Covariant helicity-coupling amplitudes

Suh-Urk Chung

Physics Department
Brookhaven National Laboratory
Upton, NY

- Introduction
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- Examples
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Spin Formalisms

S. U. Chung
CERN Yellow Report—1971

Helicity-coupling amplitudes in tensor formalism

S. U. Chung
PR D**48**, 1225 (1993)

Covariant Helicity-coupling Amplitudes: A Practical Guide

S. U. Chung
BNL-QGS94-21 (1994)

General formulation of covariant helicity-coupling amplitudes

S. U. Chung
PR D—to be published
in Dec 1997 or Jan 1998

Introduction

- **Conventional mesons:** $q\bar{q}$

$$\vec{J} = \vec{L} + \vec{S}$$

$$\left\{ \begin{array}{l} P = (-)^{L+1} \\ C = (-)^{L+S} \end{array} \right\} \quad \Rightarrow \quad CP = (-)^{S+1}$$

- **Exotic mesons**

$J^{PC} = 0^{--}, 0^{+-}, 1^{-+}, 2^{+-}, \cdots$

$\Longrightarrow (q\bar{q} \; + \; g)$ or $(q\bar{q} \; + \; q\bar{q})$

$CP = (-)^S$ in the flux-tube model

Resonances

- **Breit-Wigner form:**

  Mass= w_0 and Width= Γ_0
  Effective mass= w

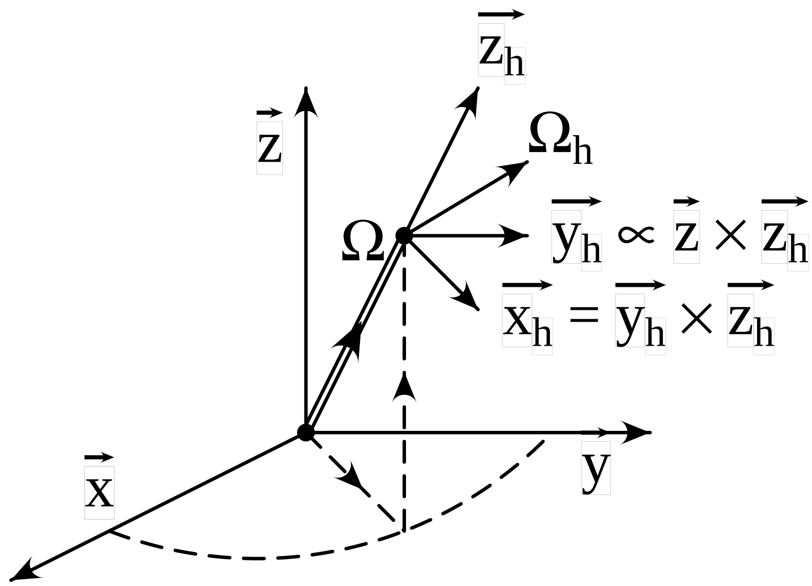
$$\Delta(w) = \frac{\Gamma_0\, w_0}{w_0^2 - w^2 - i\,\Gamma_0\, w_0}$$

$$= \mathrm{e}^{i\,\delta(w)} \sin\delta(w)$$

- **Phase Shift:**

$$\cot\delta(w) = \frac{w_0^2 - w^2}{\Gamma_0\, w_0}$$

Decay in helicity quantization



$\Omega = (\theta, \phi)$ for the primary decay

$\Omega_h = (\theta_h, \phi_h)$ for the secondary decay

For $|JM\rangle \to |s\lambda\rangle + |\sigma\nu\rangle$, the decay amplitude is

$$A^J_{\lambda\nu}(\theta, \phi, M) \propto \; D^{J\,*}_{M,\lambda-\nu}(\phi, \theta, 0) F^J_{\lambda\nu}$$

M. Jacob and G. C. Wick
Ann. Phys. (N.Y.) **7**,404 (1959)

If the angles are zero,

$$A^J_{\lambda\nu}(0, 0; \lambda - \nu) \propto \; F^J_{\lambda\nu}$$

Decay in canonical quantization

For $|JM\rangle \to |sm_1\rangle + |\sigma m_2\rangle$, the decay amplitude is

$$A^J_{\ell S}(\theta,\phi; m_1\, m_2\, M) \propto G^J_{\ell S}\; (sm_1\,\sigma m_2|Sm_s) \sum_m (\ell m\, Sm_s|JM)\, Y^\ell_m(\theta,\phi)$$

If the angles are zero,

$$A^J_{\ell S}(0,0; m_1\, m_2\, m_s) \propto \; G^J_{\ell S}\; (sm_1\,\sigma m_2|Sm_s)\,(\ell 0\, Sm_s|Jm_s)$$

and $m_1 = \lambda$ and $m_2 = -\nu$.

