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Analysis of $K\bar{K}\pi$ systems
—Version I—

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abstract

This note deals with decays of non-strange X^0 and X^- into the final state $K\bar{K}\pi$.

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1 Introduction

A complete treatment is given here of the problem of constructing I^G eigenstates of the $K\bar{K}\pi$ systems. This note relies on the results of a previous note by the author[1]. The reader is referred to this work for a thorough discussion of the I, G and C operators acting on elementary particles, quarks and the composite systems. It may be worth emphasizing that the I and C operators do not commute, and so the wave functions for anti-particles with non-zero hypercharges, such as those of K's and K^*, should be defined with care.

Section 2 is devoted to the classification of all the possible I^G eigenstates for $(K\bar{K}\pi)^0$ and $(K\bar{K}\pi)^-$ resulting from an interchange of the K with a $\bar{K}$. One may find the complete decay amplitudes for $(K\bar{K}\pi)^-$ in Sections 3 and 4, and those for $(K\bar{K}\pi)^0$ in Section 5. A comment on notation is in order; the wave functions and the corresponding parameters, whenever necessary, are specified with superscripts $\pm$ for the G-parity and with subscripts 0 or 1 for the isotopic spin I.

An important conclusion of this note is that $K^+K_s\pi^-$ and $K^-K_s\pi^+$ are fundamentally different and that the two data sets cannot be combined in general. One notable exception is the $\bar{p}p$ annihilation at rest leading to the final states $K^\pm K_s\pi^\mp$. The decay amplitudes for this process as well as that in flight are dealt with in Section 6. Conclusions are given in Section 7.

Although the final state $(K\bar{K}\pi)$ can come in $I = 0$, $I = 1$ or $I = 2$, this note deals mainly with $I = 0$ or $I = 1$ only, but it should be borne in mind that the $I = 2$ states—however small in magnitude—may nevertheless play a significant role through interference effects. In the Appendix, the states $\pi a_0(980)$ are expanded in all possible eigenstates of I, and it is shown that the states $\pi^+a_0^-(980)$ and $\pi^-a_0^+(980)$ *do* have different interference effects between I=even and I=odd states. It is demonstrated, finally, that the states of two charged pions, i.e. $\pi^+\pi^-$ and $\pi^-\pi^+$, are indistinguishable—as expected.

Finally, a comment on the symbols is in order: $\mathbb{I}^2$, $\mathbb{G}$, $\mathbb{C}$, $\mathbb{J}^2$ and $\mathbb{P}$ are used to indicate the operators for the isospin squared, the G-parity, the C-parity, total spin squared and the parity, with the corresponding eigenvalues denoted by $I(I+1)$, G (mostly written as g), C, $J(J+1)$ and P.

2 Classification of $K\bar{K}\pi$ systems

Consider production of a non-strange meson state X in

$$\begin{aligned} \pi^- p &\to X^- p \\ \pi^- p &\to X^0 n \end{aligned} \tag{1}$$

which then decay into

$$\begin{aligned} X^- &\to (K\bar{K}\pi)^- \\ X^0 &\to (K\bar{K}\pi)^0 \end{aligned} \tag{2}$$

The purpose of this note is to give a complete list of all possible decay modes and exhibit the interplay of I, C- and G-parities for each distinct final state.

For ease of reference, the actions of C and G operators[1] are collectively given here. For π's, one has

$$\mathbb{C}\begin{pmatrix}\pi^+\\ \pi^0\\ \pi^-\end{pmatrix} = \begin{pmatrix}-\pi^-\\ +\pi^0\\ -\pi^+\end{pmatrix}, \qquad \mathbb{G}\begin{pmatrix}\pi^+\\ \pi^0\\ \pi^-\end{pmatrix} = \begin{pmatrix}-\pi^+\\ -\pi^0\\ -\pi^-\end{pmatrix} \tag{3}$$

For K's, one has

$$\mathbb{C}\begin{pmatrix}K^+\\ K^0\end{pmatrix} = \begin{pmatrix}+K^-\\ -\bar{K}^0\end{pmatrix}, \qquad \mathbb{C}\begin{pmatrix}\bar{K}^0\\ K^-\end{pmatrix} = \begin{pmatrix}-K^0\\ +K^+\end{pmatrix} \tag{4}$$

and

$$\mathbb{G}\begin{pmatrix}K^+\\ K^0\end{pmatrix} = \begin{pmatrix}-\bar{K}^0\\ -K^-\end{pmatrix}, \qquad \mathbb{G}\begin{pmatrix}\bar{K}^0\\ K^-\end{pmatrix} = \begin{pmatrix}+K^+\\ +K^0\end{pmatrix} \tag{5}$$

where $\mathbb{C}^2 = 1$ and $\mathbb{G}^2 = (-)^{2I}$.

One assumes that the state X is in an eigenstate of $I^G\, J^{PC}$. If these quantum numbers are conserved in the decay, then the final state $K\bar{K}\pi$ must also be in the same eigenstate. In terms of the orbital angular momentum ℓ for the $K\bar{K}$ system, one finds readily the results given in Tables 1 and 2.

Table 1. Possible Decay Modes with a K^+

I	G	C	X	Decay Mode	ℓ *
0,1	$+$	$+,-$	X^0	$K^+K_s\pi^-$	even
0,1	$-$	$-,+$	X^0	$K^+K_s\pi^-$	odd
0,1	$+,-$	$+$	X^0	$K^+K^-\pi^0$	even
0,1	$-,+$	$-$	X^0	$K^+K^-\pi^0$	odd
1	$+,-$	$-,+$	X^-	$K^+K^-\pi^-$	even
1	$+,-$	$-,+$	X^-	$K^+K^-\pi^-$	odd

* Orbital angular momentum for the $K\bar{K}$ system

The decay modes are listed with K_s instead of K^0 or $\bar{K}^0$, as a neutral K is detected mainly through its decay $K_s \rightarrow \pi^+\pi^-$. An exotic X can come with $I = 2$, but it is not considered in this note. It is worth noting that neither the G nor the C is definite for the final state $K^+K^-\pi^-$, since the K^+K^- isospin is not known (it can be either 0 or 1).

