \times 10^5. \end{aligned} \tag{A.13}$$

This implies $a_3 = (7.0 \pm 0.8) \times 10^{-5}\, M_\pi^{-7}$.

The contribution of the F wave to all our sum rules is very small; the interest of calculating it lies in that it provides a test (by its very smallness) of the convergence of the partial wave expansions.

A.9 The G waves

For the G0 wave, we take its imaginary part to be given by the tail of the $f_4(2050)$ resonance, with its properties as given in the Particle Data Tables:

$$\text{Im}\, \hat{f}_4^{(0)}(s) = \left(\frac{k(s)}{k(M_{f_4}^2)}\right)^{18} \text{BR} \frac{M_{f_4}^2 \Gamma^2}{(s-M_{f_4}^2)^2 + M_{f_4}^2 \Gamma^2 (k(s)/k(M_{f_4}^2))^{18}}; \tag{A.14}$$

$$s^{1/2} \geq 1 \text{ GeV}. \qquad \text{BR} = 17 \pm 2, \quad M_{f_4} = 2025 \pm 8 \text{ MeV}, \Gamma = 194 \pm 13 \text{ MeV}.$$

For the wave G2, we can write, neglecting its eventual inelasticity,

$$\cot\delta_4^{(2)}(s) = \frac{s^{1/2}M_\pi^8}{2k^9}\,B, \quad B = (-7.8 \pm 3.3)\times 10^6; \quad s^{1/2} \geq 1 \text{ GeV}\,. \tag{A.15}$$

It should be noted that this last, as well as the expression above for the G0 wave, are little more than order of magnitude estimates. Moreover, at low energies they certainly fail; an expression in terms of the scattering length approximation (cf. Subsect. (6.5.4)) is more appropriate. Thus, if, in a calculation, the value of either of the two G waves is important, it means that the calculation will have a large error.

Appendix B: The conformal mapping method

Let us consider a function, $f(z)$, analytic in a domain, $\mathcal{D}$; for example, this domain may be a plane with two cuts, as for the partial wave amplitudes; see e.g. Fig. 6.3.1 in the main text. According to general theorems (see, e.g., Ahlfors, 1953), it is always possible to map the interior of this domain into the interior of the disk $\Delta(0,1)$, with center at the origin and unit radius. Let us call $w = w(z)$ to the corresponding variable. Then, in this variable, f is analytic inside $\Delta(0,1)$ and thus the ordinary Taylor expansion in terms of w is absolutely and uniformly convergent in $\Delta(0,1)$. Therefore, undoing the mapping, it follows that we can write

$$f(z) = \sum_{n=0}^{\infty} c_n w(z)^n, \tag{B.1}$$

and this expansion is absolutely and uniformly convergent inside all of $\mathcal{D}$.

It is important to realize that the representation (B.1) does not imply any supplementary assumption on $f(z)$ besides its analyticity properties; the convergence of (B.1) and the analyticity of f in $\mathcal{D}$ are strictly equivalent statements.

We next say a few words about the specific situations we encountered in the main text.[60] In some cases we have a function f analytic inside $\mathcal{D}$ except for a pole at z_0. Then the function

$$\varphi(z) = (z - z_0)f(z)$$

is analytic inside $\mathcal{D}$ and it is φ that can be expanded as in (B.1). In some other cases, we have a function $f(z)$ analytic inside $\mathcal{D}$, with a zero at z_0. Of course, this zero does not spoil the analyticity, so we could expand f itself. But, because the expansion of a function converges best if the function varies little, we have interest in extracting this zero and write

$$f(z) = (z - z_0)\psi(z),$$

expanding then ψ, which has the same analyticity properties as f.

[60] Further discussion (with references) of the present method, and also of other similar ones (for example, mapping into an ellipse and expanding in a Legendre or Tchebycheff series there, the second in principle the more efficient procedure), applied in particular to $\pi\pi$ scattering, may be found in the reviews of Pišut (1970) and Ciulli (1973). The question of the stability of extrapolations is also discussed there.

