A Description of the Initial Value Formulation of Vacuum General Relativity for the Non-Specialist [1]

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Introduction

In any specialized scientific endeavor, the workers in that specific area develop their own language, or jargon, that can be confusing for the non-specialist (and, at times, for the specialist, too!). This makes it difficult for scientists to "latch–on" to fields that are outside their area of expertise. General Relativity is certainly no exception. In light of the Computational Grand Challenge Program "Black Hole Binaries: Coalescence and Gravitational Radiation," there is a need for a "jargonless" description of the formulation of General Relativity that is used in computer simulations (i.e. its initial value formulation). This paper is an attempt at meeting this need.

It should be stressed, however, that the main goal of this paper is to give a precise mathematical statement (sans jargon) of the initial value formulation of general relativity. The first section of the paper introduces the reader to a few major conceptual issues in General Relativity. In no way does this paper attempt to give a full physical interpretation of General Relativity. Interested readers are referred elsewhere [1, 2].

Note: This paper itself is a work in progress, and can be added to/revised at any time . (Date of last revision: Oct 10, 1994).

Conventions and Notation

In this paper, standard index notation will be used. The convention will be that greek indices run from 0 to 3 (i.e. $\mu = 0, 1, 2, 3$), and latin indices run from 1 to 3 (i.e. $i = 1, 2, 3$). For example, v^i denotes the three numbers (v^1, v^2, v^3) (**not** v to the first, second, and third power). Also, we will differentiate between superscripted indices and subscripted indices (v^i does **not** equal v_i in general).

The "Einstein summation convention" will also be employed. This convention dictates that indices that are repeated, once as a superscript and once as a subscript, are summed over. For example, $v_i w^i = v_1 w^1 + v_2 w^2 + v_3 w^3$. Another example is the standard matrix equation $M\vec{x} = \vec{b}$, where M is a (4 x 4) matrix and $\vec{x}$ and $\vec{b}$ are vectors. This equation, using index notation, reads $M^\mu_\nu x^\nu = b^\mu$.

The speed of light, c, is set equal to 1. Therefore, the time coordinate, $x^0 = t$, is measured in meters. To convert to seconds, use the conversion defined by $c = 1$: 1 second = 3×10^8 meters).

1 The Spirit of General Relativity

General Relativity is a theory of gravitational interaction. In a real sense, it is a more accuarate theory than Newtonian gravity ($F_g = G\frac{m_1 m_2}{{r_{12}}^2}$), because it has better agreement with observations. This is not to say that the Newtonian theory of gravity is a bad theory of gravity. It does very well in predicting how fast a rock should fall or predicting how long it should take the moon to circle the earth. In simple cases like these, the Newtonian theory of gravity and General Relativity agree to within obervational uncertainty. In fact, Newtonian gravity is the "weak-field limit" of General Relativity. However, very accurate measurements of other gravitational phenomena reveal a deviation from the Newtonian theory of gravity. To date all of these deviations are explained by General Relativity [3]. In addition to explaining weak-field deviations from Newtonian gravity, General Relativity has recently (in the past decades) passed a "strong-field" test, that of the prediction of the rate of energy loss due to gravitational waves emitted from a binary neutron star system.