Table 2. Possible Decay Modes with no K^+

I	G	C	X	Decay Mode	ℓ *
0,1	$+,-$	$+$	X^0	$K_sK_s\pi^0$	even
1	$+,-$	$-,+$	X^-	$K_sK_s\pi^-$	even
0,1	$+$	$+,-$	X^0	$K_sK^-\pi^+$	even
0,1	$-$	$-,+$	X^0	$K_sK^-\pi^+$	odd
1	$+$	$-$	X^-	$K_sK^-\pi^0$	even
1	$-$	$+$	X^-	$K_sK^-\pi^0$	odd

* Orbital angular momentum for the $K\bar{K}$ system

It should be understood that the quantum number ℓ applies even when the isobars do not involve the $K\bar{K}$ systems. For example, if the decay amplitude containing an isobar K^* or a $\bar{K}^*$ is even (odd) under the interchange of K and $\bar{K}$, then one should have ℓ=even (odd).

When one searches for an exotic meson, it is critical that one determines unambiguously its C-parity (along with its J^P). In this regard, there are four decay modes at our disposal

$$\begin{aligned} X^0 &\to K^+K^-\pi^0 \\ &\to K_s K_s \pi^0 \\ X^- &\to K_s K_s \pi^- \\ &\to K_s K^- \pi^0 \end{aligned} \tag{6}$$

Note, however, that the decay modes involving two K_s's suffer a visibility factor of 1/9 and that it must necessarily have ℓ=even.

3 K^* Isobar in the decay of X^-

In this section, a treatment is given of a state X^- decaying into a $K\bar{K}\pi$ system via K^* intermediate states.

Once again, one uses a convention in which ordering of particles signifies different momenta, so that one must keep track of it with care. A K^* decays into a πK. For K^*'s with positive strangeness, one has

$$\begin{aligned} K^{*+} &= \sqrt{\frac{2}{3}}\pi^+ K^0 - \sqrt{\frac{1}{3}}\pi^0 K^+ \\ K^{*0} &= \sqrt{\frac{1}{3}}\pi^0 K^0 - \sqrt{\frac{2}{3}}\pi^- K^+ \end{aligned} \tag{7}$$

and for negative strangeness

$$\begin{aligned} \bar{K}^{*0} &= \sqrt{\frac{2}{3}}\pi^+ K^- - \sqrt{\frac{1}{3}}\pi^0 \bar{K}^0 \\ K^{*-} &= \sqrt{\frac{1}{3}}\pi^0 K^- - \sqrt{\frac{2}{3}}\pi^- \bar{K}^0 \end{aligned} \tag{8}$$

It is seen that the C and G operators act on K^*'s in the following way:

$$\mathbb{C}\begin{pmatrix} K^{*+} \\ K^{*0} \end{pmatrix} = \begin{pmatrix} -K^{*-} \\ +\bar{K}^{*0} \end{pmatrix}, \qquad \mathbb{C}\begin{pmatrix} \bar{K}^{*0} \\ K^{*-} \end{pmatrix} = \begin{pmatrix} +K^{*0} \\ -\bar{K}^{*+} \end{pmatrix} \tag{9}$$

and

$$\mathbb{G}\begin{pmatrix} K^{*+} \\ K^{*0} \end{pmatrix} = \begin{pmatrix} +\bar{K}^{*0} \\ +K^{*-} \end{pmatrix}, \qquad \mathbb{G}\begin{pmatrix} \bar{K}^{*0} \\ K^{*-} \end{pmatrix} = \begin{pmatrix} -K^{*+} \\ -K^{*0} \end{pmatrix} \tag{10}$$

A state X^- decaying into a $K\bar{K}\pi$ system via K^* intermediate states can be represented by an amplitude

$$\begin{aligned} A(K^*) &= \sqrt{\frac{1}{2}}\left(K^{*0}K^- + g\,K^{*-}K^0\right) \\ &= \sqrt{\frac{1}{6}}\left[(\pi^0 K^0)_* K^- + g(\pi^0 K^-)_* K^0\right] - \sqrt{\frac{1}{3}}\left[(\pi^- K^+)_* K^- + g\,(\pi^- \bar{K}^0)_* K^0\right] \end{aligned} \tag{11}$$

where $g = \pm 1$ is the G-parity eigenvalue for the $K\bar{K}\pi$ system and the subscripts $*$ signify the K^* isobars. It is easy to see that, as expected,

$$\mathbb{G}\,A(K^*) = g\,A(K^*) \tag{12}$$

using (5) and (10).

It should be noted that the amplitude A does *not* possess symmetry under the interchange of a K with a $\bar{K}$, since neither $\pi^- K^-$ nor $\pi^- K^0$ can form a K^*. Therefore, one cannot apply—indiscriminately—the classification scheme of the previous section for the $K\bar{K}\pi$ system in which the final states are divided into those with either ℓ=even or with ℓ=odd. An examination of (11) shows that the decay channel $K_S K^- \pi^0$ can be either ℓ=even or ℓ=odd depending on whether $g = +1$ or $g = -1$.

4 Complete Decay Amplitudes for X^-

In order to write down a complete decay amplitude, one has first to specify the G-parity for X^-. Denoting the eigenvalue via a superscript, one finds, for the complete amplitude,[a]

$$A = A^+ + A^- \tag{13}$$

[a] The state X^- could in principle be in the $I = 2$ state, but this possibility is ignored in this note. It is assumed that X^- is an $I = 1$ state.

where

$$A^+ = x_1^+ \, A^+(K^*) + y_1^+ \, A(a)$$
$$A^- = x_1^- \, A^-(K^*) + y_1^- \, A(f) \tag{14}$$

where $A^\pm(K^*)$ is the amplitude given in the previous section for $g = \pm 1$, and $x_1^\pm$ and $y_1^\pm$ are arbitrary complex constants in general (the subscripts refer to I.) The arguments a's refer to $a_0(980)$, $a_2(1320)$ and other $I^G = 1^-$ objects, and f's stand for either $f_0(980)$, $f_2(1270)$ or other $I^G = 0^+$ states.[b] The corresponding amplitudes for X^- are given by

$$A(a) = \sqrt{\frac{1}{2}} \left[\pi^0 a^- - \pi^- a^0\right] \tag{15}$$

The a's have the following expansion:

$$\begin{aligned} a^0 &= \frac{1}{2} \left[K^+ K^- + K^0 \bar{K}^0 + \bar{K}^0 K^0 + K^- K^+\right] \\ a^- &= \sqrt{\frac{1}{2}} \left[K^0 K^- + K^- K^0\right] \\ a^+ &= \sqrt{\frac{1}{2}} \left[K^+ \bar{K}^0 + \bar{K}^0 K^+\right] \end{aligned} \tag{16}$$