The gain in convergence and stability obtained by expanding in the conformally transformed variable is enormous. The reader may verify this with the simple example of the function $\log(1+x)$. Here the region $\mathcal{D}$ is the complex plane cut from $-\infty$ to -1. If we expand in powers of x,

$$\log(1+x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n} \tag{B.2}$$

then for e.g. $x \sim 1/2$ we need five terms for a 1% percent accuracy.

In this case the expansion in the conformal variable can be made explicitly. The transformation that maps $\mathcal{D}$ into $\Delta(0,1)$ is

$$w(x) = \frac{(1+x)^{1/2}-1}{(1+x)^{1/2}+1},$$

with inverse $x = 2w/(1-w)$. Substituting this into $\log(1+x)$, we get the expansion in the conformally transformed variable

$$\log(1+x) = 4\sum_{n=\text{odd}}^{\infty} \frac{1}{n}\left[\frac{(1+x)^{1/2}-1}{(1+x)^{1/2}+1}\right]^n. \tag{B.3}$$

If using this expansion, only two terms are necessary for an accuracy of a part in a thousand for $x \sim 1/2$. Even for $x = 25$, very far from the region of convergence of (B.2), the expansion (B.3) still represents the function closely: only three terms in (B.3) are necessary to get a precision better than 2%.[61]

This economy is also apparent in our parametrizations of the partial waves or the Omnès auxiliary function $G(t)$, where only two, or in one case three terms, are necessary. Indeed, the simplicity and economy of our parametrization contrasts with some of the complicated ones found in the literature. Thus, for example, Colangelo, Gasser and Leutwyler (2001), who take it from Schenk (1991), write

$$\cot\delta = \frac{2s^{1/2}}{k^{2l+1}}\frac{s-s_R}{4\mu^2-s_R}\frac{1}{A+Bk^2+Ck^4+Dk^6}. \tag{B.4}$$

For the P wave, they need these four parameters A, B, C, D (apart from the squared mass of the resonance, s_R) when we only require two. Moreover, (B.3) only converges in the shaded disk in Fig. 3.1.2 (but it is used in the whole range, which is a recognized cause of unstability) and, in general, presents complex singularities, hence violating causality.

It s true that (B.4) is only used by Schenk and by Colangelo *et alii* in the physical region; this, in fact, is one of its disadvantages: our parametrization can be used in all the cut complex plane and is therefore suited to discuss effects such as location of the poles associated with the resonances or the Adler zeros for the S waves. As a graphical example (Fig. B.1), consider the phase $\delta_0^{(0)}$ as given by Colangelo, Gasser and Leutwyler (2001); for $s^{1/2} \leq 0.8$ GeV it agrees, at

[61] In this example we compare the virtues of (B.1), (B.3) as *expansions*, for simplicity; in the main text, they are, however, used to *fit*. Thus, we should really give ourselves the values of $\log(1+x)$ at a series of points, x_1, x_2, ... x_n and fit with (B.1), (B.3). The improvement is less spectacular than before, but it is still substantial.

