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DYNAMIC EXPONENTS FOR POTTS MODEL CLUSTER ALGORITHMS

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We have studied the Swendsen-Wang and Wolff cluster update algorithms for the Ising model in 2, 3 and 4 dimensions. The data indicate simple relations between the specific heat and the Wolff autocorrelations, and between the magnetization and the Swendsen-Wang autocorrelations. This implies that the dynamic critical exponents are related to the static exponents of the Ising model. We also investigate the possibility of similar relationships for the Q-state Potts model.

The Monte Carlo cluster update algorithms of Swendsen and Wang (SW) [1] and Wolff [2] can dramatically reduce critical slowing down in computer simulations of spin models (for reviews of cluster algorithms, see refs. [3,4]). There is little theoretical understanding of the dynamics of these algorithms, although a rigorous bound on the dynamic critical exponent z for the SW algorithm for Potts models has been derived [5]. In order to better understand these algorithms, we are doing numerical simulations of the Ising model in 2, 3 and 4 dimensions using the SW and Wolff algorithms. We measured the time correlation function $\rho(t)$ for the energy and extracted the integrated autocorrelation time $\tau_{int,E}$ [3]. The dynamic critical exponent $z_{int,E}$ is given by $\tau_{int,E} \sim L^{z_{int,E}}$, where $\tau_{int,E}$ for the different lattice sizes is measured at the infinite volume critical point. A detailed account of the methods we use to do the measurements, fits and error estimates, is given in Ref. [6].

Results for $\tau_{int,E}$ are shown in Fig. 1. For $d = 3$ we have used a log-log plot, with the straight lines representing χ^2 fits to a power law, while for $d = 2$ and 4 we show a log-linear plot, with straight lines representing χ^2 fits to a logarithm. Note that for $d = 4$ we plot $\log \tau^{SW}_{int,E}$ rather than $\tau^{SW}_{int,E}$, since the SW autocorrelations increase as a power of L. In Fig. 1 we also include the measured value of the specific heat C_H, scaled by an appropriate factor, in order to show that the bound of Li and Sokal [5]

$$\tau_{int,E} \geq constant \times C_H, \qquad z_{int,E} \geq \alpha/\nu, \quad (1)$$

is indeed satisfied by the SW algorithm. Here α is the critical exponent for the specific heat, and ν is the exponent for the correlation length. No such bound has been proven for the Wolff algorithm, although it appears from Fig. 1 that not only does the bound hold, but that there may actually be equality in the exponents, i.e. the Wolff algorithm for the Ising model seems to satisfy the surprisingly simple relations

$$\tau^{W}_{int,E} = a + b \times C_H, \qquad z^{W}_{int,E} = \alpha/\nu, \qquad (2)$$

where a and b are constants.

In Fig. 2 we plot the scaled difference

$$(\tau^{W}_{int,E} - (a + b \times C_H)) \quad / \quad \tau^{W}_{int,E} \tag{3}$$

with a and b chosen to minimize χ^2 (smaller values of L are excluded from the fit). In all cases values of a and b can be found such that the difference is zero within the errors, which are generally of the order of 1%. All the errors shown here are purely statistical (one standard deviation).

The surprising simplicity of this result led us to look for a similar relation for the SW algorithm. The relative average size of the largest SW cluster, $m =< |c^{max}_{SW}| > /L^d$, is an estimator of the magnetization [7], and the exponent β/ν for the divergence of the magnetization has values which are similar to our values for $z^{SW}_{int,E}$. We thus scaled the SW autocorrelations by multiplying them by m. This is also shown in Fig. 1 (note that the results are also scaled by an additional arbitrary constant, so that these points are not entangled with others in the plots). For $d = 4$ the results are very close to a constant, while for $d = 3$ they seem to approach a constant as L increases. In two dimensions the scaled autocorrelations are not constant, but they fit very well (much better than the unscaled data) to a logarithm, as can be seen in Fig. 1(a). The results therefore support the assertion that

$$m \;\; \tau^{SW}_{int,E} = a + b \times \log L, \qquad z^{SW}_{int,E} = \beta/\nu. \tag{4}$$

