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Relativistic isobar model: Spinless particles

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Three-body unitarity imposes constraints on isobar amplitudes when different resonance pairs interact. We develop a set of relativistic integral equations which incorporate these constraints together with analyticity. The subenergy dependence of the isobar amplitudes predicted by these equations could be an important ingredient of phenomenological analyses using the isobar model. We point out possible applications and discuss the danger of imposing unitarity without analyticity.

I. INTRODUCTION

Very little is understood about the structure of many-particle final states and how two-body information is distributed over these states. This problem is particularly acute when strong pairwise final-state interactions overlap in the final-state phase space. Yet there are a great deal of data on such systems and the analysis of these data to obtain information on the interaction of unstable particles or to study how the pair information reflects reaction mechanisms is an important part of particle physics. In this paper we develop the simplest relativistic theory of three-body final states consistent with unitarity and analyticity. We show how these principles can be used to establish the domain of applicability of the usual phenomenology and how a better phenomenology can be developed which is more generally applicable. The better phenomenology leads to a set of integral equations which is nearly identical to those obtained earlier using the techniques of Blankenbecler and Sugar.[1,2]

We shall be interested in three-body final states in which the pair interactions are dominated by a few, usually resonant, partial waves. The usual method of analysis for such states is the isobar model.[3] In this model the amplitude for the reaction $a(p)+b(p')\rightarrow c(p_1)+d(p_2)+e(p_3)$ in the three-body center of mass is written

$$\langle p_1,p_2,p_3|T_{2,3}|p,p'\rangle=\sum_{\substack{\alpha,\beta,\gamma=1\\ \text{cyclic}}}^{3}\langle p_\alpha|f|p,p'\rangle\, G(p_\beta,p_\gamma), \tag{1}$$

where, for simplicity, we have suppressed internal quantum numbers and have assumed that each pair is dominated by a single isobar. The quasi-two-body amplitude $\langle p_\alpha|f|p,p'\rangle$ describes the production, from the initial state, of particle α and the (β,γ) isobar. $G(p_\beta,p_\gamma)$ describes the subsequent propagation of the isobar and its decay into $p_\beta+p_\gamma$. Up to kinematical and threshold factors, $G(p_\beta,p_\gamma)$ is the (β,γ) two-body t matrix in the isobar partial wave. The decomposition of (1) is represented graphically in Fig. 1. In applications of the isobar model, the amplitude $\langle p_\alpha|f|p,p'\rangle$ is expanded in terms of partial-wave amplitudes $f_\alpha^J(W,\sigma_\alpha,l',l)$, where J is the total angular momentum, l is the orbital angular momentum of the initial system, and l' is the orbital angular momentum of the final-particle–isobar system. In general there will also be labels for other internal quantum numbers. W is the total center-of-mass energy, $p+p'=P$, $P^2=W^2$, while $\sigma_\alpha=(P-p_\alpha)^2$ is the square of the (β,γ) isobar mass. For fixed W,J,l,l', f_α can only depend on σ_α. The usual isobar assumption is that f_α is a slowly varying function of σ_α and hence can be approximated by a constant. If the isobar resonances are very narrow, only the value of f_α at the (well-defined) isobar mass is relevant and $f_\alpha=\text{constant}$ is a reasonable choice. However, we shall show that if the isobar resonance bands are wide and if they overlap, f_α can be a rapidly varying function of σ_α. It is the purpose of this paper to develop a framework in which these ideas can be quantified, and to provide a phenomenological scheme for obtaining the variation of f_α.

In Sec. II we show that unitarity forces f_α to have a physical branch cut in σ_α. It is then convenient to decompose f_α according to

$$f_\alpha(\sigma_\alpha)=\text{Disp}f_\alpha(\sigma_\alpha)+i\,\text{Abs}f_\alpha(\sigma_\alpha), \tag{2}$$

where we define Abs $f_\alpha(\sigma_\alpha)$ as the discontinuity of f_α required by unitarity across the physical σ_α cut, i.e.,

$$\text{Abs}f_\alpha(\sigma_\alpha)=\frac{1}{2i}\,[f_\alpha(\sigma_\alpha+i\epsilon)-f_\alpha(\sigma_\alpha-i\epsilon)]. \tag{3}$$

The constraints imposed by unitarity appear as integral equations of the form

$$\mathrm{Abs} f_\alpha(\sigma_\alpha) = \sum_{\beta\neq\alpha} \int K(\sigma_\alpha, \sigma_\beta) f_\beta(\sigma_\beta), \tag{4}$$

where the kernel $K(\sigma_\alpha, \sigma_\beta)$ is made up of known kinematical factors and two-body t matrices.[4] The integral in (4) goes over the phase space allowed to σ_β for fixed W and σ_α. (2) and (4) thus form a set of integral constraints on the set of Absf_α if Disp f_α is known. Any choice of Disp f_α will generate a subenergy unitary $f_\alpha(\sigma_\alpha)$ via (4) and (2). For threshold enhancements, that is pairwise final-state interactions that are important at subenergy threshold, this procedure yields a useful phenomenolgy. However, in the case of resonant final-state interactions, when the resonances are in the final-state phase space, the use of (2) and (4) above can be very misleading. In that case Abs $f_\alpha(\sigma_\alpha)$ acquires via (4) rapid variation that comes from singularities on the second sheet of σ_α. They, therefore, show up in $f_\alpha(\sigma_\alpha - i\epsilon)$. If a fully analytic decomposition of f_α is used, there will be corresponding rapidly varying terms in Disp f_α and the spurious rapid variation will cancel in the physical amplitude $f_\alpha(\sigma_\alpha + i\epsilon)$. However, if a simple input guess, e.g., constant, is used for Disp f_α in (2) and (4), the spurious variation will propagate into the physical amplitude. To avoid this difficulty one must include analyticity as well as unitarity as a constraint on the f_α's. In Sec. III we discuss implementation of the unitarity constraint; stressing the importance of including analyticity and using a dispersion relation we derive the "minimal" integral equations satisfied by the f_α which are consistent with subenergy unitarity and analyticity. In Sec. IV we discuss possible applications to particular physical systems and give some conclusions. There are two technical appendices: Appendix A gives an explicit demonstration of the threshold behavior of the absorptive parts and Appendix B gives details on the "wrong" sheet nature of the singularities of Abs f in the case of resonant isobars and the connection of this to the so-called Peierls mechanism.[5]

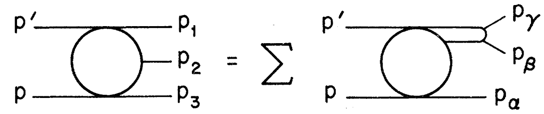

FIG. 1. Graphical representation of isobar production and decay corresponding to Eq. (1).

II. UNITARITY CONSTRAINTS

A. Two-body unitarity

Since our basic tool is unitarity, we begin with a discussion of our convention for it. In the succeeding material we shall use the following expression of unitarity:

$$\begin{aligned}\langle\alpha|T(W)|\beta\rangle - \langle\alpha|T^\dagger(W)|\beta\rangle &= i\int \langle\alpha|T(W)|n\rangle\rho(n)\langle n|T^\dagger(W)|\beta\rangle \\ &= i\int \langle\alpha|T^\dagger(W)|n\rangle\rho(n)\langle n|T(W)|\beta\rangle,\end{aligned} \tag{5}$$

where

$$\rho(n) = (2\pi)^4\delta^4\Big(P_2 - \sum_{i=1}^{n} q_i\Big) \times \prod_{i=1}^{n}\left(\frac{d^4q_i}{(2\pi)^4}\, 2\pi(2m_i)^F\delta^+(q_i^{\,2} - m_i^{\,2})\right) \tag{6}$$

is n-body phase space. $F=0$ if particle i is a boson and $F=1$ if it is a fermion. The transition (T) matrix is defined in terms of the S matrix by

$$\langle\alpha|S(W)|\beta\rangle = \langle\alpha|1|\beta\rangle + (2\pi)^4 i\delta^4(P_\alpha - P_\beta)\langle\alpha|T|\beta\rangle. \tag{7}$$