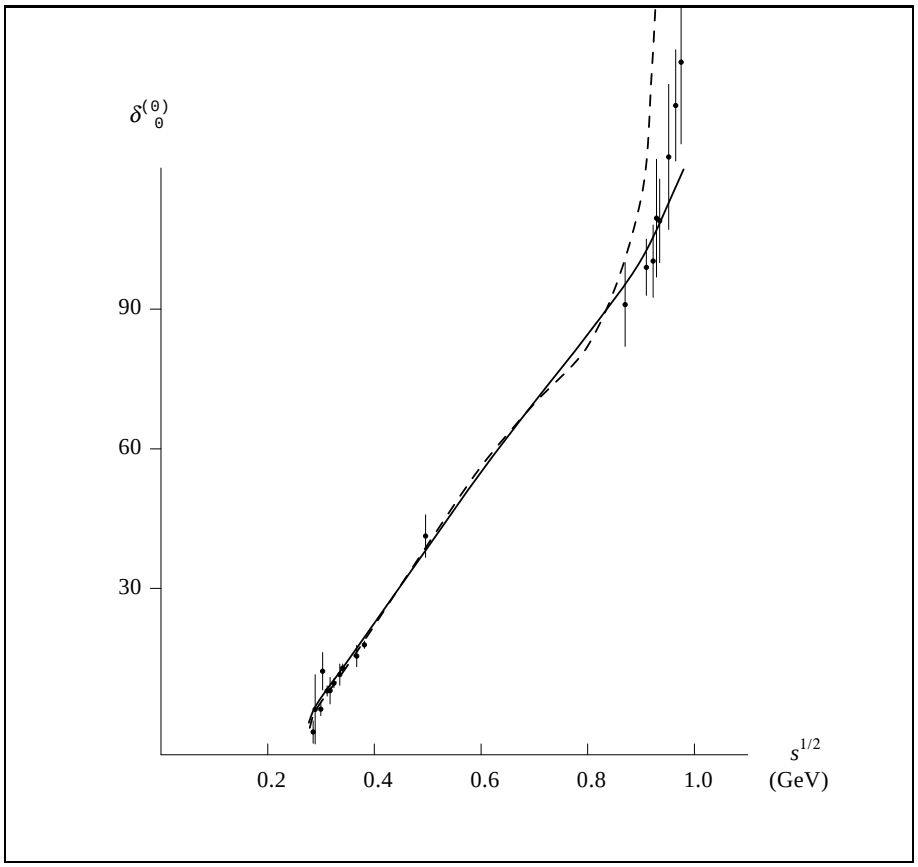

FIGURE B.1. The S0 wave phase shifts corresponding to (B.5) (continuous line) and Colangelo, Gasser and Leutwyler (2001), dashed line.

the percent level, with the phase shift given by

$$\cot\delta_0^{(0)}(s) = \frac{s^{1/2}}{2k}\,\frac{M_\pi^2}{s-\frac{1}{2}M_\pi^2}\,\frac{M_\sigma^2-s}{M_\sigma^2} \times\left\{B_0+B_1\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}+B_2\left[\frac{\sqrt{s}-\sqrt{s_0-s}}{\sqrt{s}+\sqrt{s_0-s}}\right]^2\right\}; \qquad (B.5)$$

$$s_0^{1/2}=2M_K;\quad \chi^2/\text{d.o.f.}=11.1/(19-4).$$
$$M_\sigma=836,\quad B_0=29,\quad B_1=39,\quad B_2=32.$$

[This is slightly displaced from the solution (6.4.12) in the main text]. However, the CGL phase goes berserk above 0.9 GeV, while (B.5) continues to represent it fairly well up to the $\bar{K}K$ threshold: see the accompanying figure. It is also likely that at least some of the wiggles that the CGL solution

presents (for example, the one below and around 0.8 GeV, clearly seen in the figure) are due to the unstability of the Schenk expansion.

It may be argued that, even if using Schenk's parametrization, one can get at $f_l(s)$ outside the physical region indirectly via Roy's equations. Using ours, however, you can get that *both* directly and via Roy's equations, which provides useful consistency tests. As an example, we mention that the value we obtain for the Adler zero in Eq. (7.6.3), with a simple fit to data and only three parameters, namely $z_2 = 133 \pm 4.5$ MeV, is consistent with (and the central value even slightly more accurate than) what Colangelo, Gasser and Leutwyler (2001) get with the parametrization of Schenk, with five parameters, after imposing fulfillment of the Roy equations and a large number of crossing and analyticity sum rules: $z_2 = 136$ MeV.

The fact that we manage with a smaller number of parameters is important not only as a matter of economy or consistency, but also in that we avoid spureous minima which are liable to occur when large number of parameters are present.