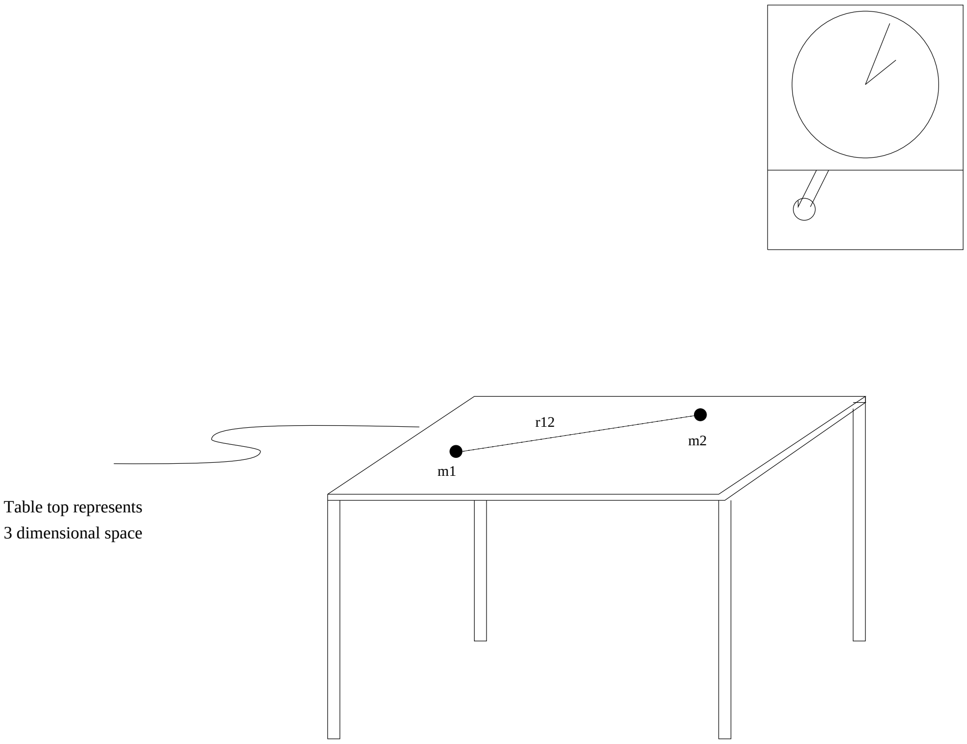
Although Newtonian gravity and General Relativity are both theories of gravitational interaction, they are conceptually very different. Newton's Universal Law of Gravitation gives the magnitude of the mutual force between two point particles of mass m_1 and m_2 seperated by a distance of r_{12} as

$$F_{12} = G\frac{m_1 m_2}{{r_{12}}^2}$$

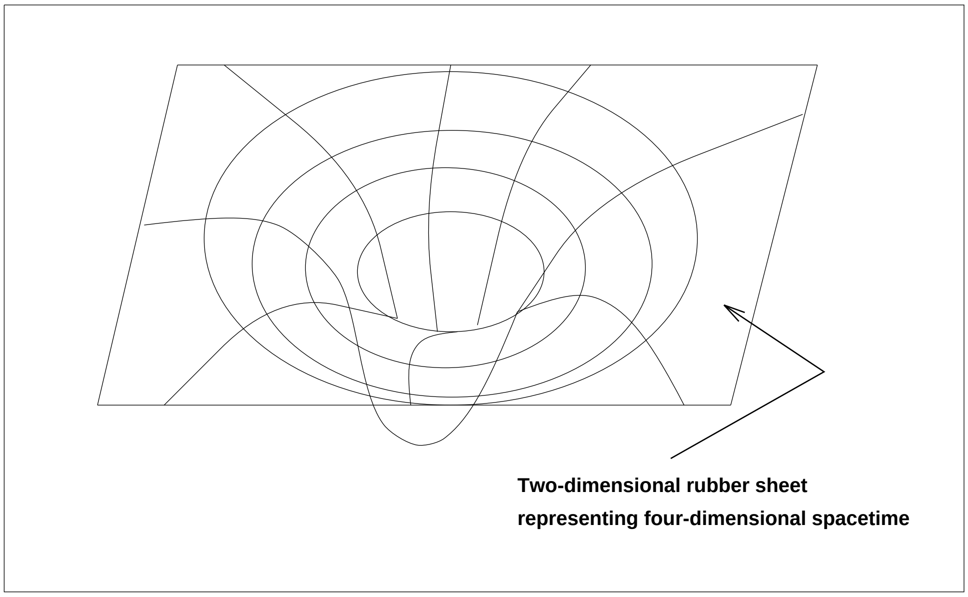
where G is a constant. If we think of 3 dimensional space as the surface of a flat 2 dimensional table, the picture is as follows:

In the diagram, note the clock on the wall. Its job is to keep track of "absolute time". In Newtonian gravity, time is seperate from space, and the wrist watches on all of the observers in space agree with the clock on the wall (absolute time).