Consider the elastic scattering of two spinless particles of masses m_1 and m_2 ($\hbar=1$) with internal quantum numbers (isospin, etc.) α_1, α_2. If we assume no coupling to inelastic states, unitarity becomes

$$\begin{aligned}\mathrm{Im}\langle q_{12}, \alpha_1\alpha_2|T(W)|q'_{12}, \alpha'_1\alpha'_2\rangle = \frac{1}{2}\sum_{\alpha''_1\alpha''_2}\int \frac{d^4p''_1 d^4p''_2}{(2\pi)^2}\,\delta^4(P_{12} - P''_{12})\delta^+(p''^{\,2}_1 - m_1^{\,2})\delta^+(p''^{\,2}_2 - m_2^{\,2}) \\ \times\langle q_{12}, \alpha_1\,\alpha_2|T^\dagger(W)|q''_{12}, \alpha''_1\alpha''_2\rangle\langle q''_{12}\alpha''_1\alpha''_2|T(W)|q'_{12}, \alpha'_1\alpha'_2\rangle,\end{aligned} \tag{8}$$

where

$$P_{12} = p_1 + p_2, \quad P_{12}^{\,2} = W^2, \quad 2q_{12} = p_1 - p_2, \quad \text{etc.} \tag{9}$$

In the above equation the momenta p_1, p_2, p'_1, and p'_2 are on their mass shells, i.e., $(p_1)_0 = (\vec{p}_1^{\,2} + m_1^{\,2})^{1/2}$. However, in general, the amplitude in (8) is completely off the energy shell in the sense that q_{12}, q'_{12}, and the center-of-mass energy W can be considered independent quantities. Changing variables to P''_{12} and q''_{12}, which transformation has unit Jacobian, we obtain

$$\mathrm{Im}\langle q_{12},\alpha_1\alpha_2|T(W)|q'_{12},\alpha'_1\alpha'_2\rangle=\frac{1}{8\pi^2}\sum_{\alpha''_1\alpha''_2}\int\frac{d^4q''_{12}}{4\omega''_1\omega''_2}\,\delta\big((q''_{12})_0+\tfrac{1}{2}W-\omega''_1\big)\delta(\omega''_1+\omega''_2-W)$$
$$\times\langle q_{12},\alpha_1\alpha_2|T^\dagger(W)|q''_{12},\alpha''_1\alpha''_2\rangle\langle q''_{12},\alpha''_1\alpha''_2|T(W)|q'_{12},\alpha'_1\alpha'_2\rangle, \qquad (10)$$

where

$$\omega''_1=(\vec{q}''^2_{12}+m_1^2)^{1/2},\quad \omega''_2=(\vec{q}''^2_{12}+m_2^2)^{1/2}. \qquad (11)$$

If we assume that there is no coupling between the internal quantum numbers and the orbital motion, we can make the following Lorentz-invariant decomposition into partial-wave amplitudes which will be useful in studying the three-body problem:

$$\langle q_{12},\alpha_1\alpha_2|T(W)|q'_{12},\alpha'_1\alpha'_2\rangle$$
$$=\sum_{lmT}Y^*_{lm}(\hat{M}_{12})C^T_{\alpha_1\alpha_2}\tau_{lT}(|\vec{M}_{12}|,|\vec{M}'_{12}|,W)\times C^T_{\alpha'_1\alpha'_2}Y_{lm}(\hat{M}'_{12}), \qquad (12)$$

where $\hat{M}_{12}$ is the unit *special vector*. This special vector $\vec{M}_{12}$, a function of $\vec{p}_1$ and $\vec{p}_2$, is defined and discussed in detail in Ref. 2, p. 2017. It is, in fact, the relative three-momentum in the (1,2) center-of-mass system expressed in terms of the three-vectors $\vec{p}_1$ and $\vec{p}_2$ in an arbitrary Lorentz frame. As shown in Ref. 2 we can therefore interpret $Y_{lm}(\hat{M}_{12})$ as the angular-momentum factor associated with vertices appearing in the three-body problem. In (12), $C^T_{\alpha_1\alpha_2}$ is an element of a unitary transformation from the α representation to the T representation in which the scattering amplitude is diagonal. We shall call the diagonal on-shell t-matrix element in this (l,T) state $\tau_{lT}(W)$. T can be thought of as the channel spin or isospin, or both, etc. Substituting (12) in (10) and evaluating it in the center-of-mass system using the orthonormality of the Y's and C's gives

$$\mathrm{Im}\tau_{lT}(W)=\frac{|\vec{q}_{12}|}{32\pi^2W}|\tau_{lT}(W)|^2. \qquad (13)$$

For many purposes it is often convenient to write τ in the form

$$\tau_l(M_{12},M'_{12},W)=N(M_{12},M'_{12},W)/D(W), \qquad (14)$$

where we have suppressed the T label and where M and M' are magnitudes of three momenta. $D(W)$ has a zero at the isobar mass, carries the scattering phase, and has the unitarity cut. N has the left-hand cuts and the threshold factors $M_{12}{}^l$ and $M'_{12}{}^l$. To insure proper asymptotic behavior we want

$$\lim_{M_{12}\to\infty}N(M_{12},M'_{12},W)=0. \qquad (15)$$

Going on-shell and substituting (14) into (13) yields

$$\mathrm{Im}\,D(W)=\frac{-|\vec{q}_{12}|N(|\vec{q}_{12}|,|\vec{q}_{12}|,W)}{32\pi^2W}. \qquad (16)$$

Assuming analyticity in the Mandelstam invariant $s=W^2$, the function $D(s)$ which satisfies the above condition is

$$D(s)=P(s)-\frac{1}{2(2\pi)^3}\int_0^\infty\frac{q^2dq}{\omega_1\omega_2}\frac{(\omega_1+\omega_2)}{(\omega_1+\omega_2)^2-s-i\epsilon}\times N(q,q,\omega_1+\omega_2), \qquad (17)$$

where $P(s)$ is a polynomial in s and $\omega_1=(q^2+m_1^2)^{1/2}$, etc. By assuming analyticity in s rather than W we have insured certain known analytic properties (e.g., reflection symmetry in W). These additional properties may not be important in the phenomenological applications that we consider, but since we can include them with no additional effort, we do so.

Finally, let us examine the case of identical particles. No special care is needed with sums, etc. if the states are defined with the appropriate normalization. In particular, an n-body state of identical particles $|\alpha\beta\gamma\cdots\rangle$ is constructed according to

$$|\alpha\beta\gamma\cdots\rangle=\frac{1}{\sqrt{N}}\psi^\dagger_\alpha|\beta\gamma\cdots\rangle=\frac{1}{(N!)^{1/2}}\psi^\dagger_\alpha\psi^\dagger_\beta\psi^\dagger_\gamma\cdots|0\rangle, \qquad (18)$$

where the $\psi^\dagger$ are the creation operators and they obey the appropriate commutation or anticommutation relations. In this case the states $|\alpha\beta\gamma\cdots\rangle$ will have the correct symmetry as well as normalization. Using this fact, the unitarity relation will still be (13). The only restriction is that in the decomposition (12) we maintain the appropriate symmetry. Since interchanging the particles sends $\vec{q}$ to $-\vec{q}$ and since $Y_{lm}(-\hat{q})=(-1)^lY_{lm}(\hat{q})$, we need to take T's such that $C^T_{\alpha_1,\alpha_2}=\pm C^T_{\alpha_2,\alpha_1}$, the plus sign going with even l for bosons and odd l for fermions, and the minus sign going with odd l for bosons and even l for fermions.