Appendix C: Sum rules and asymptotic behaviour

Long time ago it was remarked by Pennington (1975) that one can use sum rules, based on crossing symmetry, to relate low energy $\pi\pi$ physics to the high energy behaviour of the $\pi\pi$ scattering amplitudes. In Pennington's work, experimental phase shifts were used up to $s^{1/2} = 2$ GeV and the conclusion was drawn that the Regge behaviour of $\pi\pi$ scattering was very different from what one could expect on the basis of factorization.

This conclusion could perhaps be maintained in 1974, when Pennington wrote his paper. First of all, the phase shifts used were those of the Cern–Munich experiment, which, as discussed in Sect. 6.6 here, bear little resemblance to reality –as we now know. Secondly, the QCD theory of strong interactions (in which factorization is automatic) was not established at the time, when indeed it competed with other, very different ones; string theories, for example. And finally (in both senses of the word), experimental data on various $\pi\pi$ cross sections between 1.2 and 6 GeV (Cohenet al., 1973; Robertson, Walker and Davis, 1073; Hanlan et al., 1976 and Abramowicz et al., 1980) have fully confirmed the standard Regge picture, as described in Sect. 2.4 in the main text.

Unfortunately, Ananthanarayan, B., et al. (2001) et al have, without looking carefully enough at the foundations of the paper of Pennigton (1975), accepted its conclusions. And in this they have been followed by a number of modern authors, quoting the result uncritically. In view of this we have considered useful to give a brief discussion of the matter in this Appendix, in which we will show that standard Regge theory, as reviewed here in Sect. 2.4, is perfectly compatible (within errors) with low energy $\pi\pi$ scattering provided we consider "low energy" to mean $s^{1/2}$ less than 1.42 GeV, and we apply Regge formulas consistently above this energy.

First of all, we remark that the experimental cross sections, as deduced from the fits to experiment carried in Chapter 6 here, produce cross sections that, at $s^{1/2} \sim 1.4$ GeV, agree with what we get from the Regge formulas. A complete set of figures may be seen in Sect. 2.4 and in Peláez and Ynduráin (2003); here we only add that for isospin zero exchange, a fairly impressive one.

We next turn to the sum rules which, as explained, relate high ($s^{1/2} \geq 1.42$ GeV) and low energy, with the low energy given by the P, D, F waves in the region $s^{1/2} \leq 1.42$ GeV and where the high energy is dominated by, respectively, the rho and Pomeron Regge trajectories. We will explore explicitely only two of them; we remark that the equality of the determinations of the parameters a_1, b_1 from the pion form factor and from the Froissart–Gribov representations (Subsect. 7.3.6)

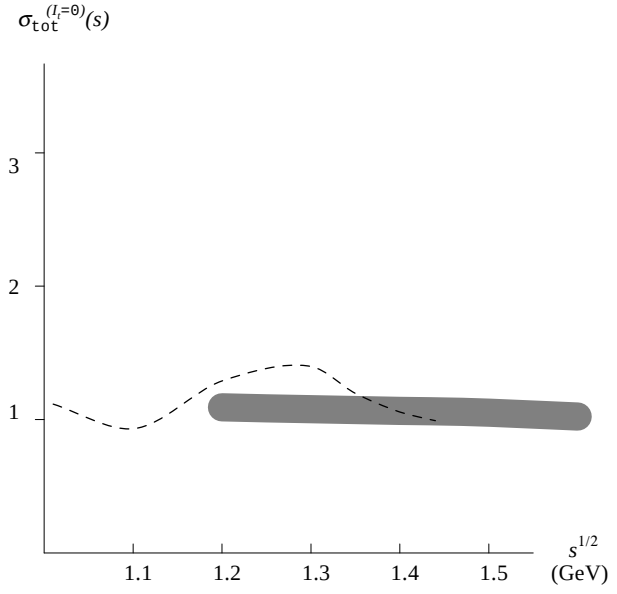

FIGURE C.1. The average cross section $\frac{1}{3}[2\sigma_{\pi^0\pi^+} + \sigma_{\pi^0\pi^0}]$, which is pure $I_t = 0$, arbitrarily normalized. Broken line: experimental cross section. Note that the bump here is due to the coincidence of two resonances, $f_0(1270)$, $f_2(1370)$, mostly elastic, around $s^{1/2} \sim 1.3$ GeV. Thick gray line: Regge formula. The thickness of the line covers the error in the theoretical value of the Regge residue.

provide other, highly nontrivial tests of the compatibility of Regge behaviour (in that case, of rho exchange) with s, u and t crossing, which all three enter into the Froissart–Gribov projection.