In contrast to Newtonian gravity, with General Relativity, there **is no** force due to gravity!! Also, the "table top" that the particles sit in is not flat, nor is it simply 3 dimensional space. Pictorally, we can think of 4 dimensional spacetime as a stretched 2 dimensional rubber sheet. If we place a bowling ball on the sheet, the bowling ball deforms the rubber sheet:

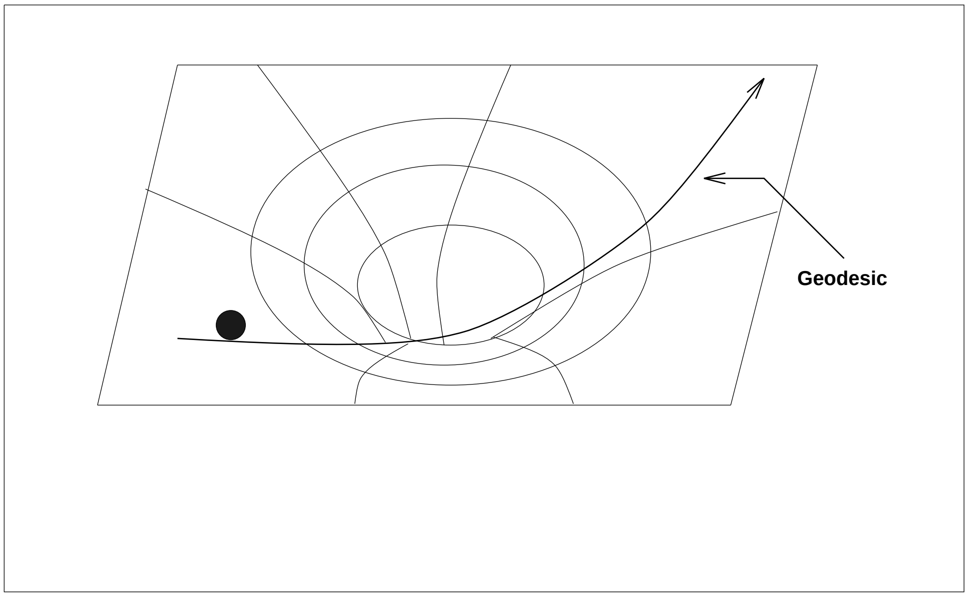


Newtonian Gravity



Just as the bowling ball deforms the rubber sheet, General Relativity says that mass (or energy) will deform, or curve, the spacetime around it. Also, there is no "absolute time" in General Relativity. Different observers will disagree as to how much time passes between events.

Now, in addition to a bowling ball sitting on the rubber sheet, imagine a marble rolling along the rubber sheet. General Relativity says that there is *no* gravitational force between the marble and the bowling ball! The marble will therefore travel in a straight line, since there are no forces acting on it. But what does it mean to travel in a straight line when the spacetime you are sitting in is curved? In flat space, a straight line between two points can be thought of as the unique path connecting the two points that has the shortest length. We take this concept of "extremal distance" over to the case of curved spacetime. We now say that the marble follows the path with extremal length (the path with shortest distance). These curves are called geodesics, and can be intuitively thought of as the "straightest possible path" in curved spacetimes.



So, General Relativity is not a law which gives the gravitational force between a bowling ball and a marble. Instead, it is a theory that predicts how the bowling ball curves the spacetime around it. The marble in "free fall" is simply following a geodesic (path of shortest distance) in this curved spacetime. Therefore, the dynamical variables in General Relativity are variables that determine the distance between any two (infinitesimally) nearby points. These variables are referred to as "the metric."

2 Vacuum General Relativity

Mathematically, the theory of vacuum General Relativity (General Relativity without matter) is a set of 10 coupled, nonlinear, second–order, partial differential equations involving ten functions of 4 independent variables or coordinates. The coordinates, x^μ, can be thought of as numbers labeling the "where" and "when" of events:

$$x^\mu = (\underbrace{t}_{when}, \underbrace{x, y, z}_{where})$$

The set of all events is the arena of General Relativity, and is referred to as "spacetime."

The dynamical variables of the theory can be thought of as functions arranged as entries in a symmetric 4x4 matrix:

$$\begin{bmatrix} g_{00}(t,x,y,z) & g_{01}(t,x,y,z) & g_{02}(t,x,y,z) & g_{03}(t,x,y,z) \\ g_{01}(t,x,y,z) & g_{11}(t,x,y,z) & g_{12}(t,x,y,z) & g_{13}(t,x,y,z) \\ g_{02}(t,x,y,z) & g_{12}(t,x,y,z) & g_{22}(t,x,y,z) & g_{23}(t,x,y,z) \\ g_{03}(t,x,y,z) & g_{13}(t,x,y,z) & g_{23}(t,x,y,z) & g_{33}(t,x,y,z) \end{bmatrix}$$

where each entry is a function of the coordinates, x^μ.