Better data is needed to test these conjectures, since it is possible that they are merely good approximations. Clearly a theoretical understanding of these results is also very desirable. The relations 2 and 4 are certainly not general results, since for the 2-d 3-state Potts model we find that $z^W > \alpha/\nu$ and $z^{SW} > \beta/\nu$ [5,6]. Also, it is quite surprising that these empirical relations imply that z^{SW} is not equal to z^W for the 2-d Ising model, whereas the two appear to be equal for the 2-d 3-state Potts model. We have therefore attempted to find similar simple relations for the 3-state Potts model. The measured value of z for this model is approximately 0.55(1) [6], which is very close to the value of $(\alpha + \beta)/\nu = 8/15 \approx 0.533$. It is possible that this result also holds for the 4-state model, although there the situation is less clear, since there are large logarithmic corrections to scaling [5].

In order to test this conjecture, we have scaled the autocorrelations (for both the Wolff and SW algorithms) by the magnetization m, and then attempted to find a linear relation between these scaled autocorrelations and the specific heat. In both cases we find that the data is consistent with such a relation for lattices of size 16^2 to 256^2 within the errors (which are of the order of 1%). Again, better data is required to properly check this conjecture, and a similar analysis of data for the 4-state model would be very useful.

References

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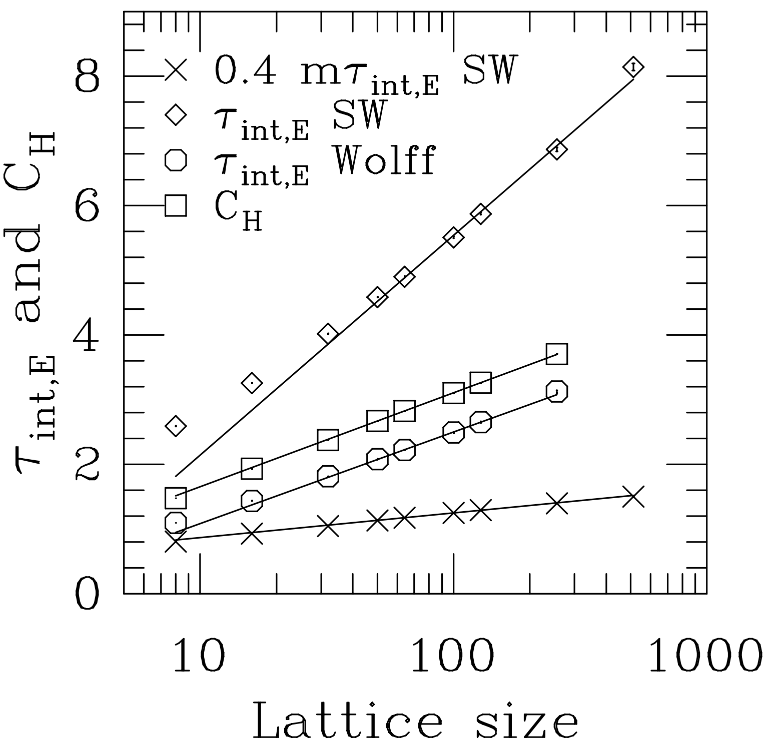
Fig. 1. $\tau_{int,E}$ for the Wolff and SW algorithms for the Ising model in (a) 2-d, (b) 3-d and (c) 4-d. Also shown is the specific heat C_H and $\tau^{SW}_{int,E}$ scaled by the average maximum cluster size m.

Fig. 2. The scaled difference between the Wolff autocorrelations and a linear function of the specific heat for the Ising model in (a) 2-d, (b) 3-d and (c) 4-d. The values of a and b are chosen so as to minimize χ^2.