Note that these amplitudes have the correct G-parity, i.e.

$$\mathbb{G}\, a^0 = -a^0, \qquad \mathbb{G}\, a^- = -a^-, \qquad \mathbb{G}\, a^+ = -a^+ \tag{17}$$

The C operator acts on a in exactly the same way as on π, see (3). Substituting (16) into (15), one finds for the amplitude

$$\begin{aligned} A(a) = \frac{1}{2} & \left[\pi^0 (K^0 K^-)_a + \pi^0 (K^- K^0)_a\right] \\ & - \frac{1}{2\sqrt{2}} \left[\pi^- (K^+ K^-)_a + \pi^- (K^0 \bar{K}^0)_a + \pi^- (\bar{K}^0 K^0)_a + \pi^- (K^- K^+)_a\right] \end{aligned} \tag{18}$$

where the subscripts a have been added to indicate that the isovector isobars a are being treated. As expected, one finds

$$\mathbb{G}\, A(a) = +A(a) \tag{19}$$

[b] Only even spins for the isobars a and f are treated here. The states $\rho(1450)$ and $\rho_3(1690)$ have very little decay modes into $K\bar{K}$. The $\phi(1020)$ should be handled separately as if it were a stable particle because of its narrow width. The $\phi_3(1850)$ is omitted from the list of potential $K\bar{K}$ isobars, because of its high spin and its high mass.

for a decay of X^- into $\pi + a$.

By applying the same technique, one first writes down the wave function for f in an eigenstate of positive G-parity

$$f = \frac{1}{2}\left[K^+K^- - K^0\bar{K}^0 - \bar{K}^0K^0 + K^-K^+\right] \tag{20}$$

so that $Gf = +f$ and $Cf = +f$. One then finds

$$A(f) = \frac{1}{2}\left[\pi^-(K^+K^-)_f - \pi^-(K^0\bar{K}^0)_f - \pi^-(\bar{K}^0K^0)_f + \pi^-(K^-K^+)_f\right] \tag{21}$$

which obeys

$$\mathbb{G}\,A(f) = -A(f) \tag{22}$$

as required for a decay of X^- into $\pi^- + f$.

One is now ready to examine the complete decay amplitudes for all the decay modes of $X^- \to (K\bar{K}\pi)^-$. For example, the appropriate amplitude for the final state $\pi^0K_sK^-$ should read

$$\begin{aligned} A = \sqrt{\frac{1}{6}}\Big\{ x_1^+ \ \left[(\pi^0K_s)_*\,K^- + (\pi^0K^-)_*\,K_s\right]_1 + x_1^- \ \Big[(\pi^0K_s)_*\,K^- - (\pi^0K^-)_*\,K_s\Big]_1 \Big\} \\ + \left(\frac{1}{2}\right)\ y_1^+ \ \left[\pi^0(K_sK^-)_a + \pi^0(K^-K_s)_a\right]_1 \end{aligned} \tag{23}$$

where the subscripts at the square bracket indicate isotopic spin. For the final state $\pi^-K^+K^-$, one has

$$\begin{aligned} A = -\sqrt{\frac{1}{3}}\left\{x_1^+ \ \left[(\pi^-K^+)_*\,K^-\right]_1^+ + x_1^- \ \left[(\pi^-K^+)_*\,K^-\right]_1^- \right\} \\ - \left(\frac{1}{2\sqrt{2}}\right)\ y_1^+ \ \left[\pi^-(K^+K^-)_a + \pi^-(K^-K^+)_a\right]_1 \\ + \left(\frac{1}{2}\right)\ y_1^- \ \left[\pi^-(K^+K^-)_f + \pi^-(K^-K^+)_f\right]_1 \end{aligned} \tag{24}$$

where the superscript at the square bracket indicates the G-parity. [Note that the final state $(\pi^-K^+)_*\,K^-$ is not an eigenstate of G-parity.] In table 3 are listed several sample decay modes corresponding to the parameters $x_1^\pm$ and $y_1^\pm$.

Table 3. Possible Decay Modes for $X^- \to (K\bar{K}\pi)^-$

I^G	J^{PC}	X^-	Decay Mode	L of decay [a]	coeffs.
1^+	1^{--}	$\rho(1700)$	$K^*(892)\bar{K}$	P-wave	x_1^+
1^-	2^{-+}	$\pi_2(1670)$	$K^*(892)\bar{K}$	P-wave	x_1^-
1^+	3^{--}	$\rho_3(1690)$	$a_2(1320)\pi$ [b]	D-wave	y_1^+
1^-	2^{-+}	$\pi_2(1670)$	$f_2(1270)\pi$	D-wave	y_1^-

[a] Orbital angular momentum between the isobar and the bachelor particle

[b] Hypothetical decay mode

Finally, one finds, for the final state $\pi^- K_S K_S$,

$$\begin{aligned} A = -\sqrt{\frac{1}{3}} \left\{ x_1^+ \left[(\pi^- K_S)_* K_S\right]_1^+ - x_1^- \left[(\pi^- K_S)_* K_S\right]_1^- \right\} \\ + \sqrt{\frac{1}{2}} y_1^+ \left[\pi^-(K_S K_S)_a\right]_1 - y_1^- \left[\pi^-(K_S K_S)_f\right]_1 \end{aligned} \tag{25}$$

The final state $(\pi^- K_S)_* K_S$ requires a comment. Since there are two identical particles K_S, the orbital angular momentum ℓ between them has to be even always. But the state $K_S K_S$ is a superposition of $I = 0$ and $I = 1$, so that the state $(\pi^- K_S)_* K_S$ is in turn a superposition of two G-parity eigenstates.

The above formulas show that one can combine the data samples, $\pi^0 K_S K^-$, $\pi^- K^+ K^-$ and $\pi^- K_S K_S$ at the stage of the partial-wave analysis. One has to be scrupulous about keeping track of the Clebsch-Gordan coefficients, as well as the visibility factor of 1/3 for $K_S \to \pi^+\pi^-$. In addition, one must remember that the experimental acceptances are different for each data sample. The partial waves are to be classified according to J^P and $G = -C$ ($I = 1$ in this case), and for each wave one must determine the parameters $x_1^\pm$ and $y_1^\pm$.

