λ-calculus: A Basis for Functional Languages

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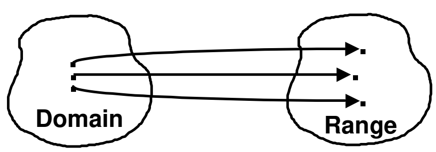
Functions



$f : D \rightarrow R$

f **may be viewed as**

- **a set of ordered pairs** $< d , r >$ **where** $d \; \varepsilon \; D$ **and** $r \; \varepsilon \; R$

- **a *method of computing* value *r* corresponding to argument *d*.**

some important notations:

- **λ-calculus (Church)**
- **Turing machines (Turing)**
- **Partial recursive functions**

The λ-calculus: a simple type-free language

- to express *all computable functions*
- to directly *express higher-order functions*
- to study *evaluation orders*, *termination*, *uniqueness of answers*...
- to study various *typing systems*
- to serve as *a kernel language for functional languages*

  However, λ-calculus extended with constants and let-blocks is more suitable

λ-notation

- a way of writing and applying functions without having to give them names
- a syntax for making a function expression from any other expression
- the syntax distinguishes between the *integer "2"* and the *function "always_two"* which when applied to any integer returns 2

```
always_two x = 2;
```

Pure λ-calculus: Syntax

E = x | λx.E | E E

variable abstraction application

1. *application* E_1 E_2

function argument

application is left associative

$E_1\ E_2\ E_3\ E_4 \equiv (((\ E_1\ E_2\ )\ E_3\ )\ E_4\ )$

2. *abstraction* λx.E

bound variable or formal parameter body

the scope of the dot in an abstraction extends as far to the right as possible

$\lambda x.x\ y \equiv \lambda x.(x\ y) \equiv (\lambda x.(x\ y)) \equiv (\lambda x.x\ y) \neq (\lambda x.x)\ y$

Free and Bound Variables

λ-calculus follows *lexical scoping* rules

Free variables of an expression

$FV(x) = \{x\}$

$FV(E_1\ E_2) = FV(E_1) \cup FV(E_2)$

$FV(\lambda x.E) = FV(E) - \{x\}$

A variable occurrence which is not free in an expression is said to be a *bound variable* of the expression

combinator: a λ-expression without free variables, aka *closed λ-expression*

β-substitution

$(\lambda x.E)\ E_a \rightarrow E\ [E_a / x]$

replace all free occurences of x in E with E_a

E[A/x] is defined as follows by case on E:

variable	$y\ [E_a / x]$	=	E_a	if $x \equiv y$
	$y\ [E_a / x]$	=	y	otherwise
application	$(E_1\ E_2)\ [E_a / x]$	=	$(E_1\ [E_a / x]\quad E_2\ [E_a / x])$	
abstraction	$(\lambda y.E_1)\ [E_a / x]$	=	$\lambda y.E_1$	if $x \equiv y$
	$(\lambda y.E_1)\ [E_a / x]$	=	$\lambda z.((E_1\ [z/y])\ [E_a / x])$ where $z \notin FV(E_1) \cup FV(E_a) \cup FV(x)$	otherwise

β-substitution: an example

$(\lambda p.p\ (p\ q))\ [(a\ p\ b)\ /\ q]$

$\rightarrow\quad (\lambda z.z\ \ (z\ \ q))\ [(a\ p\ b)\ /\ q]$

$\rightarrow\quad \lambda z.(z\ \ (z\ \ (a\ p\ b)))$

λ-Calculus as a Reduction System

Syntax

$E = x \mid \lambda x.E \mid E\ E$

Reduction Rule

α ***-rule:*** $\lambda x.E \rightarrow \lambda y.E\,[y/x]$ **if** $y \notin FV(E)$

β ***-rule:*** $(\lambda x.E)\ E_a \rightarrow E\,[E_a/x]$

η ***-rule:*** $(\lambda x.E\ x) \rightarrow E$ **if** $x \notin FV(E)$

Redex

$(\lambda x.E)\ E_a$

Normal Form

An expression without redexes

α and η Rules

α**-rule says that the bound variables can be renamed systematically:**

$$(\lambda x.x\ \ (\lambda x.a\ \ x))\ \ b \equiv (\lambda y.y\ \ (\lambda x.a\ \ x))\ \ b$$