In index notation, this matrix is referred to as $g_{\mu\nu}$ (its symmetric property can be stated as $g_{\mu\nu} = g_{\nu\mu}$). In the literature, this matrix is known as "the metric," since it is used to calculate the distance between nearby points in the physical spacetime. Note that, although a 4x4 matrix has 16 entries, the matrix is symmetric so that there are only 10 independent functions.

Remember, $g_{\mu\nu}$ is a function of the coordinates, $x^\mu = (t, x, y, z)$. The Einstein field equations are 10 coupled, nonlinear, second order, partial differential equations involving the metric, $g_{\mu\nu}$. These equations would take tens of pages to explicitly list as derivatives of the metric. This fact partially explains the need for jargon in General Relativity (relativists refer to these equations as $R_{\mu\nu} = 0$. Perhaps this is "jargon-overkill").

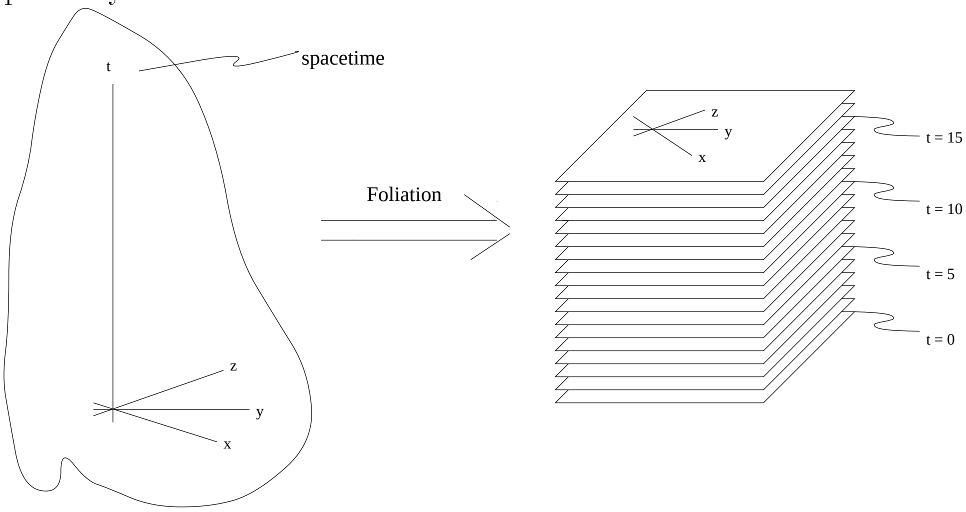
It should be stated that in General Relativity, one is free to use whatever coordinates (cartesian, spherical, hyperbolic, etc.) one wishes. However, the physical predictions of the theory are independent of the choice of coordinates. It is important to use this coordinate freedom to ones advantage when attacking a problem. In fact, this coordinate freedom introduces some ambiguity in the initial value formulation which must be dealt with.

3 General Relativity Admits an Initial Value Formulation

The Einstein equations as described above are too complicated to solve directly, except in cases of high symmetry. In a realistic problem (i.e. the coalescence of spiraling black holes), what one requires is an initial value formulation of the theory. The analogy in classical mechanics is well known: once the position and velocity of each particle is known, Newton's second law (along with rules for the interaction between the particles) completely determines the trajectory of each particle. In that case, the initial data are the position and velocity of each particle at an instant of time, t_0.

It turns out that General Relativity admits an initial value formulation that is similar in spirit to that in classical mechanics. What follows is a description of that formulation. The details of the calculations can be found in [1, 2].

In the initial value formulation, spacetime ($x^\mu = (t, x, y, z)$) is separated into space ($x^i = (x, y, z)$) and time ($x^0 = t$). This "foliation" can be viewed pictorally as follows:



Space-time is sliced up into "spatial surfaces," and each different surface is labeled by the (continuous) parameter, t. These "spatial surfaces" can have any topology (R^3 ,T^3 ,$S^2 \times R$, etc.). For our purposes, we will assume that each surface is coordinatized by $x^i = (x, y, z)$.