5 Complete Decay Amplitudes for X^0

Much of the work on the neutral $K\bar{K}\pi$ system has been given in a previous note[1]. Here one has reproduced some of the work, with emphasis on writing down the complete decay amplitudes for a neutral $K\bar{K}\pi$ system.[c]

One treats first the problem of introducing the K^* isobars. From (42) of the previous note[1], one sees that an amplitude for X^0 in a given eigenstate of I (I is equal to 1 or 0.) is given by

$$A_I^g(K^*) = \frac{1}{2}\left[\left(K^{*+}K^- + g\,\bar{K}^{*0}K^0\right) - (-)^I\left(K^{*0}\bar{K}^0 + g\,K^{*-}K^+\right)\right] \tag{26}$$

where g is the eigenvalue of the G-parity for the neutral $K\bar{K}\pi$ system, i.e.

$$\mathbb{G}\,A_I^g(K^*) = g\,A_I^g(K^*), \qquad \mathbb{C}\,A_I^g(K^*) = g(-)^I\,A_I^g(K^*) \tag{27}$$

Using (7) and (8), one finds

$$\begin{aligned} A_I^g(K^*) =& \sqrt{\frac{1}{6}}\Big\{\,\left[(\pi^+K^0)_*\,K^- + g\,(\pi^+K^-)_*\,K^0\right] \\ &\qquad + (-)^I\,\left[(\pi^-K^+)_*\,\bar{K}^0 + g\,(\pi^-\bar{K}^0)_*\,K^+\right]\Big\} \\ &- \sqrt{\frac{1}{12}}\Big\{\,\left[(\pi^0K^+)_*\,K^- + g\,(\pi^0\bar{K}^0)_*\,K^0\right] \\ &\qquad + (-)^I\,\left[(\pi^0K^0)_*\,\bar{K}^0 + g\,(\pi^0K^-)_*\,K^+\right]\Big\} \end{aligned} \tag{28}$$

or, equivalently,

$$\begin{aligned} A_I^g(K^*) =& \sqrt{\frac{1}{6}}\Big\{\,\left[(\pi^+K^0)_*\,K^- + g(-)^I\,(\pi^-\bar{K}^0)_*\,K^+\right] \\ &\qquad + (-)^I\,\left[(\pi^-K^+)_*\,\bar{K}^0 + g(-)^I\,(\pi^+K^-)_*\,K^0\right]\Big\} \\ &- \sqrt{\frac{1}{12}}\Big\{\,\left[(\pi^0K^+)_*\,K^- + g(-)^I\,(\pi^0K^-)_*\,K^+\right] \\ &\qquad + (-)^I\,\left[(\pi^0K^0)_*\,\bar{K}^0 + g(-)^I\,(\pi^0\bar{K}^0)_*\,K^0\right]\Big\} \end{aligned} \tag{29}$$

[c] The state X^0 is in general an admixture of $I = 0$, $I = 1$ and $I = 2$ eigenstates, but the $I = 2$ state is ignored in this note.

Note that the two wave functions above have been classified according to their G-parity eigenvalues given by g or to their C-parity eigenvalues given by $g(-)^I$. It should be emphasized that (28) is *different* from (44) of the previous note[1], which explored the consequence of interchanging K and $\bar{K}$; the resulting wave function there breaks up into two parts, the one an eigenstate of G and the other of C, whereas (28) is is constructed to be in an eigenstate of G only.

One may introduce next the isobars a and f, so that the complete amplitude may now be written

$$A = A_0^+ + A_1^+ + A_0^- + A_1^- \tag{30}$$

where

$$\begin{aligned} A_0^+ &= x_0^+\, A_0^+(K^*) + y_0^+\, A_0(a) \\ A_1^+ &= x_1^+\, A_1^+(K^*) + y_1^+\, A_1(a) \\ A_0^- &= x_0^-\, A_0^-(K^*) \\ A_1^- &= x_1^-\, A_1^-(K^*) + y_1^-\, A_1(f) \end{aligned} \tag{31}$$

where the superscripts $\pm$ once again specifies $g = \pm 1$ and the subscripts 0 or 1 stand for I. Note that an isoscalar X^0 cannot couple to $\pi^0 + f$, so that one must set $y_0^- = 0$. The amplitudes with a and f isobars are

$$\begin{aligned} A_0(a) &= \sqrt{\frac{1}{3}}\,(\pi^+\, a^- - \pi^0\, a^0 + \pi^-\, a^+) \\ &= \sqrt{\frac{1}{6}}\,[\pi^+(K^0K^-)_a + \pi^+(K^-K^0)_a + \pi^-(K^+\bar{K}^0)_a + \pi^-(\bar{K}^0K^+)_a] \\ &- \sqrt{\frac{1}{12}}[\pi^0(K^+K^-)_a + \pi^0(K^0\bar{K}^0)_a + \pi^0(\bar{K}^0K^0)_a + \pi^0(K^-K^+)_a] \\ A_1(a) &= \sqrt{\frac{1}{2}}\,(\pi^+\, a^- - \pi^-\, a^+) \\ &= \frac{1}{2}[\pi^+(K^0K^-)_a + \pi^+(K^-K^0)_a - \pi^-(K^+\bar{K}^0)_a - \pi^-(\bar{K}^0K^+)_a] \end{aligned} \tag{32}$$

and

$$A_1(f) = \frac{1}{2}\left[\pi^0(K^+K^-)_f - \pi^0(K^0\bar{K}^0)_f - \pi^0(\bar{K}^0K^0)_f + \pi^0(K^-K^+)_f\right] \tag{33}$$

One sees that $CA_0(a) = +A_0(a)$, $CA_1(a) = -A_1(a)$ and $CA_1(f) = +A_1(f)$.

Consider, as an example, a sample of $\pi^0 K^+ K^-$ events. The appropriate amplitude is

$$A = -\sqrt{\frac{1}{12}}\Big\{ x_0^+ \left[(\pi^0 K^+)_* K^- + (\pi^0 K^-)_* K^+\right]_0 + x_1^- \left[(\pi^0 K^+)_* K^- + (\pi^0 K^-)_* K^+\right]_1$$
$$+ x_0^- \left[(\pi^0 K^+)_* K^- - (\pi^0 K^-)_* K^+\right]_0 + x_1^+ \left[(\pi^0 K^+)_* K^- - (\pi^0 K^-)_* K^+\right]_1 \Big\}$$
$$-\sqrt{\frac{1}{12}}\, y_0^+ \left[\pi^0 (K^+ K^-)_a + \pi^0 (K^- K^+)_a\right]_0 + \left(\frac{1}{2}\right) y_1^- \left[\pi^0 (K^+ K^-)_f + \pi^0 (K^- K^+)_f\right]_1 \tag{34}$$

where the subscripts at the square brackets indicate isotopic spin. Sample decay modes for the isovector states are already given in Table 3; those for the isoscalar states are listed in Table 4.