B. Three-body unitarity

Let us now turn to the three-particle case and, in particular, to the pair-subenergy dependence of a three-body final-state amplitude required by unitarity. We wish to exploit the fact that a three-body amplitude is a function of many variables, and we are interested in the subenergy dependence for fixed total three-body energy. Each term in unitarity has a threshold which signals the existence of a singularity at that threshold. What unitarity yields is the discontinuity of the total amplitude across the singularity that begins at that threshold and in the variable with that threshold.[6] As we discuss Sec. III, a dispersion relation is needed to obtain the full subenergy dependence implied by that discontinuity. Here we are only interested in obtaining the discontinuity in the subenergy. Consider an amplitude $T_{2,3}$ for "two particles goes to three" (breakup). Assuming that only two- and three-body channels are open, unitarity for $T_{2,3}$ can be written

$$2\,\mathrm{Im}T_{2,3}=\int T_{2,2'}\rho(2')T^{\dagger}_{2',3}+\int T_{2,3'}\rho(3')T^{\dagger}_{3',3}\,, \tag{19}$$

where $T_{2,2'}$ and $T_{3,3'}$ are the elastic two-body and three-body amplitudes, respectively. $T_{3,3'}$ can be decomposed into a connected part $T^{c}_{3,3'}$ and a sum of disconnected parts $T^{d}_{3,3'}$ which represents one particle going by while the other two scatter. These disconnected pieces of $T_{3,3'}$ are a correct and necessary consequence of our definition of the S and T matrices [Eq. (3)]. Hence Eq. (19) can be rewritten

$$2\,\mathrm{Im}T_{2,3}=\int T_{2,2'}\rho(2')T^{\dagger}_{2',3}+\int T_{2,3'}\rho(3')T^{c\dagger}_{3',3}+\int T_{2,3'}\rho(3')T^{d\dagger}_{3',3}. \tag{20}$$

The equation is represented diagrammatically in Fig. 2. As we noted, each term in unitarity implies a singularity at the threshold of that term and in the variable with that threshold. Strictly speaking, each term contributes the discontinuity across the singularity beginning at that threshold.[6] We are interested in exploiting (20) and the implied analyticity to obtain the dependence of $T_{2,3}$ on the

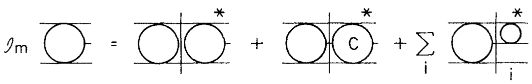

FIG. 2. Graphical representation of the unitarity relation, Eq. (20).

pair subenergies for fixed total energy. We are particularly interested in its physical region singularities since a singularity represents rapid dependence. As we mentioned, only terms in unitarity having pair-subenergy thresholds will be related to these subenergy singularities. Clearly the $T_{2,2'}T_{2',3}$ term has a threshold in W, the total energy, at $W=W_{2,0}$, the minimum two-body energy. Similarly, the $T_{2,3'}T^{c}_{3',3}$ term has a threshold in W at $W=W_{3,0}$, the minimum three-body energy. So apparently does the $T_{2,3'}T_{3',3}$ (disc) term from (20), but in fact, as is clear from Fig. 2 and as will become clear in our development, the δ function in $T^{d}_{3',3}$ coming from the fly-by particle will give it a threshold in the subenergy of the interacting pair. Hence this is the only term we need keep to study the discontinuity across the subenergy singularities. This point often causes confusion because the other terms in unitarity have parts with singular subenergy dependence, but in fact the entire *discontinuity* across this subenergy cut is given by the third term.[6] Keeping this term alone, we no longer have $\mathrm{Im}T_{2,3}$, but only the discontinuity of $T_{2,3}$ across the subenergy cut. This shall be called the absorptive part of $T_{2,3}$ ($\mathrm{Abs}T_{2,3}$) to stress the fact that we are no longer dividing the amplitude into real and imaginary parts, but into absorptive and dispersive parts, each of which can be complex, while the absorptive part contains the appropriate physical threshold. We now write

$$2\,\mathrm{Abs}T_{2,3}=\int T_{2,3'}\rho(3')T^{d\dagger}_{3',3}. \tag{21}$$

Let us apply (21) to three final spinless particles of masses m_1, m_2, and m_3. We use the kinematic conventions previously developed. Working in the three-body center-of-mass system, we have for $T^{d}_{3,3'}$

$$\langle \vec{p}_1\alpha_1,\vec{p}_2\alpha_2,\vec{p}_3\alpha_3|T^{d}_{3,3'}(W)|\vec{p}_1{}'\alpha_1',\vec{p}_2{}'\alpha_2',\vec{p}_3{}'\alpha_3'\rangle=\sum_{\substack{ijk\\ \text{cyclic}}}(2\pi)^3 2\omega_i\,\delta(\vec{p}_i-\vec{p}_i{}')\delta_{\alpha_i,\alpha_i'}\langle\vec{M}_{jk},\alpha_j\alpha_k|T(\sigma_i)|\vec{M}'_{jk},\alpha_j'\alpha_k'\rangle \tag{22}$$

in terms of the two-body T matrix defined in Sec. II A, with

$$\sigma_i=(P-p_i)^2 = W^2-2W\omega_i+m_i{}^2. \tag{23}$$

For $T_{2,3}$ we take one of two forms of the isobar model (called the sequential decay model in nuclear physics). These forms are also suggested by the Faddeev equations and are represented schematically by Eq. (1) and in Fig. 1. We take either

$$\langle \vec{k}\rho | T_{2,3}(W) | \vec{p}_1\alpha_1, \vec{p}_2\alpha_2, \vec{p}_3\alpha_3 \rangle = \sum_{\substack{ijk\,\mathrm{cyclic} \\ l_{jk}, m, T_{jk}}} \langle \vec{k}\rho | F(W) | \vec{p}_i\alpha_i, l_{jk}, m, T_{jk} \rangle \frac{\tau_{l_{jk}T_{jk}}(\sigma_i) Y_{l_{jk},m}(\hat{M}_{jk}) C^{T_{jk}}_{\alpha_j\alpha_k}}{M_{jk}{}^{l_{jk}}} \tag{24a}$$

or

$$\langle \vec{k}\rho | T_{2,3}(W) | \vec{p}_1\alpha_1, \vec{p}_2\alpha_2, \vec{p}_3\alpha_3 \rangle = \sum_{\substack{ijk\,\mathrm{cyclic} \\ l_{jk}, m, T_{jk}}} \langle \vec{k}\rho | f(W) | \vec{p}_i\alpha_i, l_{jk}, m, T_{jk} \rangle \frac{M_{jk}{}^{l_{jk}} Y_{l_{jk}m}(\hat{M}_{jk}) C^{T_{jk}}_{\alpha_j\alpha_k}}{D_{l_{jk}T_{jk}}(\sigma_i)}. \tag{24b}$$

In Eq. (24) and in all subsequent equations, unless stated otherwise, momentum labels refer to three-momenta. In the above equation k is the relative momentum of the particles in the initial state, and ρ represents internal quantum numbers. τ, M, and D are the two-body quantities defined in (12) and (14). F and f are defined by (24), but may be thought of as the quasi-two-body amplitudes for yielding the final state of particle i with momentum p_i and internal quantum numbers α_i, and a correlated j-k pair state with orbital angular momentum l_{jk} and internal quantum numbers T_{jk}. The factors

$$\frac{\tau_{l_{jk}T_{jk}}(\sigma_i) Y_{l_{jk}m}(\hat{M}_{jk}) C^{T_{jk}}_{\alpha_j\alpha_k}}{M_{jk}{}^{l_{jk}}},$$

$$\frac{M_{jk}{}^{l_{jk}} Y_{l_{jk}m}(\hat{M}_{jk}) C^{T_{jk}}_{\alpha_j\alpha_k}}{D_{l_{jk}T_{jk}}(\sigma_i)}$$