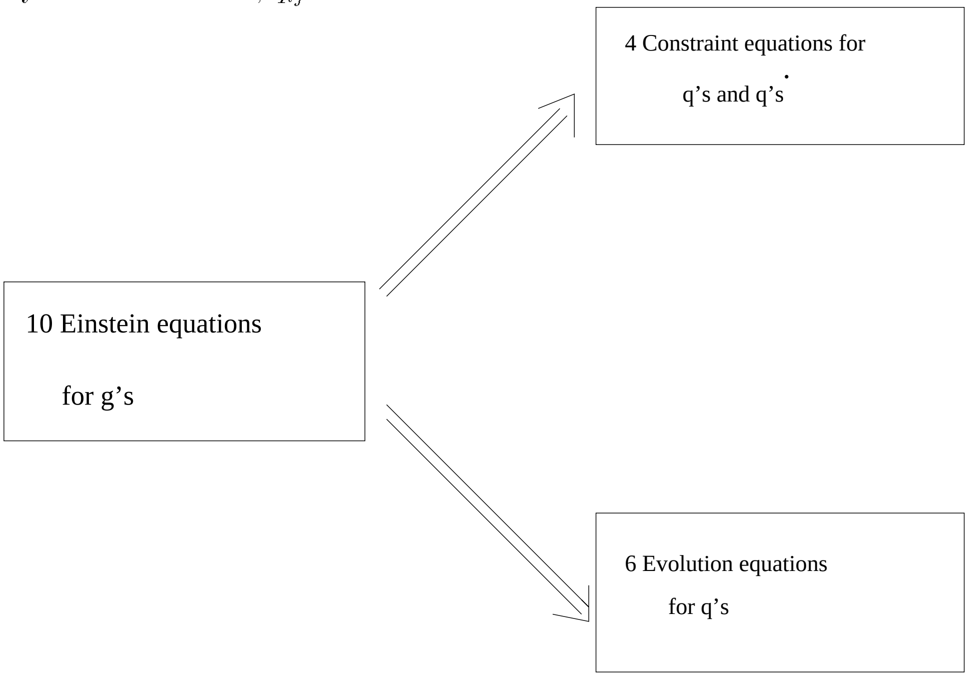
The 10 Einstein equations are then "projected" onto this foliation. What happens is that natural dynamical variables appear: the so–called 3–metric, q_{ij}. Like the dynamical variables for the full Einstein equations $g_{\mu\nu}$, q_{ij} can be thought of as a symmetric matrix of functions:

$$\begin{bmatrix} q_{11}(t,x,y,z) & q_{12}(t,x,y,z) & q_{13}(t,x,y,z) \\ q_{12}(t,x,y,z) & q_{22}(t,x,y,z) & q_{23}(t,x,y,z) \\ q_{13}(t,x,y,z) & q_{23}(t,x,y,z) & q_{33}(t,x,y,z) \end{bmatrix}$$

Unlike $g_{\mu\nu}$, q_{ij} is only a 3x3 symmetric matrix, so there are only 6 independent functions.

At first glance, it might appear that the system is over-determined. Afterall, there are 10 independent equations for 6 functions. A closer look reveals that 4 of the equations involve only first–order time derivatives of

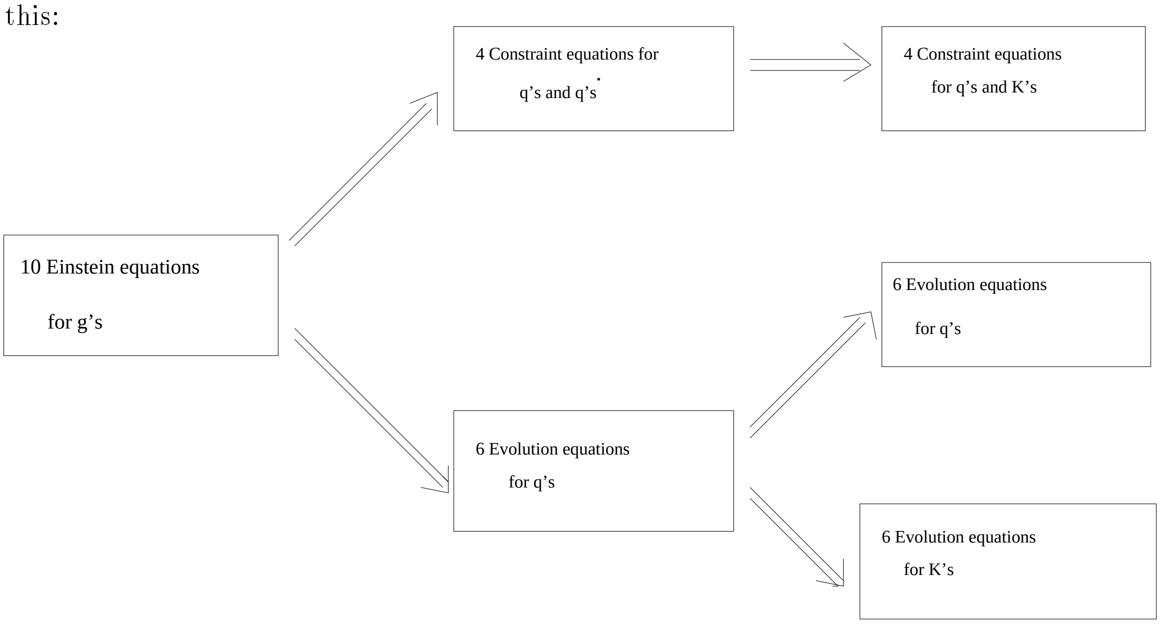
q_{ij}. In the analogy with classical mechanics, these equations involve only positions and velocities. These equations are therefore constraints on the initial data. This leaves $10 - 4 = 6$ equations left over, which are called the evolution equations. They completely determine the evolution of the 6 dynamical variables, q_{ij}.