$\times$ 0.4 $m\tau_{int,E}$ SW
$\diamond$ $\tau_{int,E}$ SW
$\circ$ $\tau_{int,E}$ Wolff
$\square$ C_H
$\tau_{int,E}$ and C_H
0
2
4
6
8
10
100
1000
Lattice size

Fig. 1(a)

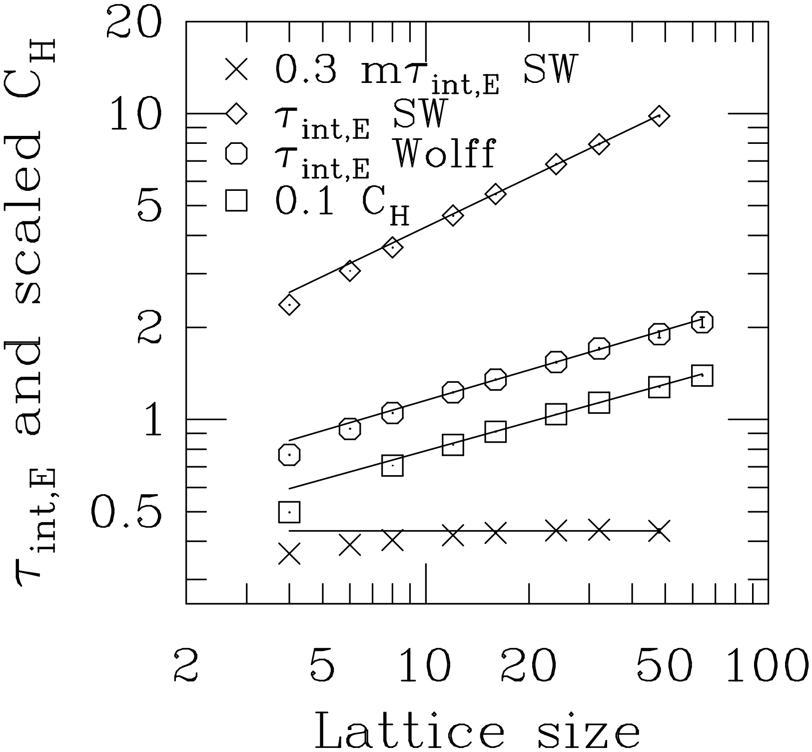
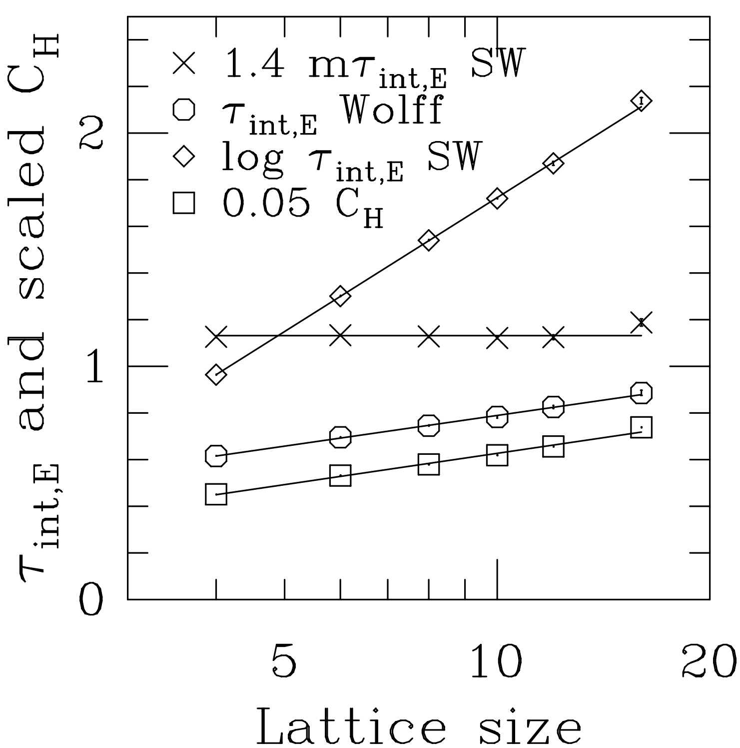
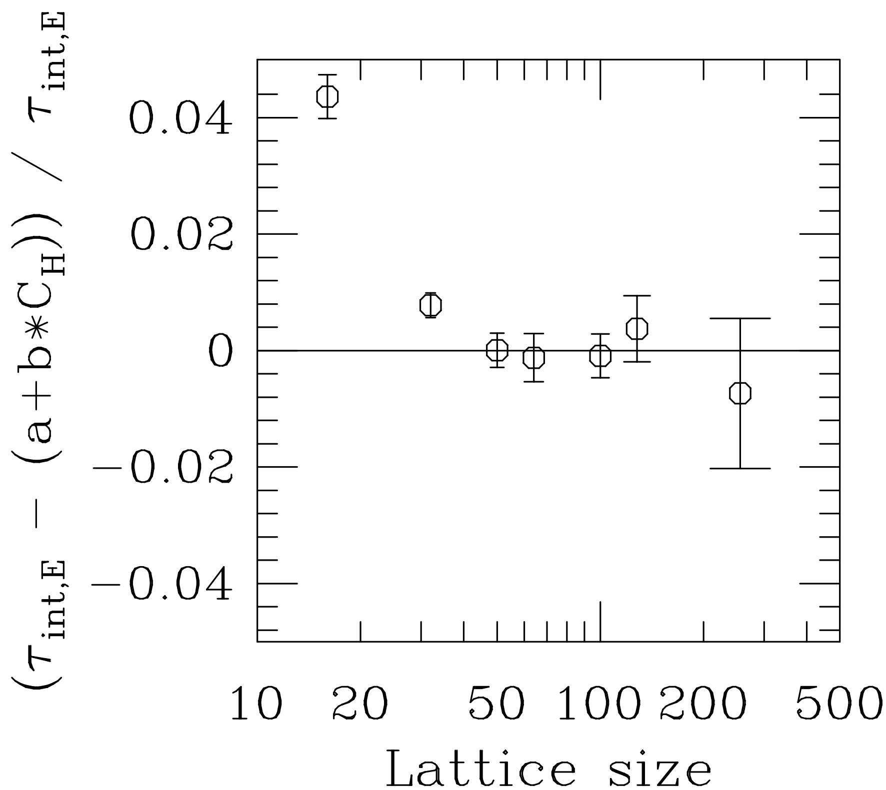