Therefore, with $\delta = \lambda - \nu$,

$$F^J_{\lambda\,\nu} = \sum_{\ell\, S} \left(\frac{2\ell+1}{2J+1}\right)^{1/2} (\ell 0\; S\delta|J\delta)\,(S\lambda\;\sigma\;-\nu|S\delta)\, G^J_{\ell\, S}$$

M. Jacob and G. C. Wick
Ann. Phys. (N.Y.) **7**,404 (1959)

Two-body decay: $J \to s + \sigma$

	Parent	Daughter 1	Daughter 2
Spin	J	s	σ
Parity	η_J	η_s	η_σ
Helicity		λ	ν
Momentum	$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{k}$
Energy	p_0	q_0	k_0
Mass	W	m	μ
Energy/Mass		γ_s	γ_σ
Velocity		β_s	β_σ
Wave function	$\phi^*(\lambda - \nu)$	$\omega(\lambda)$	$\varepsilon(-\nu)$

where

$$\gamma_s = \frac{q_0}{m}, \quad \gamma_s\beta_s = \frac{q}{m}, \quad \gamma_\sigma = \frac{k_0}{\mu} \quad \text{and} \quad \gamma_\sigma\beta_\sigma = \frac{k}{\mu}$$

and

$$\delta = \lambda - \nu \quad \text{and} \quad \boldsymbol{r} = \boldsymbol{q} - \boldsymbol{k}$$

Momenta and Wave functions

$$
\begin{aligned}
p^\alpha &= (&& W; && 0, && 0, && 0 &&) \\
q^\alpha &= (&& q_0; && 0, && 0, && q &&) \\
&= (&& \gamma_s m; && 0, && 0, && \gamma_s\beta_s m &&) \\
k^\alpha &= (&& k_0; && 0, && 0, && -q &&) \\
&= (&& \gamma_\sigma \mu; && 0, && 0, && -\gamma_\sigma\beta_\sigma \mu &&) \\
r^\alpha &= (&& q_0 - k_0; && 0, && 0, && 2q &&)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi^\alpha(\pm) &= \mp\tfrac{1}{\sqrt{2}} && (&& 0; && 1, && \pm i, && 0 &&) \\
\phi^\alpha(0) &= && (&& 0; && 0, && 0, && 1 &&) \\
\omega^\alpha(\pm) &= \mp\tfrac{1}{\sqrt{2}} && (&& 0; && 1, && \pm i, && 0 &&) \\
\omega^\alpha(0) &= && (&& \gamma_s\beta_s; && 0, && 0, && \gamma_s &&) \\
\varepsilon^\alpha(\pm) &= \mp\tfrac{1}{\sqrt{2}} && (&& 0; && 1, && \pm i, && 0 &&) \\
\varepsilon^\alpha(0) &= && (&& -\gamma_\sigma\beta_\sigma; && 0, && 0, && \gamma_\sigma &&)
\end{aligned}
$$

Transversality condition:

$$p_\alpha \phi^\alpha(\lambda) = q_\alpha \omega^\alpha(\lambda) = k_\alpha \varepsilon^\alpha(\lambda) = 0$$

Wave functions—continued

$$
\begin{aligned}
p_\alpha \phi^\alpha(m) &= 0 \\
\phi^*_\alpha(m)\phi^\alpha(m') &= -\delta_{mm'} \\
\sum_m \phi_\alpha(m)\phi^*_\beta(m) &= \tilde{g}_{\alpha\beta}(W)
\end{aligned}
$$

where the 'rest-frame metric' is given by

$$
\tilde{g}_{\alpha\beta}(W) = -\bar{g}_{\alpha\beta} + \frac{p_\alpha p_\beta}{W^2} \equiv P_{\alpha\beta}(W)
$$

and $P_{\alpha\beta}(W)$ is the spin-1 projection operator.

The wave functions ω and ε obey similar relationships but with their own rest-frame metric:

$$
\begin{aligned}
\tilde{g}_{\alpha\beta}(m) &= -\bar{g}_{\alpha\beta} + \frac{q_\alpha q_\beta}{m^2} \\
\tilde{g}_{\alpha\beta}(\mu) &= -\bar{g}_{\alpha\beta} + \frac{k_\alpha k_\beta}{\mu^2}
\end{aligned}
$$

and similarly $P_{\alpha\beta}(m)$ and $P_{\alpha\beta}(\mu)$.