The first sum rule is obtained by profiting from the threshold behaviour to write an unsubtracted forward dispersion relation for the quantity $F^{(I_s=1)}(s,0)/(s-4M_\pi^2)$. One gets

$$\frac{6M_\pi}{\pi}a_1 = \frac{1}{\pi}\int_{M_\pi^2}^{\infty} \mathrm{d}s\, \frac{\mathrm{Im}\, F^{(I_s=1)}(s,0)}{(s-4M_\pi^2)^2} + \frac{1}{\pi}\sum_{I_s} C_{1I_s}^{(su)} \int_{M_\pi^2}^{\infty} \mathrm{d}s\, \frac{\mathrm{Im}\, F^{(I_s)}(s,0)}{s^2}, \tag{C.1}$$

which is known at times as the (second) Olsson sum rule; see e.g. Martin, Morgan and Shaw (1976). $C_{1I_s}^{(su)}$ are the $s-u$ crossing matrix elements. Canceling a_1 with the Froissart–Gribov expression for this quantity and substituting the $C_{1I_s}^{(su)}$ we find the result

$$\begin{aligned} I \equiv I_1 + I_2 \equiv & \int_{M_\pi^2}^{\infty} \mathrm{d}s\, \frac{\mathrm{Im}\, F^{(I_t=1)}(s,4M_\pi^2) - \mathrm{Im}\, F^{(I_t=1)}(s,0)}{s^2} \\ & - \int_{M_\pi^2}^{\infty} \mathrm{d}s\, \frac{8M_\pi^2[s-2M_\pi^2]}{s^2(s-4M_\pi^2)^2}\, \mathrm{Im}\, F^{(I_s=1)}(s,0) = 0. \end{aligned} \tag{C.2}$$

The second term, I_2, can also be expressed in terms of amplitudes with fixed isospin in the t channel, writing

$$\mathrm{Im}\, F^{(I_s=1)} = \tfrac{1}{3} F^{(I_t=0)} + \tfrac{1}{3} F^{(I_t=1)} - \tfrac{5}{6} F^{(I_t=2)}. \tag{C.3}$$

The contributions of the S waves cancel in (C.2), so only the P, D and F waves contribute (we systematically neglect waves G and higher). At high energy, I_2 contributes little since the corresponding integral converges rapidly: most of the high energy contribution comes from the first term, dominated by rho exchange. We will use units so that $M_\pi = 1$ and obtain the following results:

$$\begin{aligned} I(\text{low energy}) &= -3.5 \times 10^{-2}, \\ I(\text{high energy (Regge)}, \rho) &= 3.46 \times 10^{-2}, \\ I(\text{high energy (Regge)}, I=0) &= -0.19 \times 10^{-2}. \end{aligned} \tag{C.4}$$

By "low energy" we understand, as usual, the contributions from energies below 1.42 GeV, where we use phase shifts and inelasticities to calculate the scattering amplitudes, and "high energy" is above 1.42 GeV, where a Regge description is employed. Our Regge parameters are those in Sect. 2.4.

Adding the (small) contributions of the $I_t = 1$ background and the (also small) $I_t = 2$ piece we find

$$I = (0.02 \pm 0.4) \times 10^{-2}, \tag{C.5}$$

that is to say, perfect consistency.