$\tau_{\rm int,E}$ and scaled C_H
20
10
5
2
1
0.5
2
5
10
20
50
100
Lattice size
0.3 $m\tau_{\rm int,E}$ SW
$\tau_{\rm int,E}$ SW
$\tau_{\rm int,E}$ Wolff
0.1 C_H

Fig. 1(b)



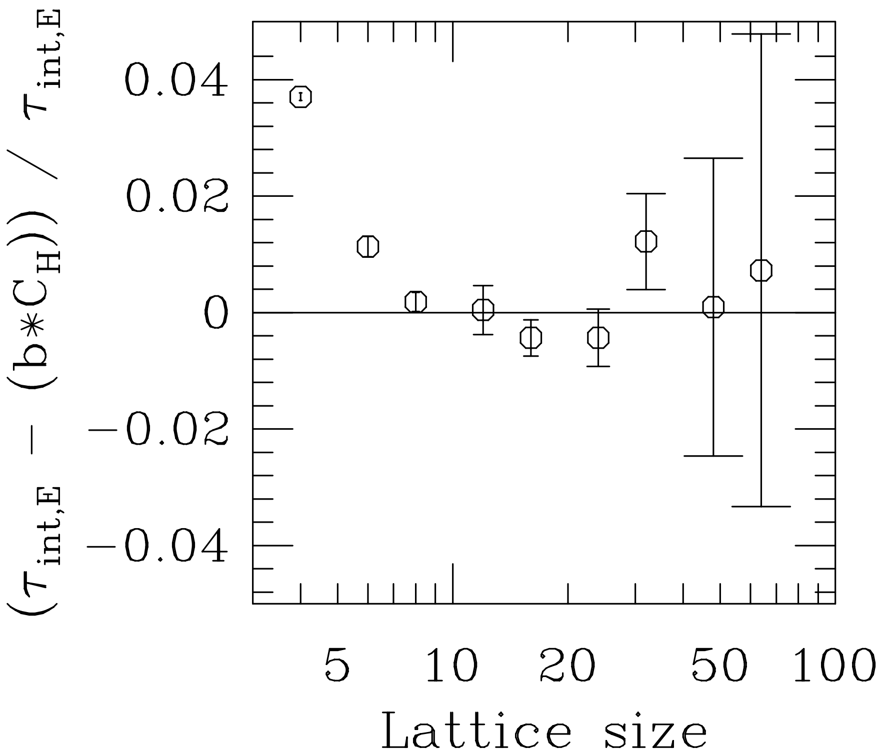
$\tau_{int,E}$ and scaled C_H
Lattice size
$\times$ 1.4 m$\tau_{int,E}$ SW
$\tau_{int,E}$ Wolff
log $\tau_{int,E}$ SW
0.05 C_H
0
1
2
5
10
20