Table 4. Possible Decay Modes for $X^0 \to (K\bar{K}\pi)^0$

I^G	J^{PC}	X^0	Decay Mode	L of decay [a]	coeffs.
0^+	1^{++}	$f_1(1420)$	$K^*(892)\bar{K}$	S, D-wave	x_0^+
0^-	1^{--}	$\phi(1680)$	$K^*(892)\bar{K}$	P-wave	x_0^-
0^+	0^{-+}	$\eta(1440)$	$a_0(980)\pi$	S-wave	y_0^+

[a] Orbital angular momentum between the isobar and the bachelor particle

One sees that the first and the second lines above have been grouped together corresponding to the C-parity eigenstates. If the $K\bar{K}$ intermediate states happen to be $a_0(980)$ and $f_0(980)$, then the last two terms, both of which have a positive C-parity, assume the same form and hence could not be distinguished.

Consider next the amplitude corresponding to $\pi^- K_S K^+$. The complete amplitude is

$$A = \sqrt{\frac{1}{6}}\Big\{ x_0^+ \ \left[(\pi^- K^+)_* K_S + (\pi^- K_S)_* K^+\right]_0 - x_1^+ \ \left[(\pi^- K^+)_* K_S + (\pi^- K_S)_* K^+\right]_1$$
$$+x_0^- \ \left[(\pi^- K^+)_* K_S - (\pi^- K_S)_* K^+\right]_0 - x_1^- \ \left[(\pi^- K^+)_* K_S - (\pi^- K_S)_* K^+\right]_1 \Big\}$$
$$+ \left(\frac{1}{2}\right)\Big\{ \sqrt{\frac{2}{3}} y_0^+ \ \left[\pi^-(K^+ K_S)_a + \pi^-(K_S K^+)_a\right]_0 - \ y_1^+ \ \left[\pi^-(K^+ K_S)_a + \pi^-(K_S K^+)_a\right]_1 \Big\} \tag{35}$$

Similarly one finds, for the $\pi^+ K_S K^-$ amplitude,

$$A = \sqrt{\frac{1}{6}}\Big\{ x_0^+ \ \left[(\pi^+ K^-)_* K_S + (\pi^+ K_S)_* K^-\right]_0 + x_1^+ \ \left[(\pi^+ K^-)_* K_S + (\pi^+ K_S)_* K^-\right]_1$$
$$+x_0^- \ \left[(\pi^+ K_S)_* K^- - (\pi^+ K^-)_* K_S\right]_0 + x_1^- \ \left[(\pi^+ K_S)_* K^- - (\pi^+ K^-)_* K_S\right]_1 \Big\}$$
$$+ \left(\frac{1}{2}\right)\Big\{ \sqrt{\frac{2}{3}} y_0^+ \ \left[\pi^+(K^- K_S)_a + \pi^+(K_S K^-)_a\right]_0 + y_1^+ \ \left[\pi^+(K^- K_S)_a + \pi^+(K_S K^-)_a\right]_1 \Big\} \tag{36}$$

or, equivalently,

$$A = \sqrt{\frac{1}{6}}\Big\{ x_0^+ \ \left[(\pi^+ K^-)_* K_S + (\pi^+ K_S)_* K^-\right]_0 + x_1^+ \ \left[(\pi^+ K^-)_* K_S + (\pi^+ K_S)_* K^-\right]_1$$
$$-x_0^- \ \left[(\pi^+ K^-)_* K_S - (\pi^+ K_S)_* K^-\right]_0 - x_1^- \ \left[(\pi^+ K^-)_* K_S - (\pi^+ K_S)_* K^-\right]_1 \Big\}$$
$$+ \left(\frac{1}{2}\right)\Big\{ \sqrt{\frac{2}{3}} y_0^+ \ \left[\pi^+(K^- K_S)_a + \pi^+(K_S K^-)_a\right]_0 + y_1^+ \ \left[\pi^+(K^- K_S)_a + \pi^+(K_S K^-)_a\right]_1 \Big\} \tag{37}$$

The final states $\pi^- K_S K^+$ and $\pi^+ K_S K^-$ are fundamentally different, if the initial state consists of an arbitrary admixture of I^G and C eigenstates. One must bear in mind, therefore, that even the cross sections for the two final states could be different. An examination of both (35) and (36) shows that the two data sets, $\pi^- K_S K^+$ and $\pi^+ K_S K^-$, could be combined at the analysis stage, if the final states consist entirely of $I = 0$ or $I = 1$ but *not* both.[d] But one should be careful to combine them according to their strangeness; in a Dalitz plot analysis, one must match the variables $M^2(\pi^- K^+)$ with $M^2(\pi^+ K_S)$ and $M^2(\pi^- K_S)$ with $M^2(\pi^+ K^-)$.

[d] It turns out that a short discussion on this topic in Ref. 1 did not properly spell out the necessary assumptions. The new note with this correction is now called Version III.

Note that (36) and (37) differ only in the sign of the second line, but this leads to a profound change in the interpretation of the two data sets. More specifically, the signs of x_1^+, x_0^- and y_1^+ have changed from (35) to (37)—note that all these coefficients correspond to the $C = -1$ eigenstates, while the coefficients x_0^+, x_1^- and y_0^+, which does *not* change sign, all belong to $C = +1$ eigenstates. Once again, the two data sets, $\pi^- K_S K^+$ and $\pi^+ K_S K^-$, could be combined for analysis, if it consisted entirely of $C = +1$ or $C = -1$ states but *not* both; or, if the two C eigenstates do not interfere, as given in an example in the next section. It should be emphasized that the two data sets are to be combined in accordance with the C conjugation; in a Dalitz plot analysis, one must therefore match the variables $M^2(\pi^- K^+)$ with $M^2(\pi^+ K^-)$ and $M^2(\pi^- K_S)$ with $M^2(\pi^+ K_S)$.

