Foundations Qualifying Examination

January 10, 1997

1. Consider a context-free grammar in which the right hand side of every production rule consists either of a single terminal symbol or a single terminal symbol followed by a variable.

   (a) Show that the language generated by such a grammar must be regular.

   (b) Show further that for any regular language not containing the empty string, there is a grammar of the above form.

2. Fix some encoding of finite automata as strings over the binary alphabet $\{0, 1\}$, and let $\langle A \rangle$ denote the encoding of the automaton A. Furthermore, assume that all strings are encoded (by a fixed encoding) over the alphabet $\{0, 1\}$, and let $\langle w \rangle$ denote the encoding of the string w.

   Prove that it is impossible to have a universal finite automaton, i.e. an automaton that accepts the language $\{\langle A \rangle \# \langle w \rangle \mid A \text{ accepts } w\}$, where $\#$ is a character different from 0 and 1.

3. Consider a Turing machine M that generates, one after the other, programs in your favorite programming language with the following guarantees: every program printed by M computes a function on the natural numbers, and every program printed by M halts on all inputs.

   Show that there must be some computable total function f such that M does not print a program computing f.

4. Which of the following properties of Turing machines are decidable? Justify your answer in each case. Clearly state any theorem you apply.

   (a) Given a machine M, does it halt on all inputs?

   (b) Given a machine M, is there a machine with fewer states than M accepting the same language as M?

   (c) Given a machine M, is there a machine with more states than M accepting the same language as M.

5. Show that a non-deterministic Turing machine that uses only a logarithmic amount of work space can be simulated by a deterministic machine using only a polynomial amount of time.

6. (a) Define what it means for a decision problem L to be $\mathtt{NP}$-complete.

   (b) Let $G = (V, E)$ be a graph. An *Hamiltonian cycle* is a cycle of G containing every vertex of V.

   Given that the Hamiltonian cycle problem for *directed* graphs is $\mathtt{NP}$-complete, prove that the Hamiltonian cycle problem for *undirected* graphs is $\mathtt{NP}$-complete.

7. Let S be a *finite* set of sentences in first order predicate logic. For each of the following statements, state whether it is always true, true for some but not all S, or always false. In each case, justify your answer.

   (a) The set of logical consequences of S is finite.

   (b) The set of logical consequences of S is decidable.

   (c) The set of logical consequences of S is recursively enumerable.