The spin-2 wave function can be written

$$\phi^{\alpha\beta}(m) = \sum_{m_1 m_2} (1m_1\, 1m_2|2m)\phi^{\alpha}(m_1)\phi^{\beta}(m_2)$$

where $m = m_1 + m_2$.

Sinilarly, the spin-3 wave function is

$$\begin{aligned}\phi^{\alpha\beta\gamma}(m) &= \sum_{n_1 m_2} (2n_1\, 1m_3|3m)\phi^{\alpha\beta}(n_1)\phi^{\gamma}(m_3) \\ &= \sum_{m_1 m_2 m_3} (1m_1 1m_2|2n_1)(2n_1 1m_3|3m) \\ &\qquad \phi^{\alpha}(m_1)\phi^{\beta}(m_2)\phi^{\gamma}(m_3)\end{aligned}$$

where $m = m_1 + m_2 + m_3$.

Note that it is orthogonal to p, symmetric and tracelss,i.e.

$$g_{\alpha\beta}\, \phi^{\alpha\beta}(m) = \tilde{g}_{\alpha\beta}(W)\, \phi^{\alpha\beta}(m) = 0$$

P. R. Auvil and J. J. Brehm
PR **145**, 1152 (1966)

The spin-J wave function is

$$\phi^{\delta_1\cdots\delta_J}(m) = [a^J(m)]^{1/2} \sum_{m_0} 2^{m_0/2} \sum_P \phi^{\alpha_1}(+)\cdots\phi^{\beta_1}(0)\cdots\phi^{\gamma_1}(-)\cdots$$

where

$$a^J(m) = \frac{(J+m)!(J-m)!}{(2J)!}$$

The number of terms in the summation is

$$b^J(m, m_0) = \frac{J!}{m_+! m_0! m_-!}$$

where $2m_\pm = J \pm m - m_0$ and

$$m_0 : \quad 0(1), 2(3), \cdots, J-m = \text{even(odd)}$$

S. U. Chung
PR—to be published

Invariant ℓS-coupling amplitudes

Define a projection operator for orbital angular momenta. It should be a tensor of rank 2ℓ

$$P^\ell = \sum_m \tau^\ell(m)\tau^{\ell\,*}(m)$$

Relevant invariant amplitude

$$\left[\tau^{\ell *}(m) \otimes r\, r \cdots\right] = c_\ell\, r^\ell\, \delta_{m\,0}, \qquad c_\ell = \ell! \left[\frac{2^\ell}{(2\ell)!}\right]^{1/2}$$

Consider now a wave function $\chi^S(m_s)$ defined in the J rest frame corresponding to $|Sm_s\rangle$

$$\chi^S_{\alpha_1\cdots\alpha_s;\beta_1\cdots\beta_\sigma}(m_s) = \sum_{m_a m_b} (s m_a\, \sigma m_b | S m_s) \chi^s_{\alpha_1\cdots\alpha_s}(m_a) \chi^\sigma_{\beta_1\cdots\beta_\sigma}(m_b)$$

This is a tensor of rank $s+\sigma$. The projection operator for the instrinsic spin S is a tensor of rank $2(s+\sigma)$

$$P^S = \sum_{m_s} \chi^S(m_s)\chi^{S\,*}(m_s)$$

Relevant invariant amplitudes are

$$\begin{aligned}\left[\chi^{s\,*}(m_a) \otimes \omega(\lambda)\right] &= f^s_\lambda(\gamma_s)\ \delta_{m_a\,\lambda} \\ \left[\chi^{\sigma\,*}(m_b) \otimes \varepsilon(-\nu)\right] &= f^\sigma_\nu(\gamma_\sigma)\ \delta_{m_b\,-\nu}\end{aligned}$$

and the overall invariant amplitude for pure spin S

$$\left[\omega(\lambda) \otimes \chi^{S\,*}(m_s) \otimes \varepsilon(-\nu)\right] = (s\lambda\ \sigma\text{-}\nu|S\delta) f^s_\lambda(\gamma_s)\ f^\sigma_\nu(\gamma_\sigma)\ \delta_{m_s\delta}$$

