Foundations Qualifying Examination

August 28, 1998

Notation. Σ and Γ denote grammars, and x and y denote strings.

1. The *reverse* of a string x over Σ, denoted $\mathbf{rev}(x)$, is defined as follows:

$$\begin{aligned}&\mathbf{rev}(\epsilon) = \epsilon\\ &\mathbf{rev}(xa) = a\ \mathbf{rev}(x), \text{ where } a \in \Sigma\end{aligned}$$

   Prove that for any $x \in \Sigma^*$, $\mathbf{rev}(\mathbf{rev}(x)) = x$.

2. Prove that for any $A \subseteq \Sigma^*$, if A is regular, then $\mathbf{rev}(A)$ is regular, where $\mathbf{rev}(A)$ is the set $\{\mathbf{rev}(x) \mid x \in A\}$.

3. For each of the following languages, show that it is either regular, context-free but not regular, or decidable but not context-free ($\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$).

$$\begin{aligned}&\{x \in \Sigma^* \mid x = \mathbf{rev}(x)\}\\ &\{\mathtt{a}^{3n} \mid n \geq 0\}\\ &\{\mathtt{a}^n\mathtt{b}^m \mid n - m \leq 3\}\\ &\{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}\end{aligned}$$

4. (a) Let M and N denote Turing machines, and let $\langle M\rangle$ and $\langle N\rangle$ denote their respective encodings (according to a fixed encoding scheme for Turing machines). Argue whether or not the following sets are recursively enumerable.
      i. $\{(\langle M\rangle, \langle N\rangle) \mid L(M) \cap L(N)$ is a context-free language$\}$.
      ii. $\{\langle M\rangle \mid M$ takes more than 20 steps on some input$\}$.
      iii. $\{\langle M\rangle \mid M$ takes fewer than 20 steps on all inputs$\}$.

   (b) Argue whether or not your answer to Question 4a changes if M and N denote deterministic pushdown automata?

5. (a) Show that the class of recursive languages is closed under Kleene star.

   (b) Show that the class of recursively enumerable languages is not closed under complementation.

   (c) Show that not every recursively enumerable language is a recursive language.

6. (a) Define the complexity class LOGSPACE and prove that it is a subset of PTIME.

(b) Show that the language $\{x \in \{0,1\}^* \mid x \text{ has more 0's than 1's}\}$ is in LOGSPACE.

7. A *homomorphism* is a function h from Σ^* to Γ^* such that for any two strings x and y, $h(xy) = h(x)h(y)$. Thus a homomorphism is determined by its value on single symbols: if $x = a_1 a_2 \ldots a_n$ with each a_i in Σ, then $h(x) = h(a_1)h(a_2)\ldots h(a_n)$. A homomorphism is *nonerasing* if it maps no symbol to ϵ.

   Show that the complexity class NP is closed under nonerasing homomorphism.

8. The `HAMILTONIAN GRAPH` problem is the following: Given an undirected graph G, does G have a cycle that visits each node of G **exactly once**? `HAMILTONIAN GRAPH` is an NP-complete problem. Use this knowledge to prove that the `SUBGRAPH ISOMORPHISM` problem is NP-complete.

   The `SUBGRAPH ISOMORPHISM` problem is the following: Given two undirected graphs G and H, is G a subgraph of H?

9. Construct a first order theory that has exactly one model (up to isomorphisms of models).

10. A first-order formula is in prenex normal form if it is of the form

$$Q_1 x_1 \cdots Q_k \ \varphi(x_1, \ldots, x_k, x_{k+1}, \ldots x_n),$$

   where for each i, $1 \leq i \leq k$, Q_i is an existential or universal quantifier, and φ is a quantifier-free formula.

   Give a set of transformation rules that can be used to transform each first-order formula into an equivalent prenex normal form formula. Argue that this set of rules is sufficient.