When working with second order differential equations, it is usually convenient (in analytical work, as well as in numerical work) to write each second order (in time) differential equation as two first order (in time) differential equations, treating the time derivative of the dynamical variable as a separate dynamical variable. In our case, the new dynamical variable is K_{ij}, a symmetric 3x3 matrix of functions:

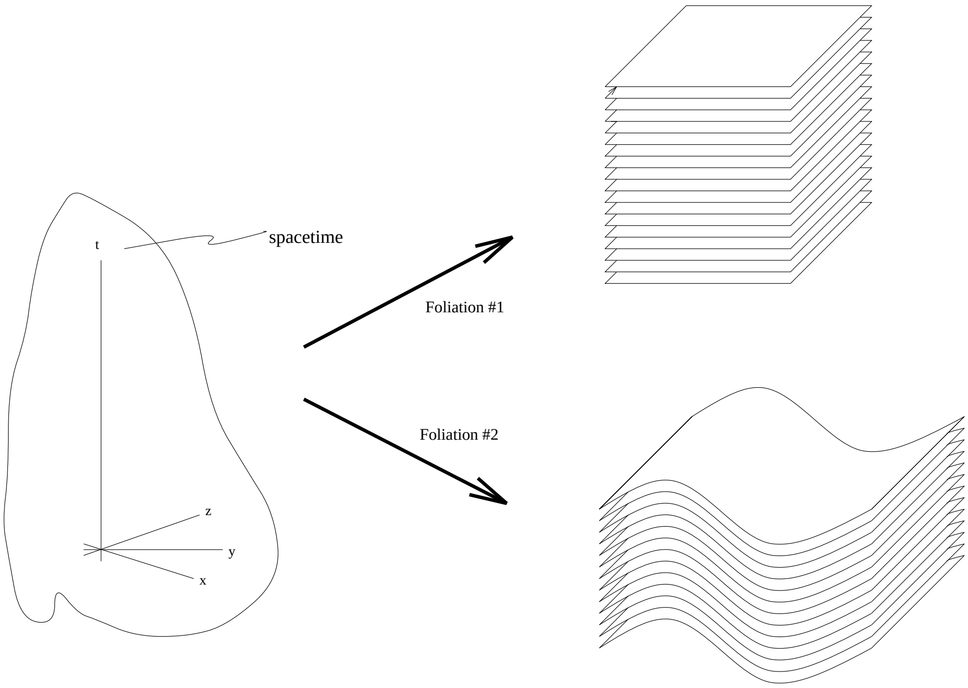
$$\begin{bmatrix} K_{11}(t,x,y,z) & K_{12}(t,x,y,z) & K_{13}(t,x,y,z) \\ K_{12}(t,x,y,z) & K_{22}(t,x,y,z) & K_{23}(t,x,y,z) \\ K_{13}(t,x,y,z) & K_{23}(t,x,y,z) & K_{33}(t,x,y,z) \end{bmatrix}$$

and is called the “extrinsic curvature.” It derives its name from its geometrical interpretation: it contains information on how each “spatial surface” is “embedded” in the spacetime. For our purposes, we will think of K_{ij} as “the time derivative of q_{ij}” (although, K_{ij} does **not** strictly equal $\frac{\partial}{\partial t}(q_{ij}(t,x,y,z))$, as we shall see explicitly). The picture so far looks like this:



4 The Final Step

Before writing down the constraint and evolution equations, let's take a look back. The way we "slice up," or foliate, spacetime is quite arbitrary:

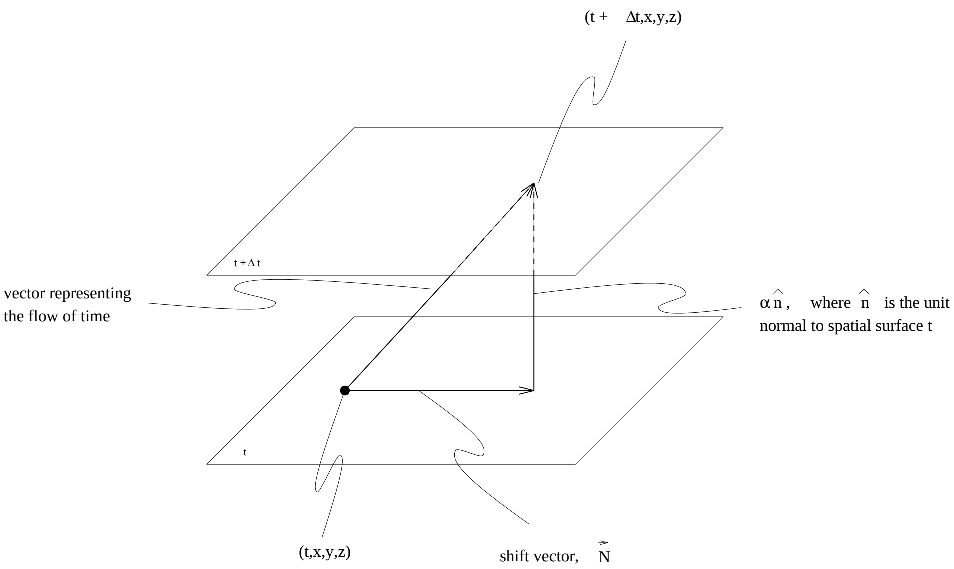


This is actually a restatement of the freedom to choose coordinates in General Relativity. This freedom results in ambiguity in the evolution equations for q_{ij} and K_{ij}. Specifically, there are 4 undetermined functions in the evolution equations. These functions are given the names "lapse" (1 function) and "shift" (3 functions):

Lapse: $\alpha(t, x, y, z)$ (1 function)

Shift: $N^i(t, x, y, z)$ (3 functions. Remember: $i = 1, 2, 3$)

The geometrical interpretation of the lapse and shift is given in the picture below:



As shown in the diagram, the shift functions (N^i) are the components of the vector that describe the "tangential shift" of coordinates as one moves from the t_0 spatial surface to the $t_0 + \delta t$ spatial surface. The lapse describes how much physical time elapses between the two spatial surfaces. It should be emphasized again that the physical predictions of the theory are completely independent of the choice of the lapse and shift. In numerical work, this freedom (if used wisely) can be used to accomplish one or more of the following:

- avoid singularities (regions of spacetime that generate infinite answers to physical questions)
- keep numerical algorithm stable
- insure the consistency of the constraints

5 Summary

The initial value formulation of general relativity consists of 12 first order (in time), coupled, nonlinear differential equations for the 12 dynamical variables, $q_{ij}(t, x, y, z)$ and $K_{ij}(t, x, y, z)$. The initial data for the theory is specified by $q_{ij}(t_0, x, y, z)$ and $K_{ij}(t_0, x, y, z)$ (that is, by assigning values to q_{ij} and K_{ij} at an instant of time, t_0) that satisfy 4 constraint equations. The constraint equations are consistent in the sense that if they are solved on the initial spatial surface and that spatial surface is evolved via the evolution equations, then the constraints are solved automatically on any future spatial surface. In addition, the evolution equations contain 4 arbitrary functions, α (lapse) and N^i (shift). The physical predictions of General Relativity are independent of the choice of these functions.