represent the subsequent decay of the correlated pair. They are called G in Eq. (1). The factors $M_{jk}^{l_{jk}}$ are in (24) to give the correct threshold behavior. If there are Coulomb forces present, or if (24) is to be used far from the two-body thresholds, these factors should be replaced by the appropriate penetrability factors. In most applications only one or a few (l, T)'s are important for each pair. It is therefore convenient to use the partial-wave expansion (12) for the two-body t matrix in (22). Remember that the arguments of the Y_{lm}'s, wherever they appear, are always a unit *special vector* as defined earlier. We shall first work with (24a): Substituting (22) and (24a) into (21) with this expansion, and equating coefficients of $C^{T_{jk}}_{\alpha_j\alpha_k} Y_{l_{jk}m}(\hat{M}_{jk})$ on both sides we obtain

$$2\,\mathrm{Abs}\left[\frac{\langle \vec{k}\rho | F(W) | p_i\alpha_i, l_{jk}, m, T_{jk} \rangle \tau_{l_{jk}T_{jk}}(\sigma_i)}{M_{jk}{}^{l_{jk}}}\right] = \int \frac{d^4p'_j d^4p'_k}{(2\pi)^2} \delta^4(p_j + p_k - p'_j - p'_k) \delta^+(p'^2_j - m_j{}^2) \delta^+(p'^2_k - m_k{}^2)$$
$$\times \sum_{\substack{rst\,\mathrm{cyclic} \\ l'st, m', T'st, \beta's}} \langle \vec{k}\rho | F(W) | \vec{p}_r{}'\beta_r, l'_{st}, m', T'_{st} \rangle \tau_{l'_{st}T'_{st}}(\sigma'_T)$$
$$\times Y_{l'_{st}m'}(\hat{M}'_{st}) C^{T'_{st}}_{\beta_s\beta_t} \tau^*_{l_{jk}T_{jk}}(\sigma_i) Y^*_{l_{jk}m}(\hat{M}'_{jk}) C^{T_{jk}}_{\beta_j\beta_k}, \tag{25}$$

where we have used the fact that various δ functions force $\sigma_i = \sigma'_i$. On the left-hand side of (25) we use

$$\begin{aligned} \mathrm{Abs}(F\tau) &= (\mathrm{Abs}F)\tau^* + F\,\mathrm{Abs}\tau, \\ &= (\mathrm{Abs}F)\tau^* + F\,\mathrm{Im}\tau, \end{aligned} \tag{26}$$

which follows from the definition of Abs f as

$$\mathrm{Abs}F(\sigma) = \frac{1}{2i}[F(\sigma + i\epsilon) - F(\sigma - i\epsilon)] \tag{27}$$

and from the relation for the imaginary (or absorptive) part of a product, i.e.,

$$\begin{aligned} 2i\,\mathrm{Im}AB &= AB - A^*B^* \\ &= AB - A^*B + A^*B - A^*B^* \\ &= 2i(B\,\mathrm{Im}A + A^*\mathrm{Im}B). \end{aligned} \tag{28}$$

In (26) we have also used Abs$\tau \to$ Imτ because the only singular part of τ comes from its one threshold and that gives its imaginary part. There are two types of terms on the right-hand side of (25). In the first $r = i$, $st = jk$. In this term Y_{lm} integrals are easily done by orthonormality since they are of the same argument, and the $C^T_{\beta\beta'}$ sums are similarly done. One then finds that the $F\,\mathrm{Im}\tau$ term on the left-hand side cancels exactly with an $F|\tau|^2$ term on the right-hand side by two-body unitarity (13). This result that the $F\,\mathrm{Im}\tau$ term must cancel with a corresponding term on the right-hand side by two-body unitarity is general in all such calculations, identical particle or not, relativistic or not, and serves as a useful check on these calculations. One is now left only with the $(\mathrm{Abs}F)\tau^*$ term on the left-hand side and the term on the right-hand side where $r=i$. Cancelling the τ^* in both of these one finally gets

$$\begin{aligned}
&\mathrm{Abs}\,\langle \vec{k}\rho|F(W)|\vec{p}_i\alpha_i, l_{jk}, m, T_{jk}\rangle \\
&\quad= \frac{1}{2(2\pi)^2} M_{jk}{}^{l_{jk}} \\
&\quad\times\Bigg[\sum_{\substack{\beta_j,\beta_k \\ l'_{ik}T'_{ik}m'_{ik}}} \int \frac{d^3p'_j}{2\omega'_j}\frac{\langle \vec{k}\rho|F(W)|\vec{p}'_j\beta_j, l'_{ik}, m'_{ik}, T'_{ik}\rangle \tau_{l'_{ik}T'_{ik}}(\sigma'_j)}{M'_{ik}{}^{l'_{ik}}} C^{T'_{ik}}_{\alpha_i\beta_k}\delta^+(p'^2_k - m_k{}^2) Y^*_{l_{jk},m}(\hat{M}'_{jk}) C^{T_{jk}}_{\beta_j\beta_k} Y_{l'_{ik}m'_{ik}}(\hat{M}'_{ik}) \\
&\qquad + j \leftrightarrow k\Bigg].
\end{aligned} \tag{29a}$$

If we start with (24b) for the breakup amplitude, a set of steps similar to the previous ones leads to

$$\begin{aligned}
&\mathrm{Abs}\,\langle \vec{k}\rho|f(W)|\vec{p}_i\alpha_i, l_{jk}, m, T_{jk}\rangle \\
&\quad= \frac{N_{l_{jk}T_{jk}}(M_{jk}, M_{jk}, (\sigma_i)^{1/2})}{2(2\pi)^2 M_{jk}{}^{l_{jk}}} \\
&\quad\times\Bigg[\sum_{\substack{\beta_j\beta_k \\ l'_{ik}T'_{ik}m'_{ik}}} \int \frac{d^3p'_j}{2\omega'_j}\langle \vec{k}\rho|f(W)|\vec{p}'_j\beta_j, l'_{ik}, m'_{ik}, T'_{ik}\rangle \frac{M'_{ik}{}^{l'_{ik}}}{D_{l'_{ik}T'_{ik}}(\sigma'_j)} Y_{l'_{ik}m'_{ik}}(\hat{M}'_{ik}) C^{T'_{ik}}_{\alpha_i\beta_k}\delta^+(p'^2_k - m_k{}^2) Y^*_{l_{jk},m}(\hat{M}'_{jk}) C^{T_{jk}}_{\beta_j\beta_k} \\
&\qquad + (j \leftrightarrow k)\Bigg].
\end{aligned} \tag{29b}$$

Equation (29) is represented graphically in Fig. 3. In (29) we see that unitarity alone forces the quasi-two-body amplitude F or f to have a branch cut in σ. As is clear from Fig. 3 and as we make more explicit in Appendix B, this is a two-body cut with a square-root branch point. The strength of the singularity in f_α or F_α is determined by the other isobar groupings as we already indicated in the schematic equations (4). It is clearly a manifestation of the coherence or interference required by the quantum mechanics when there is more than one isobar parentage for the final three-body state.

It is straightforward to include spin and fermions in the discussion given here, but they only complicate an already extremely complicated formalism.

III. IMPLEMENTATION

We have seen that unitarity alone forces the quasi-two-body amplitudes F or f of Eq. (24) to have important (singular) dependence on the isobar mass—a dependence normally neglected in empirical applications of the isobar formalism. The next step is to know when these unitarity effects are important, and how to implement them in phenomenology if they are important. If we decompose the f's as in (2), Eq. (4) [which is the very schematic form of (29)] becomes

$$\mathrm{Abs}\, f_\alpha(\sigma_\alpha) = \sum_{\beta\neq\alpha}\int K(\sigma_\alpha,\sigma_\beta)\mathrm{Disp} f_\beta(\sigma_\beta) + \sum_{\beta\neq\alpha}\int K(\sigma_\alpha,\sigma_\beta)\mathrm{Abs} f_\beta(\sigma_\beta). \tag{30}$$

From (29) we see that if we make a three-body partial-wave decomposition, (30) will be a one-variable set of Fredholm integral equations for $\mathrm{Abs}f_\alpha$ with $\mathrm{Disp}f_\alpha$ required for input. [They are Fredholm equations because the kernel K is finite and the δ function in (29) will confine the integral in (30) to a finite domain of phase space.] The usual phenomenological assumption that f_α has no absorptive part (that is, no branch cut in suben-

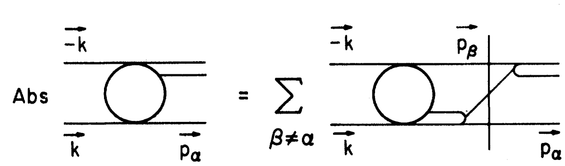

FIG. 3. Graphical representation of Eq. (29).

ergy) is easily checked in the context of (30). One assumes that $\mathrm{Abs} f_\alpha \ll \mathrm{Disp} f_\alpha$ so that the second integral in (30) may be neglected. If the integral over the dispersive part (assumed known, e.g., a constant) does indeed maintain the inequality, the assumption is consistent. If it does not, one must find a way to implement (30).[7] Clearly, any choice of $\mathrm{Disp} f_\alpha$ in (30) will generate an Abs f_α that satisfies subenergy unitarity. In a sense, then, (30) considered as an integral equation is the implementation one needs. The problem is that one must give Disp f_α to use (30).