η **-rule can turn any expression, including a constant, into a function:**

$$\lambda x.a\ x \quad \rightarrow_{\eta} \quad a$$

η **-rule does not work in the presence of types**

A Sample Reduction

$C \equiv \lambda x.\lambda y.\lambda f.\ f\ x\ y$
$H \equiv \lambda f.\ f\ (\lambda x.\lambda y.\ x)$
$T \equiv \lambda f.\ f\ (\lambda x.\lambda y.\ y)$

What is H (C a b) ?

$\rightarrow (\lambda f\ .\ f\ (\lambda x.\lambda y.x))\ (C\ a\ b)$
$\rightarrow (C\ a\ b)\ (\lambda x.\lambda y.x)$
$\rightarrow (\lambda x.\lambda y.x)\ a\ b$
$\rightarrow (\lambda y.a)\ b$
$\rightarrow a$

$H\ (C\ a\ b) \twoheadrightarrow a$
$T\ (C\ a\ b) \twoheadrightarrow b$

Integers: Church's Representation

$0 \equiv \lambda x.\lambda y.\ y$
$1 \equiv \lambda x.\lambda y.\ x\ y$
$2 \equiv \lambda x.\lambda y.\ x\ (x\ y)$
M
$n \equiv \lambda x.\lambda y.\ x\ (x\ldots(x\ y)\ldots)$

succ ?

If *n* is an integer, then *(n a b)* gives *n* nested *a*'s followed by *b*.

⇒ the successor of *n* should be *a (n a b)*

$\text{succ} \equiv \lambda n.\lambda a.\lambda b.a\ (n\ a\ b)$
$\text{plus} \equiv \lambda m.\lambda n.m\ \text{succ}\ n$
$\text{mul} \equiv$?

Booleans and Conditionals

True $\equiv$ λx.λy.x
False $\equiv$ λx.λy.y

zero? $\equiv$ λn. n (λy.False) True
zero? 0 $\twoheadrightarrow$?
zero? 1 $\twoheadrightarrow$?

cond $\equiv$ λb.λx.λy. b x y
cond True E_1 E_2 $\twoheadrightarrow$?
cond False E_1 E_2 $\twoheadrightarrow$?

Recursion ?

```
fact n = if (n == 0) then 1
                     else n * fact (n-1)
```

Assuming suitable combinators, fact can be rewritten as:

fact = λn. cond (zero? n) 1 (mul n (fact (sub n 1)))

How do we get rid of the fact *on the RHS?*

suppose
H = λf.λn.cond (zero? n) 1 (mul n (f (sub n 1)))
then
fact = H fact
fact is a solution of this equation???
more on recursion in the next lecture

Choosing Redexes

1. $\underbrace{((\lambda x.M)\ A)}_{\rho_1}\ \underbrace{((\lambda x.N)\ B)}_{\rho_2}$

2. $\underbrace{((\lambda x.M)\ \underbrace{((\lambda y.N)B)}_{\rho_2})}_{\rho_1}$

Does ρ_1 followed by ρ_2 produce the same expression as ρ_2 followed by ρ_1?

Notice in the second example ρ_1 can *destroy* or *duplicate* ρ_2 .

Church-Rosser Property

A reduction system is said to have the *Church-Rosser property*, if E $\twoheadrightarrow$ E_1 and E $\twoheadrightarrow$ E_2 then there exits a E_3 such that $E_1 \twoheadrightarrow E_3$ and $E_2 \twoheadrightarrow E_3$.

$$
\begin{tikzcd}
E \arrow[r, two heads] \arrow[d, two heads] & E_1 \arrow[d, two heads, dotted] \\
E_2 \arrow[r, two heads, dotted] & E_3
\end{tikzcd}
$$

also known as *CR* or *Confluence*

Theorem: **The λ-calculus is CR.**
(Martin-Lof & Tate)