In the way of notation, I will define q^{ij} (note the superscripted indices) as the inverse of the 3–metric, q_{ij}. That is,

$$\begin{bmatrix} q^{11} & q^{12} & q^{13} \\ q^{12} & q^{22} & q^{23} \\ q^{13} & q^{23} & q^{33} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix}^{-1}$$

I will also define

$$\Gamma^k_{ij} = \frac{1}{2} q^{km} \left(\frac{\partial q_{im}}{\partial x^j} + \frac{\partial q_{jm}}{\partial x^i} - \frac{\partial q_{ij}}{\partial x^m} \right)$$

5.1 Constraints

The first constraint, $\mathcal{C}$, is called the scalar constraint (or sometimes the Hamiltonian constraint). It reads

$$0 = \mathcal{C} = q^{ij} \left(\frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{jk}}{\partial x^i} + \Gamma^m_{ij}\Gamma^k_{mk} - \Gamma^m_{jk}\Gamma^k_{im} \right) + (K_{ij}q^{ij})^2 - K_{ik}K_{jm}q^{ij}q^{km}$$

The other 3 constraints, $\mathcal{C}_i$, are called the vector constraints (or sometimes the diffeomorphism constraints). They read

$$0 = \mathcal{C}_i = q^{jk} \left(\frac{\partial K_{ik}}{\partial x^j} - \frac{\partial K_{jk}}{\partial x^i} + K_{jm}\Gamma^m_{ik} - K_{im}\Gamma^m_{jk} \right)$$

5.2 Evolution Equations

The evolution equations for the q_{ij} are

$$\frac{\partial q_{ij}}{\partial t} = 2\alpha K_{ij} + q_{jk} \left(\frac{\partial N^k}{\partial x^i} + N^m \Gamma^k_{im} \right) + q_{ik} \left(\frac{\partial N^k}{\partial x^j} + N^m \Gamma^k_{jm} \right)$$

The evolution equations for the K_{ij} are

$$\begin{aligned} \frac{\partial K_{ij}}{\partial t} = & -\alpha \left(\frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{jk}}{\partial x^i} + \Gamma^m_{ij}\Gamma^k_{km} - \Gamma^m_{jk}\Gamma^k_{im} \right) + \alpha q^{km} (2K_{ik}K_{jm} - K_{km}K_{ij}) + \\ & + \frac{\partial^2 \alpha}{\partial x^i \partial x^j} - \frac{\partial \alpha}{\partial x^k}\Gamma^k_{ij} + N^k \frac{\partial K_{ij}}{\partial x^k} + K_{jk}\frac{\partial N^k}{\partial x^i} + K_{ik}\frac{\partial N^k}{\partial x^j} \end{aligned}$$

6 Glossary of Terms

Christoffel symbol $(\Gamma^{\mu}{}_{\alpha\beta})$ Mathematically, these are simply combinations of first derivatives of the metric:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right)$$

Einstein equations The Einstein equations are 10 coupled, nonlinear, second order, partial differential equations in the 10 metric functions:

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

where $G_{\alpha\beta}$ is the Einstein tensor and $T_{\alpha\beta}$ is the stress-energy tensor.

Einstein tensor $(G_{\alpha\beta})$ The Einstein tensor is defined as:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$$

where $R_{\alpha\beta}$ is the Ricci tensor, R is the scalar curvature, and $g_{\alpha\beta}$ is the metric.

Geodesic Geodesics are paths in spacetimes that have "extremal distance". They can be thought of intuitively as the "straightest possible paths" in curved spacetime. Physically, geodesics are trajectories of particles in free-fall (no forces acting on the particles).

Metric $(g_{\mu\nu})$ The metric is a set of 10 functions that can be thought of as entries in a symmetric matrix:

$$\begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{bmatrix}$$

Physically, the metric determines the distance between (infinitesimally) nearby points. All information concerning the geometry of the spacetime is contained in the metric, and it is therefore the metric that is the dynamical variable in General Relativity.

Ricci tensor $(R_{\alpha\beta})$ The Ricci tensor is a set of 10 independent functions of the coordinates. It is constructed by contracting the second and fourth indices of the Riemann tensor: $R_{\alpha\beta} = R_{\alpha\mu\beta}{}^{\mu}$.

Riemann tensor ($R_{\alpha\beta\mu}{}^{\nu}$) Mathematically, The Riemann tensor is a set of 20 (not 256!!) independent functions of the coordinates that completely describe the curvature of the spacetime. (if $R_{\alpha\beta\mu}{}^{\nu} = 0$, then the spacetime is said to be completely flat.) It is constructed out of various first and second derivatives of the metric.

Scalar Curvature (R) The scalar curvature is a function of the coordinates obtained by contracting the Ricci tensor with the metric: $R = R_{\alpha\beta}g^{\alpha\beta}$.

References

[1] C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (1973).

[2] R. Wald, *General Relativity* (1984).

[3] C. M. Will, *Theory and Experiment in Gravitational Physics* (1993).