There are two cases of practical interest. The first is final-state pairwise threshold enhancements. These are final-state interactions that are important at the pair thresholds. In this case we need to know the $f_\alpha(\sigma_\alpha)$ near the σ_α scattering threshold $(m_\beta + m_\gamma)^2$. It is precisely at that threshold that unitarity imposes its singularity in f. Furthermore, it is a square-root singularity and is carried entirely in the absorptive part. For this case (30) considered as an integral equation works well for any simple input for the dispersive part including a constant. There have been studies of the nonrelativistic problem for this situation, and the dominance of the square-root threshold in f as well as the fact that subenergy unitarity can get the shape but not the over-all magnitude of f has been amply demonstrated.[3,8] The most common case in particle physics is one of resonant-final-state interactions. In this case, the use of (30) with a simple Disp f can be disastrous. If the resonance bands are in the phase space, one finds spurious rapid variations in Abs $f_\alpha(\sigma_\alpha)$ generated by (30). This variation can be traced to singularities in $f_\alpha(\sigma_\alpha)$ on the second sheet and hence affecting the $f(\sigma_\alpha - i\epsilon)$ term in (3). If we had properly taken into account that $f_\alpha(\sigma_\alpha)$ is an analytic function and had written parallel to (3)

$$\mathrm{Disp} f_\alpha(\sigma_\alpha) = \tfrac{1}{2}\left[f_\alpha(\sigma_\alpha + i\epsilon) + f_\alpha(\sigma_\alpha - i\epsilon)\right], \tag{31}$$

similar rapid variation would also be present in Disp f and would cancel in the physical amplitude when it was constructed as in (2). (30) generates a subenergy unitary f for any choice of Disp f, but there is no guarantee that such a choice leads to an analytic f as well. In Appendix B we expand this point. Here we simply stress these difficulties in order to motivate including analyticity in our phenomenology.

Since (4) or (29) gives us the discontinuity of f or F across its cut in σ, we can build in analyticity by writing a dispersion relation for f or F that runs over the cut. Schematically we write

$$f_\alpha(\sigma, W) = R_\alpha(\sigma, W) + \frac{1}{\pi}\int_{\sigma_0}^{\infty} \frac{d\sigma' \,\mathrm{Abs} f_\alpha(\sigma', W)}{\sigma' - \sigma}, \tag{32}$$

where σ_0 is the threshold of the σ_α cut, $\sigma_0 = (m_\beta + m_\gamma)^2$, and $R_\alpha(\sigma)$ is any function of σ that does not have the unitarity cut. Since (20) or (4) relate Abs f back onto f, (32) is a set of linear integral equations for f_α. To study (32) in detail we note first that f or F are functions of W, σ_α, and P_α. Clearly all these variables are not independent since

$$\sigma_\alpha = (P - p_\alpha)^2 = W^2 - 2W(p_\alpha{}^2 + m_\alpha{}^2)^{1/2} + m_\alpha{}^2. \tag{33}$$

There is, therefore, the question of which variable to keep fixed while running over the σ cut in (32). There is also the question of which isobar form to use [(29a) or (29b)]. Any choice will, via (32), lead to correct σ analyticity and unitarity. However, as we have discussed in the nonrelativistic case,[9] some choices introduce extra singularities, for example, in W through the integral (32). In principle these can be compensated for by careful choice of the inhomogeneous term R in (32) so that they do not propagate into f, but (32) is only useful for phenomenology if it works with simple choices of R. This situation is reminiscent of our problems with choice of Disp f in using (2) and (30) to implement unitarity. Hence we will follow the nonrelativistic case and make choices that do not require a sophisticated R to compensate spurious singularities in W coming from the dispersion integral. This choice is to disperse in W for fixed p_α and to choose a somewhat altered form of (24b) with its unitarity constraint (29b) for f. It will be recalled that in discussing the threshold factors M^l in (24) we stressed that they should be replaced by barrier factors if we are to use the equation far from threshold. The integral in (32) will require them far from threshold, hence we make the replacement in (24b)

$$M^l \to M^l v(M^2), \tag{34}$$

with the asymptotic condition $M^l v(M^2) \to 0$ as $M \to \infty$. If we choose v so that

$$q^{2l} v^2(q^2) = N(q, q, W) \tag{35}$$

in terms of the N of Eq. (14), the unitarity constraint (29b) becomes considerably simpler. Rather than use the full (29b) in our fleshed-out form of (32), let us take a less complex and hopefully more transparent case. Consider the production of three particles of equal mass m and no internal degrees of freedom and, in addition, the presence of a single l-wave isobar in the expansion (12). Under these conditions and with (34) and (35) we obtain

$$\mathrm{Abs}\langle \vec{k}|f(W)|\vec{p}_i,l,m\rangle = \frac{v(M_{jk}{}^2)M_{jk}{}^l}{(2\pi)^2}\sum_{m'}\int\frac{d^3p_j'}{2\omega_j'}\langle\vec{k}|f(W)|\vec{p}_j',l,m'\rangle v(M_{ik}'^2)M_{ik}'^l Y_{lm'}(\hat{M}_{ik}')\frac{\delta^+(p_k'^2-m^2)}{D(\sigma_j')}Y_{lm}^*(\hat{M}_{jk}'). \tag{36}$$

The content of this equation is shown schematically in Fig. 3. The choice of W as the dispersion variable is dictated by the structure of the δ^+ function in (36) which can be written

$$\delta^+(p_k{}^2-m^2)=\delta^+((P-p_i-p_j)^2-m^2)=\delta^+((W-\omega_i-\omega_j)^2-(\vec{p}_i+\vec{p}_j)^2-m^2)=\frac{1}{2\omega_{ij}}\,\delta(W-\omega_i-\omega_j-\omega_{ij}), \tag{37}$$

where

$$\omega_{ij}=[(\vec{p}_i+\vec{p}_j)^2+m^2]^{1/2}. \tag{38}$$

We then find for (32), with the absorptive part given by (36),

$$\langle\vec{k}|f(W)|\vec{p}_i,l,m\rangle=\langle\vec{k}|R(W)|\vec{p}_i,l,m\rangle + \frac{1}{(2\pi)^4}\sum_{m'}\int\frac{d^3p_j}{2\omega_j}\frac{\langle\vec{p}_j,l,m'|B(W)|\vec{p}_i,l,m\rangle}{D_l(\sigma_k)}[\langle\vec{k}|f(\omega_i+\omega_j+\omega_{ij})|\vec{p}_j,l,m'\rangle-\mathfrak{F}(\vec{k},\vec{p}_i,W,l,m')], \tag{39}$$

where

$$\langle\vec{p}_j,l,m'|B(W)|\vec{p}_i,l,m\rangle=v(M_{ik}{}^2)M_{ik}^l Y_{lm'}(\hat{M}_{ik}')\frac{(\omega_i+\omega_j+\omega_{ij})v(M_{jk}'^2)}{\omega_{ij}[(\omega_i+\omega_j+\omega_{ij})^2-W^2]}M_{jk}'^l Y_{lm}^*(\hat{M}_{jk}'). \tag{40}$$