The second sum rule we discuss is that given in Eqs. (B.6), (B.7) of Ananthanarayan et al. (2001), which these authors use to claim a Pomeron with a residue a third of its standard value. It reads,

$$\begin{aligned} J \equiv \int_{4M_\pi^2}^{\infty} \mathrm{d}s \Bigg\{ & \frac{4\,\mathrm{Im}\, F'^{(0)}(s,0) - 10\,\mathrm{Im}\, F'^{(2)}(s,0)}{s^2(s-4M_\pi^2)^2} \\ & - 6(3s - 4m_\pi^2)\, \frac{\mathrm{Im}\, F'^{(1)}(s,0) - \mathrm{Im}\, F^{(1)}(s,0)}{s^2(s-4M_\pi^2)^3} \Bigg\} = 0. \end{aligned} \tag{C.6}$$

Here $F'^{(I)}(s,t) = \partial F^{(I)}(s,t)/\partial \cos\theta$, and the upper indices refer to isospin in the s channel.

We will separate J into a low energy and a high energy piece:

$$J = J_{\text{l.e.}} + J_{\text{h.e.}}; \quad J_{\text{l.e.}} = \int_{4M_\pi^2}^{s_h} \mathrm{d}s \ldots, \quad J_{\text{h.e.}} = \int_{s_h}^{\infty} \mathrm{d}s \ldots . \tag{C.7}$$

The low energy piece, $J_{\text{l.e.}}$, only contains contributions of waves D and higher. We will show that, if we choose (as we are doing systematically) $s_h = 1.42^2$ GeV^2, then we find cancellation, within errors.

For the low energy piece we use the parametrizations of Appendix A and get, with $M_\pi = 1$,

$$J_{\text{l.e.}} = (1.15 \pm 0.05) \times 10^{-4}. \tag{C.8}$$

For the high energy piece we first neglect the P' and $I_t = 2$ exchange pieces. Expanding in amplitudes with definite isospin in the t channel, and with the numbers in Sect. 2.4 for the Pomeron and rho contributions, we then get

$$J_{\text{h.e.}}(\text{Pomeron}) = -1.093 \times 10^{-4}, \quad J_{\text{h.e.}}((\text{Regge})\ \rho) = 0.034 \times 10^{-4},$$

i.e., including errors,

$$J_{\text{h.e.}} = (-1.06 \pm 0.17) \times 10^{-4}. \tag{C.9}$$

Thus, we have cancellation between (C.8) and (C.9), within errors: there is no reason to justify departure off the expected Regge behaviour.

We next comment a little on the P' and on the inclusion of the $I_t = 2$ contribution. Because the high energy part of the sum rule is mostly given by the t derivative of the even isospin amplitudes, a more precise evaluation than the one carried here requires an accurate formula for the P'. Unfortunately, the characteristics of this Regge pole are poorly known; see e.g. Rarita et al. (1968). If we take for the the P' trajectory a formula like that of the ρ, as discussed in Sect. 2.4, then (C.9) changes to

$$J_{\text{h.e.}}(\text{With } P') = (-1.2 \pm 0.2) \times 10^{-4}.$$

Including also the $I_t = 2$ contribution, as given in Peláez and Ynduráin (2003), we would find

$$J_{\text{h.e.}}(\text{With } P', \text{ and including PY } I_t = 2) = (-0.5 \pm 0.3) \times 10^{-4}. \tag{C.10}$$

This only cancels the low energy piece, (C.8), at the $2\,\sigma$ level. This discrepancy cannot be taken seriously, because the t slope for the $I_t = 2$ exchange term of Peláez and Ynduráin (2003) is little more than guesswork. In fact, one can reverse the argument and use (C.6) to get an idea of the parameters of isospin 2 exchange. Thus, if we take for $I_t = 2$ exchange the parameters of Sect. 2.4, Eqs. (2.4.7), we get

$$J_{\text{h.e.}}(\text{With } P', \text{ and including (2.4.7) } I_t = 2) = (-0.93 \pm 0.24) \times 10^{-4}, \tag{C.11}$$

i.e., $J_{\text{h.e.}} + J_{\text{h.e.}} = (0.22 \pm 0.24) \times 10^{-5}$: perfect cancellation, within errors.

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