Finally, the amplitude for the all-neutral channel $\pi^0 K_S K_S$ is given by

$$\begin{aligned} A = -\sqrt{\frac{1}{3}} \Big\{ & x_0^+ \left[(\pi^0 K_S)_* K_S\right]_0^+ - x_1^- \left[(\pi^0 K_S)_* K_S\right]_1^- \Big\} \\ & - \sqrt{\frac{1}{3}} y_0^+ \left[\pi^0 (K_S K_S)_a\right]_0 - y_1^- \left[\pi^0 (K_S K_S)_f\right]_1 \end{aligned} \tag{38}$$

The final state $(\pi^0 K_S)_* K_S$ needs to be treated with care. The orbital angular momentum between the two K_S's must be even, so that the state $(\pi^0 K_S)_* K_S$ is in an eigenstate of $C = +1$. But the system is a superposition of $I = 0$ and $I = 1$ states, resulting in two different eigenstates of G-parity. Once again, one gains extra information by combining, at the analysis stage, the events of the type $\pi^0 K^+ K^-$ with those of the type $\pi^\pm K_S K^\mp$ or $\pi^0 K_S K_S$, but then one must keep careful track of the coefficients. It is noted that the visibility factor of 1/3 for each $K_S \to \pi^+\pi^-$ is not included in the above formulas.

One of the most important result of this note is that one can in fact determine I of the neutral $K\bar{K}\pi$ system, if one performs a combined analysis of the data of the type $\pi^0 K^+ K^-$, $\pi^+ K_S K^-$, $\pi^- K_S K^+$ and $\pi^0 K_S K_S$. For each partial wave with a given set of $I^G J^{PC}$, there are in general seven parameters to be determined, i.e. $x_I^\pm$ and $y_I^\pm$ for $I = 0$ and $I = 1$ (note that $y_0^- = 0$).

6 $\bar{p}p$ Annihilation into $\pi^{\mp}K_S K^{\pm}$

The $\bar{p}p$ Annihilation processes at rest and in flight into $\pi^{\mp}K_S K^{\pm}$ are treated in this section.

Assume that the annihilation at rest takes place in the S wave only. Then the initial states consists of ${}^1S_0\,(J^{PC}=0^{-+})$ and ${}^3S_1\,(J^{PC}=1^{--})$, and they do not interfere, i.e. the states with $C=+1$ and $C=-1$ add incoherently. In Table 5 are listed possible decay modes for this process.

Table 5. Possible Decay Modes for $\bar{p}p \to \pi^{\mp}K_S K^{\pm}$

I^G	J^{PC}	$\bar{p}p$	Decay Mode	L of decay*	coeffs.
0^+	0^{-+}	$\eta(2000)$	$K^*(892)\bar{K}$	P-wave	x_0^+
1^-	0^{-+}	$\pi(2000)$	$K^*(892)\bar{K}$	P-wave	x_1^-
0^+	0^{-+}	$\eta(2000)$	$a_0(980)\pi$	S-wave	y_0^+
0^-	1^{--}	$\omega(2000)$	$K^*(892)\bar{K}$	P-wave	x_0^-
1^+	1^{--}	$\rho(2000)$	$K^*(892)\bar{K}$	P-wave	x_1^+
1^+	1^{--}	$\rho(2000)$	$a_2(1320)\pi$	D-wave	y_1^+

* Orbital angular momentum between the isobar and the bachelor particle

Denoting the C eigenstates by superscripts $(\pm)$ in A, one finds that the amplitudes corresponding to $\pi^- K_S K^+$ are, from (35),

$$\begin{aligned}
{}^{(+)}A = & \sqrt{\frac{1}{6}}\,x_0^+\ \left[(\pi^-K^+)_*\,K_S + (\pi^-K_S)_*\,K^+\right]_0 \\
& -\sqrt{\frac{1}{6}}\,x_1^-\ \left[(\pi^-K^+)_*\,K_S - (\pi^-K_S)_*\,K^+\right]_1 \\
& +\sqrt{\frac{1}{6}}\,y_0^+\ \left[\pi^-(K^+K_S)_a + \pi^-(K_SK^+)_a\right]_0
\end{aligned} \tag{39}$$

and

$$
\begin{aligned}
{}^{(-)}A = &-\sqrt{\frac{1}{6}}\,x_1^+\ \left[(\pi^-K^+)_*\,K_s + (\pi^-K_s)_*\,K^+\right]_1 \\
&+\sqrt{\frac{1}{6}}\,x_0^-\ \left[(\pi^-K^+)_*\,K_s - (\pi^-K_s)_*\,K^+\right]_0 \\
&\qquad -\left(\frac{1}{2}\right)\,y_1^+\ \left[\pi^-(K^+K_s)_a + \pi^-(K_sK^+)_a\right]_1
\end{aligned}
\tag{40}
$$

The differential cross section is proportional to

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}M_{12}^2\,\mathrm{d}M_{23}^2} \propto |\,{}^{(+)}A\,|^2 + |\,{}^{(-)}A\,|^2 \tag{41}
$$

where the effective masses M_{12}^2 and M_{23}^2 are the Dalitz-plot variables. Similarly, the amplitudes for $\pi^+K_sK^-$ are given by, from (36),

$$
\begin{aligned}
{}^{(+)}A = &\sqrt{\frac{1}{6}}\,x_0^+\ \left[(\pi^+K^-)_*\,K_s + (\pi^+K_s)_*\,K^-\right]_0 \\
&-\sqrt{\frac{1}{6}}\,x_1^-\ \left[(\pi^+K^-)_*\,K_s - (\pi^+K_s)_*\,K^-\right]_1 \\
&\qquad +\sqrt{\frac{1}{6}}\,y_0^+\ \left[\pi^+(K^-K_s)_a + \pi^+(K_sK^-)_a\right]_0
\end{aligned}
\tag{42}
$$

and

$$
\begin{aligned}
{}^{(-)}A = &\sqrt{\frac{1}{6}}\,x_1^+\ \left[(\pi^+K^-)_*\,K_s + (\pi^+K_s)_*\,K^-\right]_1 \\
&-\sqrt{\frac{1}{6}}\,x_0^-\ \left[(\pi^+K^-)_*\,K_s - (\pi^+K_s)_*\,K^-\right]_0 \\
&\qquad +\left(\frac{1}{2}\right)\,y_1^+\ \left[\pi^+(K^-K_s)_a + \pi^+(K_sK^-)_a\right]_1
\end{aligned}
\tag{43}
$$

This shows that the absolute value of the C eigenstates for $\pi^-K_sK^+$ and $\pi^+K_sK^-$ are the same, leading to identical cross sections. One must remember, however, that the two data sets could be combined in a Dalitz plot analysis, by matching the variables related by the C conjugation, i.e. $M^2(\pi^-K^+)$ with $M^2(\pi^+K^-)$ and $M^2(\pi^-K_s)$ with $M^2(\pi^+K_s)$.