Fig. 1(c)



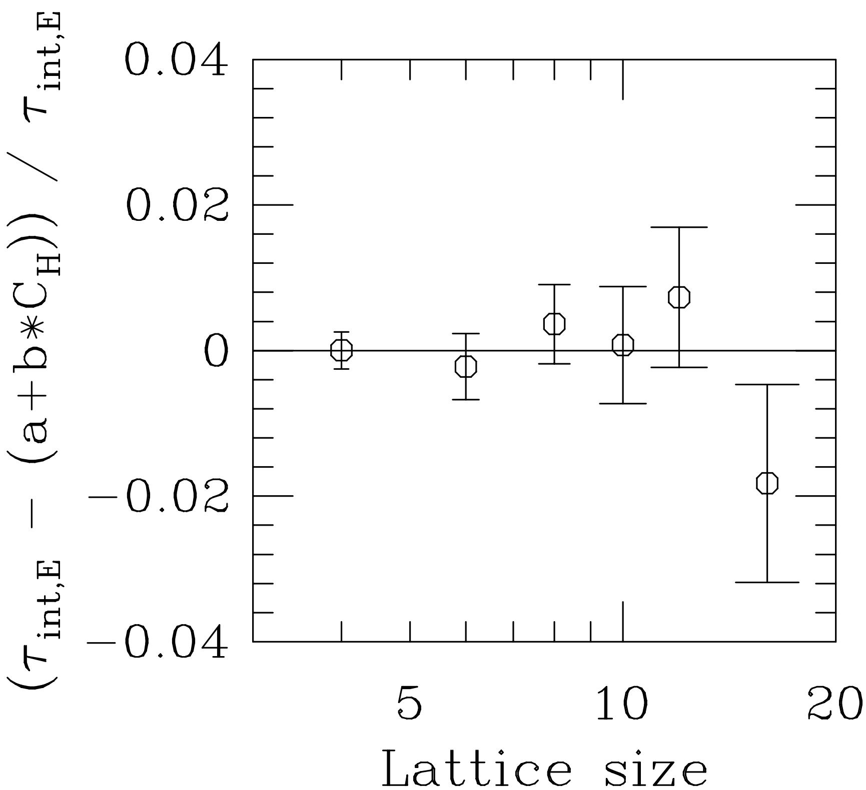
$(\tau_{\mathrm{int,E}} - (a+b*C_H)) / \tau_{\mathrm{int,E}}$
0.04
0.02
0
−0.02
−0.04
10
20
50
100
200
500
Lattice size

Fig. 2(a)



0.04
0.02
0
−0.02
−0.04
5
10
20
50
100
Lattice size
$(\tau_{\mathrm{int,E}} - (b * C_H)) \;/\; \tau_{\mathrm{int,E}}$

Fig. 2(b)



0.04
0.02
0
−0.02
−0.04
5
10
20
Lattice size
$(\tau_{\mathrm{int,E}} - (a+b*C_H)) / \tau_{\mathrm{int,E}}$

Fig. 2(c)