In obtaining (40) we have dispersed in s rather than in W as we did in (17). In (39) R and $\mathfrak{F}$ are arbitrary functions subject to the conditions that R's have no unitarity cuts in σ and that

$$\mathfrak{F}(\vec{k},\vec{p}_j,\omega_i+\omega_j+\omega_{ij},l,m')=0 \tag{41}$$

so that the discontinuity of the integral in (39), which is the residue at the $W=\omega_i+\omega_j+\omega_{ij}$ pole, agrees with unitarity, (36). We now make a *particular* choice for $\mathfrak{F}$ that, of course, satisfies (41),

$$\mathfrak{F}(\vec{k},\vec{p}_j,W,l,m')=\langle\vec{k}|f(W)|\vec{p}_j,l,m'\rangle-\langle\vec{k}|f(\omega_i+\omega_j+\omega_{ij})|\vec{p}_j,l,m'\rangle, \tag{42}$$

and we finally obtain our working integral equation,

$$\langle\vec{k}|f(W)|\vec{p}_i,l,m\rangle=\langle\vec{k}|R(W)|\vec{p}_i,l,m\rangle+\frac{1}{(2\pi)^4}\sum_{m',j}\int\frac{d^3p_j}{2\omega_j}\frac{\langle\vec{k}|f(W)|\vec{p}_j,l,m'\rangle\langle\vec{p}_j,l,m'|B(W)|\vec{p}_i,l,m\rangle}{D_l(\sigma_k)}, \tag{43}$$

where B is defined in (40). We again stress that any choice for $\mathfrak{F}$ in (39) subject to (41) will lead to an f that satisfies unitarity (36) and analyticity (32). Our particular choice is motivated by a desire not just to give an integral *representation* of f, but actually to be able to *obtain* it via an integral equation.

Equation (43) is the "minimal" implementation of subenergy unitarity and analyticity that provides a useful (in the sense of solvable for f) phenomenology without spurious W singularities. The equation is essentially the three-body scattering integral equation which Aaron, Amado, and Young derived exploiting the full content of unitarity (20).[2, 10] It is perhaps not so remarkable that we obtained it here stressing subenergy unitarity only, since we also made a number of arbitrary choices, particularly in (35), that were dictated by our knowledge of W analyticity. The alert reader will realize that (35) is a very special assumption that is essentially equivalent to separable interactions. While this observation is correct, we are forced to such an assumption if we wish to obtain from (32) a simple integral equation for f without spurious W singularities.

With appropriate choice of driving term R, (43) is the simplest form of the Blankenbecler-Sugar three-body equations.[1] Both (43) and the Blankenbecler-Sugar equations can be enriched by complicated choice of R and/or $\mathfrak{F}$, for example, to give more left-hand cut structure. The complications are required if the equations are to generate the correct (experimental) W dependence of f, but the subenergy dependence seems to depend far less on these dynamical details. Since it is only the subenergy dependence that is needed for isobar phenomenology, we hope that (43) can serve as a basis for that phenomenology, but in fact our choices made above yield an equation that also satisfies the full content of unitarity.[2]

IV. DISCUSSION

In the previous sections we have derived constraints imposed on isobar amplitudes by subenergy unitarity and obtained Fredholm integral equations for these amplitudes which incorporate both this unitarity and analyticity. Experience we have gained solving equations of this type in both relativistic and nonrelativistic situations indicates that approximate solutions will give a reasonable picture of the subenergy dependence of the isobar amplitudes, even though they do not describe very well the total energy behavior.[8] The nature of this subenergy dependence is of current interest in view of two recent final-state analyses in elementary-particle physics, i.e., $A_1 \to 3\pi$ (Ref. 11,12) and $\pi N \to \pi\pi N$.[13] In the former case the existence of the A_1 is in question and since it is predicted by the quark model and is an important ingredient of successful current algebra calculations,[14] its existence is a question of fundamental importance. Recent theoretical advances have generated considerable new interest in the latter process, $\pi N \to \pi\pi N$. In particular, a proposed connection between current and constituent quarks[15] can be tested through magnitudes and signs of amplitudes for pionic transitions between hadrons.[16] Equivalently, modified versions of $SU(6)_W$ classify resonances and at the same time predict amplitudes for reactions of the type $\pi N \to \pi\Delta$, $\pi N \to \rho N$, $\pi N \to \epsilon N$, etc.[17]

The original analysis of the A_1 by Ascoli *et al.*[11] assumed constant isobar amplitudes f_α [Eq. (2)] at fixed total energy W. More recent analyses[12] have attempted to implement subenergy unitarity without including analyticity. In these studies it is assumed that Disp $f_\alpha(\sigma_\alpha)$ can be approximated by a complex constant at fixed W and then (4) is substituted in (2) giving an inhomogeneous Fredholm integral equation for f. The resulting "unitarity" solutions are used to fit the data with the constant dispersive parts as the fitting parameters. As pointed out in Sec. III and Appendix B, this parametrization is disastrous when resonance bands overlap in the Dalitz plot (as they do in the A_1 problem), for in this case Abs f_α contains spurious rapid variation which must be canceled by identical variations in Disp f_α. This cancellation, of course, cannot occur if Disp f_α is constrained to be a constant, and the resulting amplitudes grossly violate required analyticity.[18] It is not surprising that including unitarity in the manner just described gives worse fits to the data[12] than neglecting unitarity altogether and taking the f_α's as constants. We are presently using Eq. (43), which incorporates unitarity and analyticity, to study the subenergy dependence of the f_α's in the A_1 problem.

We are also studying the behavior of the isobar amplitudes in the $\pi N \to \pi\pi N$ problem. In Fig. 4 we present previous (unpublished) results for the isobar amplitudes $\pi N \to \pi\Delta$ and $\pi N \to \rho N$ proceeding through the $D13$ πN channel. These were obtained using a dynamical scheme based on Eq. (43), incorporating both total energy and subenergy unitarity, and the results obtained for the elastic $D13$ amplitude were in reasonable agreement with experiment for energies $1400 \leq W \leq 2000$ MeV.[19] Within our model the isobar amplitudes for $\pi N \to \pi\Delta$