In the previous section, it was noted that the final states $\pi^-K_sK^+$ and $\pi^+K_sK^-$ are not the same in general. However, the $\bar{p}p$ annihilation process at rest is a special case; if C-parity is conserved and the two different C eigenstates do not interfere, then the two final states become indistinguishable. It is clear that the same conclusion must hold for the final state $(K\bar{K}\pi)^0\,(\pi\pi)^0$ as well. Because of the limited phase space in the $\bar{p}p$ annihilation at rest, the $(\pi\pi)^0$ has a very limited mass range and so it becomes predominantly the vacuum state $I^G = 0^+$ and $J^{PC} = 0^{++}$, indicating that the results of this section could be applied to this final state without modification.

However, the same argument cannot be applied to the $\bar{p}p$ annihilation in flight, where the initial state in general can consist of different I^G eigenstates. Let S and L be the total intrinsic spin and the orbital angular momentum of the $\bar{p}p$ states in its CM system. Then the $\bar{p}p$ helicity-coupling amplitude F^J can be expressed in terms of the LS-coupling amplitudes G^J_{LS},

$$F^J_{\nu_1\nu_2} = \sum_{LS} \sqrt{\frac{2L+1}{2J+1}}(L0\ S\nu|J\nu)(s_1\nu_1\ s_2{-}\nu_2|S\nu)G^J_{LS} \tag{44}$$

where ν_1 and ν_2 are nucleon helicities ($\nu = \nu_1 - \nu_2$), with the normalization given by

$$\sum_{\nu_1\nu_2} |F^J_{\nu_1\nu_2}|^2 = \sum_{LS} |G^J_{LS}|^2 \tag{45}$$

It is more appropriate to parameterize the process in terms of the F^J's, as the transition amplitude depends directly on the helicities of the initial $\bar{p}p$ system and different helicity states do not interfere.

From parity conservation in the process $|JM\rangle \ \rightarrow\ \bar{p}p$, one has

$$F^J_{-\nu_1-\nu_2} = \eta_J(-)^J F^J_{\nu_1\nu_2} \tag{46}$$

where η_J is the intrinsic parity of the $\bar{p}p$ system given by $(-)^{L+1}$. Note also that $C = (-)^{L+S}$. From (44), one sees that

$$F^J_{+-} = 0 \quad \text{if} \quad S = 0$$

and

$$F^J_{++} = 0 \quad \text{if} \quad L + S - J = \text{ odd}$$

As an example, if the $\bar{p}p$ annihilations involve all possible J^{PC} states up to $J=2$, one finds that

$$\begin{aligned} A_{++} &= \sum_{JLS} F^J_{++} = 0^{-+}(\,^1S_0) \,+\, 1^{+-}(\,^1P_1) \,+\, 2^{-+}(\,^1D_2) \\ &\qquad + 1^{--}(\,^3S_1) \,+\, 1^{--}(\,^3D_1) \,+\, 2^{++}(\,^3P_2) \,+\, 2^{++}(\,^3F_2) \\ A_{+-} &= \sum_{JLS} F^J_{+-} = 1^{++}(\,^3P_1) \,+\, 2^{--}(\,^3D_2) \\ &\qquad + 1^{--}(\,^3S_1) \,+\, 1^{--}(\,^3D_1) \,+\, 2^{++}(\,^3P_2) \,+\, 2^{++}(\,^3F_2) \end{aligned} \tag{47}$$

where each state given above stands for its amplitude and is given by G^J_{LS} in (44). Note that the state $0^{++}(\,^3P_0)$ does not contribute to a three-pseudoscalar final state. It should be emphasized that each $\bar{p}p$ state consists, in addition, of two coherent sums of I^G eigenstates for $I=0$ or $I=1$ (G is fixed, once I and C are given), and there should be complex parameters 'x' and/or 'y' for each $I^G\{J,L,S\}$ depending on the isobars under consideration.

The differential cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\Theta\,\mathrm{d}M^2_{12}\,\mathrm{d}M^2_{23}} \propto |A_{++}|^2 + |A_{+-}|^2 \tag{48}$$

where Θ specifies the angle of the beam with respect to the $K\bar{K}\pi$ plane in the overall CM system. Consider now the case in which the beam direction is integrated over and analysis is limited to the Dalitz plot. Because of the orthonormality of the D-functions, the states F^J_{++} and F^J_{+-} with different J's break up and do not interfere but the states of different P's and C's remain. One must conclude, therefore, that the final states $\pi^- K_S K^+$ and $\pi^+ K_S K^-$ should be different in general.

Finally, one must point out that the $\bar{p}p$ states listed in (47) appear in general with different coefficients in A_{++} and A_{+-}. It is best to work out a simple example, to illustrate this point. Consider the two amplitudes, $^3S_1$ and $^3D_1$, in the $J^{PC}=1^{--}$ state. Suppressing the indices J and S, both of which are equal to 1, one finds

$$\begin{aligned} \sqrt{2}\,F_{++} &= \sqrt{\frac{1}{3}}\,(G_0 - \sqrt{2}\,G_2) \\ \sqrt{2}\,F_{+-} &= \sqrt{\frac{1}{3}}\,(\sqrt{2}\,G_0 + G_2) \end{aligned} \tag{49}$$

where G_0 and G_2 correspond to the amplitudes for ${}^3S_1$ and ${}^3D_1$, respectively. This shows that the state of a given set $\{J, L, S\}$ appears in general with different strengths and signs in A_{++} and A_{+-}. Note how the normalization condition of (45) is satisfied by the equations given above. Each of the G_L's given above can be expanded in terms of the G-parity eigenstates of (35) or (36) depending on the final state being considered.

7 Conclusions

In this note is given a complete treatment of the problem of decomposing the final states $(K\bar{K}\pi)^-$ and $(K\bar{K}\pi)^0$ into I^G eigenstates.