Interpreters

An *interpreter* for the λ-calculus is a program to reduce λ-expressions to "answers". It requires:

- **the definition of an *answer:***
- **a *reduction strategy :***
  a method to choose redexes in an expression
- **a criterion for *terminating* the reduction process**

Definitions of "Answers"

- ***Normal form (NF):*** **an expression without redexes.**
- ***Head normal form (HNF):***
  x is HNF
  (λx.E) is in HNF if E is in HNF
  (x E_1 ... E_n) is in HNF
  Semantically most interesting- represents the information content of an expression
- ***Weak head normal form (WHNF):***
  x is in WHNF
  (λx.E) is in WHNF
  (x E_1 ... E_n) is in WHNF
  An expression in which the left most application is not a redex.
  Practically most interesting $\Rightarrow$ "Printable Answers"

Reduction Strategies

There are many methods of choosing redexes in an expression.

- *applicative order:* left-most innermost redex
- *normal order:* left-most (outermost) redex

$((\lambda x.M)\ \underbrace{((\lambda y.N)B)}_{\rho_2})$ with the whole expression labelled ρ_1

applicative order would require ρ_2 to be done before ρ_1.
normal order strategy would require ρ_1 to be done before ρ_2.

Facts

1. Every λ-expression does not have an answer *i.e.,* a NF or HNF or WHNF

   $(\lambda x.\ x\ \ x)\ (\lambda x.\ x\ \ x) = \Omega$

   $\Omega \rightarrow \Omega \rightarrow \Omega \rightarrow \ldots$

2. CR implies that if NF exists it is *unique*.

3. Even if an expression has an answer, not all *reduction strategies* may produce it

   $(\lambda x.\lambda y.\ y)\ \ \Omega$

leftmost redex: $(\lambda x.\lambda y.\ y)\ \ \Omega \rightarrow \lambda y.\ y$
innermost redex: $(\lambda x.\lambda y.\ y)\ \ \Omega \rightarrow (\lambda x.\lambda y.\ y)\ \ \Omega \rightarrow \ldots$

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6.827, L5-21

Normalizing Strategy

A *reduction strategy* is said to be *normalizing* if it terminates and produces an answer of an expression whenever the expression has an answer.

aka *the standard reduction*

Theorem: **Normal order (left-most) reduction strategy is normalizing for the λ-calculus.**

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A Call-by-name Interpreter

Answers: **WHNF**
Strategy: **leftmost redex**

cn(E): **Definition by cases on E**
$E \; = \; x \mid \lambda x.\, E \mid E \; E$

$$
\begin{aligned}
cn(x) &= x \\
cn(\lambda x.\, E) &= \lambda x.\, E \\
cn(E_1 \; E_2) &= \textit{let } f = cn(E_1) \\
&\quad \textit{in} \\
&\qquad \textit{case } f \textit{ of} \\
&\qquad\quad \lambda x.\, E_3 = cn(E_3[E_2/x]) \\
&\qquad\quad \_ \quad\;\; = (f \; E_2)
\end{aligned}
$$

A Call-by-value Interpreter

Answers: WHNF

Strategy: leftmost-innermost redex but not inside a λ-abstraction

cv(E): Definition by cases on E

$E \;=\; x \mid \lambda x.\,E \mid E\;E$

$$
\begin{aligned}
cv(x) &= x \\
cv(\lambda x.\,E) &= \lambda x.\,E \\
cv(E_1\;E_2) &= \textit{let}\;\; f = cv(E_1) \\
&\qquad\;\;\; a = cv(E_2) \\
&\quad \textit{in} \\
&\qquad \textit{case}\;\; f \;\textit{of} \\
&\qquad\quad \lambda x.\,E_3 = cv(E_3[a/x]) \\
&\qquad\quad \_ \qquad\;\; = (f\;a)
\end{aligned}
$$

More Facts

For computing WHNF

the call-by-name interpreter is normalizing but the call-by-value interpreter is not

e.g.

$(\lambda x.y)\;((\lambda x.x\;x)\;(\lambda x.x\;x))$