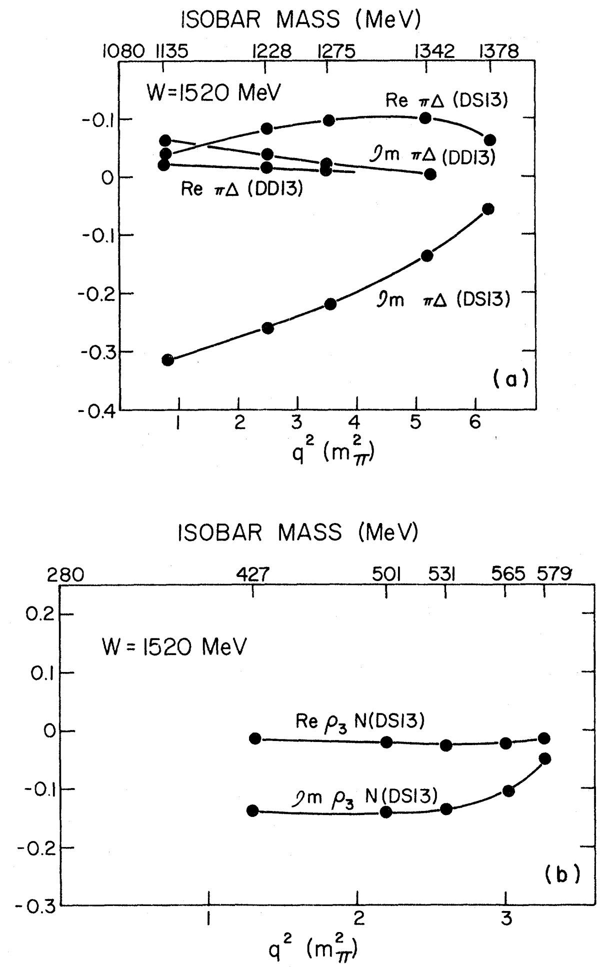

FIG. 4. (a) Isobar amplitudes for production of $\pi\Delta$ through the D13 πN partial wave vs q^2 (q is the three-momentum in the Δ c.m. system). The corresponding isobar mass is given on the upper scale. The straight lines are interpolations of the theoretical points shown as black dots. The notation is as follows: $\pi\Delta(DS13)$ means $\pi\Delta$ produced in an S state from an initial πN $D13$ state, etc. The corresponding Berkeley/SLAC amplitudes (independent of q^2) are $\pi\Delta(DS13) = 0.26 - 0.120i$ and $\pi\Delta(DD13) = -0.042 - 0.226i$. (b) Isobar amplitudes for production of $\rho_3 N$ through the $D13$ πN partial wave vs q^2 (q is the three-momentum in the π-π c.m. system). The subscript 3 on ρ refers to channel spin $\frac{3}{2}$. The corresponding Berkeley/SLAC amplitude $\rho_3 N(DS13) = 0.114 + 0.315i$.

and $\pi N \to \rho N$ (which can be produced in S waves) were predicted and are shown in the figure at $W = 1520$ MeV. (ϵN which is produced in a P wave was found to be relatively unimportant at this energy.) In our model the amplitude for $\pi N \to \pi\Delta$ in an S wave (*DS* 13) gives the major contribution to the total inelastic cross section at $W = 1520$ MeV. Note that it is a rapidly varying function of subenergy, although the variation is not of the wild type associated with the spurious behavior of absorptive parts. Finally, our isobar amplitudes differ in both relative magnitude and phase from the best Berkeley/SLAC solution (though there are problems with comparing phase conventions). We plan considerable further study of the πN system.

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APPENDIX A: THRESHOLD BEHAVIOR

We wish to show that the subenergy singularity required by unitarity (29) corresponds to a simple two-body threshold, that is,

$$\mathrm{Abs}\,\langle \vec{k}\,|f(W)|\,\vec{p}_3, l, m\rangle \propto q_{12}^{2l+1} \tag{A1}$$

for small q_{12}, the relative momentum of the 1-2 pair in its center of mass. Any form of (29) will yield (A1); we take (29a), and in order to cut down the forest of indices assume no internal degrees of freedom and that all isobars have the same l. (29a) then becomes

$$\mathrm{Abs}\,\langle \vec{k}\,|F(W)|\,\vec{p}_3, l, m\rangle = \frac{1}{2}\frac{1}{(2\pi)^2} M_{12}^{l}\left[\sum_{m'}\int \frac{d^3p_1'}{2\omega_1'}\langle \vec{k}\,|F(W)|\,\vec{p}_1', l, m'\rangle \frac{\tau_l(\sigma_1')}{M_{23}'^{\,l}} Y_{lm'}(\hat{M}_{23}') Y_{lm}^{*}(\hat{M}_{12}')\delta^{+}((P-p_1'-p_3)^2-m_2^2) + 1' \leftrightarrow 2'\right]. \tag{A2}$$

Consider only the first term; the one with 1′ and 2′ interchanged gives precisely the same result for the threshold *mutatis mutandis*. We work in the 1-2 center of mass, where

$$\begin{aligned} &\vec{P} - \vec{p}_3 = \vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2' = 0,\\ &\vec{M}_{12} = \vec{q}_{12} = \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2) = \vec{p}_1,\\ &\vec{M}_{12}' = \vec{q}_{12}' = \tfrac{1}{2}(\vec{p}_1' - \vec{p}_2') = \vec{p}_1'. \end{aligned} \tag{A3}$$

The argument of the δ^{+} function then becomes

$$2W_3(\omega_1 - \omega_1'), \tag{A4}$$

where

$$\begin{aligned} &\omega_1' = (q_{12}'^{\,2} + m_1^2)^{1/2}, \quad W_3^2 = (P-p_3)^2,\\ &\omega_1 = (q_{12} + m_1^2)^{1/2}\\ &\quad = (W_3 + m_1^2 - m_2^2)/2W_3. \end{aligned} \tag{A5}$$

Noting that

$$\begin{aligned} p_1'^2 dp_1' &= p_1'\omega_1' d\omega_1'\\ &= q_{12}'\omega_1' d\omega_1', \end{aligned} \tag{A6}$$

we then obtain for (A2)

$$\mathrm{Abs}\,\langle \vec{k}\,|F(W)|\,\vec{p}_3, l, m\rangle = \frac{1}{8}\frac{1}{(2\pi)^2}\frac{q_{12}^{l+1}}{W_3}\left[\sum_{m'}\int d\Omega q_{12}' \frac{\langle \vec{k}\,|F(W)|\,\vec{p}_1' l, m'\rangle \tau_l(\sigma_1')}{M_{23}'^{\,l}} Y_{lm'}(\hat{M}_{23}') Y_{lm}^{*}(\hat{q}_{12}')\Big|_{|q_{12}'| = |q_{12}|} + (1' \leftrightarrow 2')\right]. \tag{A7}$$

We find it useful to introduce the representation

$$\langle \vec{k}\,|F(W)|\,\vec{p}_1', l, m'\rangle = \sum_{\rho\epsilon\lambda\nu}\langle k\lambda\nu\,|F(W)|\,q_{12}\rho\epsilon, lm\rangle \times Y_{\lambda\nu}^{*}(\hat{k}) Y_{\rho\epsilon}(\hat{q}_{12}'). \tag{A8}$$

As $q_{12'} = q_{12} \to 0$, W_3 and all the various factors in the angular integral of (A7) remain finite and independent of $q_{12'}$ except $Y_{lm}(\hat{q}_{12'})$. Hence to lowest order in q_{12}, the angular integral vanishes for $\rho\epsilon \neq lm$ in (A8). In fact, the integral is proportional to q_{12}^{l} for small q_{12}. To see this, consider the angular integral in (A7) as the $\langle qlm|$ projection of some amplitude A, i.e.,

$$\int d\Omega q'_{12}\langle\vec{k}|F(W)|\vec{p}'_1,l,m\rangle \frac{\tau_l(\sigma'_1)}{M'^l_{23}} \times Y_{lm'}(\hat{M}'_{23})Y^*_{lm}(\hat{q}'_{12})\big|_{q'_{12}=q_{12}} = \langle q_{12}lm|A|\vec{p}_3,\vec{k}\rangle. \quad (A9)$$

This can be written

$$\int \langle q_{12}lm|rlm\rangle\langle rlm|A|\vec{p}_3,\vec{k}\rangle r^2 dr, \quad (A10)$$

where r is the Fourier-transform variable conjugate to q_{12} and where $\langle q_{12}lm|rlm\rangle = j_l(q_{12}r)$. Hence for small q_{12} we have

$$\langle q_{12}lm|A|\vec{p}_3,\vec{k}\rangle \propto q_{12}^l \int r^{l+2}dr\,\langle rlm|A|\vec{p}_3,\vec{k}\rangle. \quad (A11)$$

For reasonable analytic properties of the amplitudes making up A, the matrix element $\langle rlm|A|\vec{p}_3,\vec{k}\rangle$ will fall off sufficiently rapidly (exponentially) in r so that the integral in (A9) converges and (A1) is established.

It should be noted that for a three-body final state at some total energy squared s, there are two kinematic limits on a pair's subenergy squared σ. One corresponds to $q=0$ and is a genuine threshold as we just demonstrated. The other is the maximum permitted σ for a given s. Although three-body phase space vanishes at this point there is no corresponding singularity in F since that point does not correspond to a threshold in the pair system. This fact is demonstrated explicitly in the nonrelativistic case in Ref. 20.