The f-functions are

$$f^j_m(\gamma) = a^j(m) \sum_{m_0} b^j(m, m_0)\,(2\gamma)^{m_0}$$

Examples are

$$\begin{aligned}f^{(1)}_\lambda(\gamma) &= 1 \quad \text{for} \quad \lambda = \pm 1 \\ &= \gamma \quad \text{for} \quad \lambda = 0\end{aligned}$$

and

$$\begin{aligned}f^{(2)}_\lambda(\gamma) &= 1 \quad \text{for} \quad \lambda = \pm 2 \\ &= \gamma \quad \text{for} \quad \lambda = \pm 1 \\ &= \frac{2}{3}\gamma^2 + \frac{1}{3} \quad \text{for} \quad \lambda = 0\end{aligned}$$

Covariant helicity-coupling amplitudes

Nonrelativistic:

$$F^J_{\lambda\,\nu} = \sum_{\ell\,S} \left(\frac{2\ell+1}{2J+1}\right)^{1/2} (\ell 0\, S\delta|J\delta)\,(S\lambda\;\,\sigma\;\,-\nu|S\delta)\,G^J_{\ell\,S}$$

with $\delta = \lambda - \nu$.

Relativistic:

$$F^J_{\lambda\nu} = \sum_{\ell S} \left(\frac{2\ell+1}{2J+1}\right)^{1/2} (\ell 0\, S\delta|J\delta)(s\lambda\, \sigma{-\nu}|S\delta)$$

$$(W/W_0)^n\, (r/r_0)^\ell\, f^s_\lambda(\gamma_s)\;\; f^\sigma_\nu(\gamma_\sigma)\;\; G^J_{\ell s}$$

where

$$n = 1 \quad \text{if} \quad s + \sigma + \ell - J = \text{odd}$$

$$n = 0 \quad \text{if} \quad s + \sigma + \ell - J = \text{even}$$

Number of independent helicity-coupling amplitudes

Define

$$N_J = (a + b + 1)(a - s + \sigma + 1) + (s + \sigma - a)(2J + 1)$$

where

$$a = \min\{J, s + \sigma\}$$
$$b = \min\{J, s - \sigma\}$$

and

$$N_J^{(+)} = (N_J + 1)/2, \qquad F_{00}^J \neq 0,$$
$$\text{if} \quad \eta_{_J}(-)^J = +\eta_s(-)^s \, \eta_\sigma(-)^\sigma$$

$$N_J^{(-)} = (N_J - 1)/2, \qquad F_{00}^J = 0,$$
$$\text{if} \quad \eta_{_J}(-)^J = -\eta_s(-)^s \, \eta_\sigma(-)^\sigma$$

Number of Independent Amplitudes

s	σ	J	$N_J^{(-)}$	$N_J^{(+)}$	N_J
0	0	0	0	1	1
0	0	1	0	1	1
0	0	2	0	1	1
1	0	0	0	1	1
1	0	1	1	2	3
1	0	2	1	2	3
1	1	0	1	2	3
1	1	1	3	4	7
1	1	2	4	5	9
1	1	3	4	5	9
2	0	0	0	1	1
2	0	1	1	2	3
2	0	2	2	3	5
2	0	3	2	3	5
2	1	0	1	2	3
2	1	1	4	5	9
2	1	2	6	7	13
2	1	3	7	8	15
2	1	4	7	8	15

I. $\omega(1600) \to \rho(770) + \pi$

Since $1^- \to 1^- + 0^-$, one must have $\ell = 1$.
Using the notation,

$$[abcd] = [a\,b\,c\,d\,] = \epsilon_{\alpha\beta\gamma\delta}\; a^\alpha\, b^\beta\, c^\gamma\, d^\delta$$

the decay amplitude is

$$F_\lambda \propto [p\, r\, \omega(\lambda)\, \phi^*(\lambda)]$$

so that

$$F_0 = 0 \quad \text{and} \quad F_+ = -F_- \propto W\, r$$

in the rest frame of $\omega(1600)$.

II. $b_1(1320) \to \omega(782) + \pi$

Note: $1^+ \to 1^- + 0^-$ implies $\ell = 0$ or 2.

Since $(p \cdot \phi^*) = (q \cdot \omega) = 0$ and $p = q + k$,

the decay amplitude could be

$$F_\lambda \propto g_0[\omega(\lambda) \cdot \phi^*(\lambda)] + g_2[k \cdot \omega(\lambda)][q \cdot \phi^*(\lambda)]$$

where

$$[a \cdot b] = \bar{g}_{\alpha\beta}\, a^\alpha\, b^\beta = a^\alpha\, b_\alpha = a_\alpha\, b^\alpha$$