It is noted that the two neutral states $K^+\bar{K}_S\pi^-$ and $K^-\bar{K}_S\pi^+$ (and, equivalently, those in which K_S is replaced by K_L) do have fundamentally different physics because of the difference in sign of the interference terms between I=even and I=odd terms. If, however, the $(K\bar{K}\pi)^0$ system at a given mass region contains a single resonance with fixed $I^G\,J^{PC}$, then it is clear that the final states $K^+\bar{K}_S\pi^-$ and $K^-\bar{K}_S\pi^+$ are identical, and one may combine the two data sets together by pairing the $K\pi$ states according to their strangeness, i.e. K^* vs. $\bar{K}^*$. One notable exception is the $\bar{p}p$ annihilation at rest into $(K\bar{K}\pi)^0$, where there exist two non-interfering $C=+1$ and $C=-1$ eigenstates. Because of this, the final states $K^+\bar{K}_S\pi^-$ and $K^-\bar{K}_S\pi^+$ are again equivalent. Since one of the two amplitudes could be obtained by applying a C operator on the other, one should combine the two data sets together by pairing the $K\pi$ states according to those related by the C conjugation, i.e. K^*(charged) vs. K^*(neutral).

Appendix

It might be instructive to work out an example of a neutral system X^0 coupling to the final state consisting a 'π' and an 'a.' Here the symbol 'a' is a shorthand notation for, e.g. $a_0(980) \to K\bar{K}$.

Assume that X^0 is produced in reaction (1). One follows the technique of the preceding sections and display the I eigenstates of the $(\pi + a)$ system

$$\begin{aligned} A_0 &= \sqrt{\frac{1}{3}}(\pi^+ a^- - \pi^0 a^0 + \pi^- a^+) \\ A_1 &= \sqrt{\frac{1}{2}}(\pi^+ a^- - \pi^- a^+) \\ A_2 &= \sqrt{\frac{1}{6}}(\pi^+ a^- + 2\,\pi^0 a^0 + \pi^- a^+) \end{aligned} \tag{50}$$

The complete amplitude may be written

$$A = y_0^+\, A_0 + y_1^+\, A_1 + y_2^+\, A_2 \tag{51}$$

where y_0^+, y_1^+ and y_2^+ are complex parameters fixed by the production process (1). The amplitude for each final state can now be read off from the general formula above

$$\begin{aligned} A_{+-} &= \sqrt{\frac{1}{3}} y_0^+\, (\pi^+ a^-)_0 + \sqrt{\frac{1}{2}} y_1^+\, (\pi^+ a^-)_1 + \sqrt{\frac{1}{6}} y_2^+\, (\pi^+ a^-)_2 \\ A_{00} &= -\sqrt{\frac{1}{3}} y_0^+\, (\pi^0 a^0)_0 + \sqrt{\frac{2}{3}}\, y_2^+\, (\pi^0 a^0)_2 \\ A_{-+} &= \sqrt{\frac{1}{3}} y_0^+\, (\pi^- a^+)_0 - \sqrt{\frac{1}{2}} y_1^+\, (\pi^- a^+)_1 + \sqrt{\frac{1}{6}} y_2^+\, (\pi^- a^+)_2 \end{aligned} \tag{52}$$

where the two subscripts in A specify the charge states of π and a. The amplitudes for a given I, denoted by $(\pi a)_I$ above, are superpositions of partial waves ℓ, if the state X^0 itself is a superposition of many neutral states. It is clear that the amplitudes A_{+-}, A_{00} and A_{-+} are all different in general, and one needs to measure all of them in order to determine the parameters y_0^+, y_1^+ and y_2^+.

It is instructive to re-derive (52) by a different approach. For the purpose, one may resort to the ket notation and write

$$\begin{aligned} |\pi^+ a^-\rangle &= \sqrt{\frac{1}{3}}|(\pi a)00\rangle + \sqrt{\frac{1}{2}}|(\pi a)10\rangle + \sqrt{\frac{1}{6}}|(\pi a)20\rangle \\ |\pi^0 a^0\rangle &= -\sqrt{\frac{1}{3}}|(\pi a)00\rangle + \sqrt{\frac{2}{3}}|(\pi a)20\rangle \\ |\pi^- a^+\rangle &= \sqrt{\frac{1}{3}}|(\pi a)00\rangle - \sqrt{\frac{1}{2}}|(\pi a)10\rangle + \sqrt{\frac{1}{6}}|(\pi a)20\rangle \end{aligned} \tag{53}$$

where the ket states on the right-hand side stand for $|II_z\rangle$. The transition amplitudes for reaction (1) may be written

$$\begin{aligned} A_{+-} &= \langle \pi^+ a^- \, n|T|\pi^- p\rangle \\ A_{00} &= \langle \pi^0 a^0 \, n|T|\pi^- p\rangle \\ A_{-+} &= \langle \pi^- a^+ \, n|T|\pi^- p\rangle \end{aligned} \tag{54}$$

suppressing the nucleon helicities. Similarly, the amplitude for production and decay of a definite isotopic spin with a positive G-parity is

$$y_I^+ \, (\pi a)_I = \langle (\pi a) I0, \, n|T|\pi^- p\rangle \tag{55}$$

and the decay amplitude of the (πa) system in $|II_z\rangle = |I0\rangle$ could be written

$$(\pi a)_I = \sum_{\ell m} c_I(\ell m) \, Y_\ell^m(\Omega) \tag{56}$$

where $c_I(\ell m)$ is the coefficient in an expansion in the spherical harmonics $Y_\ell^m(\Omega)$. It is clear that, by substituting (53) and (55) into (54), one obtains the desired result given in (52). Note that the charges appearing in $(\pi a)_I$ in (52) are *superfluous*. For example, all three expressions $(\pi^+ a^-)_0$, $(\pi^0 a^0)_0$ and $(\pi^- a^+)_0$ are given by a single equation (56) with $I = 0$.

Suppose now that a is replaced by another π. Then, from the Bose symmetrization of a system of two pions[1], one must have $I + \ell$=even, so that ℓ=even for both $I = 0$ and $I = 2$, while ℓ=odd for $I = 1$. Now switch π^+ and π^- in A_{+-} or in A_{-+} ; each of the amplitudes $(\pi a)_I$ acquire a phase factor $(-)^\ell$, and the term with $I = 1$ picks up a negative sign and those with $I = 0$ or $I = 2$ do not, so that the amplitude A_{+-} turns into A_{-+} and *vice versa*, i.e. the final states $\pi^+\pi^-$ and $\pi^-\pi^+$ are indistinguishable—a familiar result.

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[1] S. U. Chung, 'C- and G-parity: a New Definition and Applications (Version III),' BNL-QGS-98-1001.