APPENDIX B: SPURIOUS BEHAVIOR IN Abs F

We wish to show that the unitarity constraint (29) leads to rapid dependence in Abs F if there are resonant pair final-state interactions in the phase space, but that this variation is spurious because it is not on the physical sheet. This behavior will occur in any of the forms of (29) but to keep the algebra simple we concentrate on (A2) and only on the term explicitly given there. For resonant interactions, $\tau_l(\sigma'_1)$ in (A2) has a pole at $\sigma'_1=\sigma_{\rm res}$, the resonance position. If that pole comes at a value of p'_1 such that the argument of the δ^+ in (A2) is zero, Abs $\langle k|F(W)|p'_1,lm\rangle$ will itself have a pole. For fixed W this will occur at some value of p'_3. Of course there is no true pole in (A2) since $\sigma_{\rm res}$ is complex, but for a narrow width, the pole is nearby and Abs F will be large ($\sim E_{\rm res}/\Gamma_{\rm res}$). Since in most isobar applications it is not $\langle k|F(W)|p'_\alpha,lm\rangle$ that is studied but rather the three-body Jth partial-wave projection $F^J(W,\sigma_\alpha,ll')$ defined in the Introduction (note that the l's mean something different in this case) the pole or quasipole becomes a pair of logarithmic singularities in F^J. Let us now trace explicitly the analytic origin of the rapid dependence of Abs F.

As seen in Fig. 3, (A2) gives the singularity in $\langle k|F(W)|p'_\alpha,lm\rangle$ due to the propagation of particle 2 from the configuration-free particle 1 and (23) isobar to the configuration-free particle 3 and (12) isobar. The δ^+ in (A2) puts particle 2 on its mass shell in this propagation while particles 1 and 3 are already on their mass shells. $\tau_l(\sigma'_1)$ represents the propagation of the (23) isobar previous to particle 2 going across. The singularity in Abs F occurs because this propagation and the propagation represented by the δ^+ can seemingly both occur on shell. That is, the intermediate particle 2 in the exchange graph of Fig. 5 can propagate on-shell because the resonant isobars are unstable [i.e., $m_{23}{}^2>(m_2+m_3)^2$]. This singularity was discussed by Peierls,[21] but it was subsequently pointed out that the exchange graph of Fig. 5 does not, in fact, have a pole on the physical sheet.[5] Coleman and Norton[22] proved that physical-region singularities occur in graphs that can be interpreted as classical, real space-time processes. For Fig. 5 this would require that isobar (23) emit particle 2 backwards with enough speed to "catch" particle 1. Simple kinematics shows that this cannot happen and hence that the resonant pole of τ and the propagator pole for particle 2 [represented by the δ^+ in (A2)] cannot coincide on the physical sheet. The coincidence of these poles in (A2) must therefore be occurring on an unphysical sheet and the singularity in Abs F must be coming from the $f(\sigma-i\epsilon)$ term in (3) and hence not be present in the physical amplitude.

To illustrate this general argument, let us consider a specific example. In order to be able to do all integrals explicitly we take the particularly simple nonrelativistic example of three identical bosons ($\hbar=2m=1$) interacting via S-wave resonant isobars. Unitarity corresponding to (29a), in this case, takes the form [Ref. 20, Eq. (21)]

$$\text{Abs}\,\langle\vec{k}|f(E)|\vec{p}\rangle = -\tfrac{1}{2}\int\frac{d^3p'}{(2\pi)^3}\langle\vec{k}|f(E)|\vec{p}'\rangle \times\tau(E-\tfrac{3}{2}p'^2)\delta(E-2p^2-2p'^2-2\vec{p}\cdot\vec{p}') \quad (B1)$$

while the dispersion form corresponding to (44) (without the refinements needed to get the left-hand E cut structure correct) is

$$\langle\vec{k}|f(E)|\vec{p}\rangle = \langle\vec{k}|R(E)|\vec{p}\rangle + \int\frac{d^3p'}{(2\pi)^4}\,\frac{\langle\vec{k}|f(E)|\vec{p}'\rangle\tau(E-\frac{3}{2}p'^2)}{E-2p^2-2p'^2-2\vec{p}\cdot\vec{p}'}\,. \quad (B2)$$

For the two-body t matrix we take a Breit-Wigner form

$$\tau(E)=\frac{16\pi^2(2/E_0)^{1/2}\Gamma}{E-E_0+\frac{1}{2}i\Gamma}, \tag{B3}$$

where E_0 is the resonance position and Γ the width, which, for simplicity, we approximate by a constant. Consider the S-wave projection of (B2) and further assume that we need only consider the first iterate of the equation and can take $\langle k|R(E)|p\rangle$ to be a constant, R_0. We get

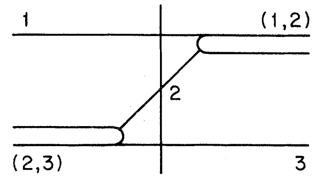

FIG. 5. Graphical representation of the Peierls mechanism.

$$\langle k|f_0(E)|p\rangle=R_0\left[1+\frac{\Gamma}{\pi p}\left(\frac{2}{E_0}\right)^{1/2}\int_0^\infty p'dp'\,\frac{1}{E-\frac{3}{2}p'^2-E_0+\frac{1}{2}i\Gamma}\ln\left(\frac{E-2p^2-2p'^2+2pp'}{E-2p^2-2p'^2-2pp'}\right)\right]. \tag{B4}$$

The integral can be evaluated to yield

$$\langle k|f_0(E)|p\rangle=R_0\left[1+\frac{2i\Gamma}{3p}\left(\frac{2}{E_0}\right)^{1/2}\ln\left(\frac{p_0+p_{++}}{p_0+p_{-+}}\right)\right], \tag{B5}$$

where

$$p_0^{\,2}=\tfrac{2}{3}(E-E_0+\tfrac{1}{2}i\Gamma) \tag{B6}$$

and

$$p_{\pm\pm}=\{\pm p\pm[2(E-\tfrac{3}{2}p^2)]^{1/2}\}\tfrac{1}{2}. \tag{B7}$$

$p'=\pm p_0$ are the roots of the Breit-Wigner denominator (B4) while $p'=p_{\pm\pm}$ are the roots of the arguments of the logarithm. $E-\frac{3}{2}p^2$ is the subenergy of the isobar associated with a free particle of momentum p. We see that (B5) has singularities at only two of the possible four places where the Breit-Wigner pole and the singularities of the denominator can coincide. Recall that the logarithm is the S-wave projection of the particle-exchange propagator of (B2), $(E-2p^2-2p'^2-2\vec{p}\cdot\vec{p}')^{-1}$.

To evaluate (B1) in the same approximation we can either approximate f under the integral by constant R_0 and do the S-wave projection or note from (B7) that to get across the subenergy cut associated with $(E-\frac{3}{2}p^2)$ we need only replace $p_{\pm\pm}$ by $p_{\pm\mp}$. Either way we get

$$\mathrm{Abs}\,\langle k|f_0(E)|p\rangle=\frac{iR_0}{3p}\Gamma\left(\frac{2}{E_0}\right)^{1/2}\left[\ln\left(\frac{p_0+p_{++}}{p_0+p_{-+}}\right)-\ln\left(\frac{p_0+p_{+-}}{p_0+p_{--}}\right)\right]. \tag{B8}$$

Simple inspection of the kinematics will show that the arguments of the first logarithm never vanish in the physical region even as $\Gamma\to 0$, but p_0+p_{+-} and p_0+p_{--} both can vanish. Hence the second term in (B8), the term coming from the second sheet, can and will generate rapid variation in Abs f that is not in the physical amplitude (B5). Equation (B5) does have subenergy dependence, but it is not so rapid as that of (B8) and of course not spurious. In fact, preliminaty investigation based on comparison of (B5) with a full calculation indicates that (B5) may form a useful base for a phenomenological parametrization of the subenergy dependence.

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