so that

$$F_+ = F_- \propto g_0$$
$$F_0 = -\gamma_s[g_0 + g_2(W/q_0)q^2]$$

IIa. $b_1(1320) \to \omega(782) + \pi$

The decay amplitude can be written

$$F_\lambda \propto g_0 \sum_m [\omega(\lambda) \cdot \chi^*(m)][\chi(m) \cdot \phi^*(\lambda)]$$
$$+ g_2 \sum_m [r \cdot \tau^*(m) \cdot r][\omega(\lambda) \cdot \tau(m) \cdot \phi^*(\lambda)]$$
$$\propto g_0 f_\lambda(\gamma_s)\,[\chi(\lambda) \cdot \phi^*(\lambda)] + g_2 c_2\, r^2\, [\omega(\lambda) \cdot \tau(0) \cdot \phi^*(\lambda)]$$

so that, with $c_2 = \sqrt{2/3}$,

$$F_+^J = g_0 - \frac{1}{3} g_2 r^2$$
$$F_0^J = \gamma_s \left(g_0 + \frac{2}{3} g_2 r^2 \right)$$

where $J = 1$. Or, one can use the general formula, to find

$$\sqrt{2} F_+^J = \sqrt{\frac{2}{3}} g_0 + \sqrt{\frac{1}{3}} g_2 r^2$$
$$F_0^J = \left(\sqrt{\frac{1}{3}} g_0 - \sqrt{\frac{2}{3}} g_2 r^2 \right) \gamma_s$$

Note

$$g_0 \to \sqrt{\frac{1}{3}}\, g_0 \quad \text{and} \quad g_2 \to -\sqrt{\frac{3}{2}}\, g_2$$

IIa. Example continued

Consider the angular distribution in $\cos\vartheta$

$$I(\vartheta) \propto \sum_{\lambda} [d^J_{0\lambda}(\vartheta)]^2 \; |F^J_\lambda|^2$$

where $J = 1$.

This leads to a distribution

$$I(\vartheta) \propto |F^J_0|^2 \cos^2(\vartheta) + |F^J_+|^2 \sin^2(\vartheta)$$

so that

$$\begin{aligned} I(\vartheta) \propto g_{_0}^2[(\gamma_s^2 - 1)\cos^2(\vartheta) + 1] \\ + \frac{2}{3} g_{_0} g_{_2}\, r^2[(2\gamma_s^2 + 1)\cos^2(\vartheta) - 1] \\ + \frac{1}{9} g_{_2}^2\, r^4[(4\gamma_s^2 - 1)\cos^2(\vartheta) + 1] \end{aligned}$$

Note

$$r = 2q \quad \text{and} \quad \gamma_s^2 = \left(\frac{q^2 + m^2}{m^2}\right)$$

in the J rest frame.

III. Higgs boson decay

Consider a decay $H \to W^+W^-$.
Since H is a scalar particle, one must have $\ell = S = 0$, $\ell = S = 1$ or $\ell = S = 2$. Assuming parity nonconservation,

$$F^J_{\pm\pm} = \sqrt{\frac{1}{3}} g_{00} \pm \frac{1}{\sqrt{2}} g_{11} W_{_H} r + \sqrt{\frac{1}{6}} g_{22} r^2$$

$$F^J_{00} = \gamma^2 \left(-\sqrt{\frac{1}{3}} g_{00} + \sqrt{\frac{2}{3}} g_{22} r^2 \right)$$

where $W_{_H}$ is the Higgs mass.

In the limit

$$W_{_H} \to \infty$$

F^J_{00} dominates over $F^J_{\pm\pm}$, i.e. W's behave like spin-zero particles (Goldstone bosons).

III. Conclusions on $J \to s + \sigma$

- **Rank-J tensor with a given m**

  Derived a general form for a rank-J tensor with a definite m, tensor analogue of the ket state $|Jm\rangle$.

- **Relativistic ℓ-S coupling**

  Introduce relativistic concept of total intrinsic spin S and orbital angular momentum ℓ, with two intermediate wave functions $\chi(m_s)$ and $\tau(m)$ (tensor analogues of the ket states $|Sm_s\rangle$ and $|\ell m\rangle$.

- **Covariant helicity-coupling amplitudes**

  Covariant helicity-coupling amplitudes $F^J_{\lambda\nu}$ depend in general on W, r, γ_s and γ_σ:

  1. W^n, n=0 or 1.
  2. r^ℓ, where ℓ is the orbital angular momentum.
  3. $F^J_{\lambda\nu}$ is a polynomial of order $s - |\lambda|$ in γ_s or of order $\sigma - |\nu|$ in γ_σ.
  4. Photons can be treated on an equal footing (Coulomb gauge).