Quantum Gravity, Random Geometry and Critical Phenomena

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Abstract

We discuss the theory of non-critical strings with extrinsic curvature embedded in a target spa d greater than one. We emphasize the analogy between $2d$ gravity coupled to matter and non liquid-like membranes with bending rigidity. We first outline the exact solution for strings in dime via the double scaling limit of matrix models and then discuss the difficulties of an extension to $d >$ from recent and ongoing numerical simulations of dynamically triangulated random surfaces indica is a non-trivial crossover from a crumpled to an extended surface as the bending rigidity is incre cross-over is a true second order phase transition corresponding to a critical point there is the excitin of obtaining a well defined continuum string theory for $d > 1$.

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String Theory is a powerful model, capable of unifying the Yang-Mills interactions of matter with the universal interaction of gravity. It softens the short distance (ultraviolet) divergences of Einstein Hilbert gravity by smearing out points to one-dimensional extended *strings*. These strings sweep out two-dimensional Riemann surfaces as they evolve in Euclidean time. In the first quantized description of string theory one may view the string coordinates describing the embedding of the worldsheet in the target spacetime as a collection of scalar fields living on the worldsheet. The worldsheet, however, must fluctuate as one is required to integrate over all admissible metrics to enforce diffeomorphism (reparametrization) invariance. In this way new intrinsic degrees of freedom (the conformal modes of the metric) enter the theory. From the statistical mechanics viewpoint one is thus dealing with an exciting class of models described by certain order fields living on a fluctuating substrate. Averaging over metrics corresponds to being in the universality class of translationally and orientationally disordered fluctuating surfaces or membranes. These are often called liquid-like membranes, as opposed to crystalline or hexatic membranes

that are translationally or orientationally order
able fact is that these statistical mechanical mo
are, in a sense, easier to solve than the convent
regular lattice. This is because diffeomorphism
of effective degrees of freedom. It is even possi
change the topology of the surface (growth or

This is certainly of great interest as a mo
the basis for an exploration of membranes. Rec
corresponding to certain types of conformal ma
been exactly solved including the sum over all
of the solution together with the relation to $2d$
realistic random surfaces (flexible liquid-like m

Let us start by considering the extreme cas
all. All that remains is the smile on the Cheshi
This is clearly two-dimensional gravity. Since
coordinates it is also a model of strings in zero d
action with a cosmological constant term for 2

$$S[g] = \frac{-1}{16\pi G}\int_{\Sigma} d^2\xi\sqrt{g}R +$$

where $g_{\alpha\beta}(\xi_1, \xi_2)$ is the $2d$ metric of the Riem
ξ_1 and ξ_2.

The partition function Z then depends on two variables, Newton's constant G and the cosmological constant μ

$$Z[G,\mu] = \int [\mathcal{D}g] \; e^{-S[g]} \;, \tag{2}$$

where the path integral is over all admissible metrics of Riemann surfaces Σ. In two dimensions the action (1) is simple since the first term is a topological invariant by the Gauss-Bonnet theorem

$$S = \frac{-\chi(\Sigma)}{4G} + \mu A(\Sigma) \;, \tag{3}$$

where χ is the Euler characteristic of Σ and A is the area. χ is related to the number of handles, or genus h, by $\chi = 2 - 2h$, where for simplicity we are assuming Σ to be closed (without boundaries). The partition function thus reduces to

$$Z[G,\mu] = \sum_h \int dA e^{\frac{\chi}{4G}} e^{-\mu A} \Omega_h(A) \;, \tag{4}$$

where $\Omega_h(A)$ is the density of states of Riemann surfaces Σ of fixed area A and genus h,

$$\Omega_h(A) \equiv \int_{(h;A)} \mathcal{D}g_{\alpha\beta}. \tag{5}$$

$\Omega_h(A)$ is very difficult to calculate as h increases and the sum over genus in (4) diverges [2]. The above expressions are all, in fact, ill-defined. To give them meaning we must regularize the path integrals. One approach is to discretize by replacing Σ by a lattice. A particularly concrete and appealing discretization is to consider all triangulations (or more generally cellular decompositions) of Σ. The surface is thus replaced by a discrete set of n points (vertices) labelled by an index i. The connectivity of the lattice is described by the adjacency matrix

$$C_{ij} = \begin{cases} 1 \text{ if } i \text{ and } j \text{ are connected by a link} \\ 0 \text{ otherwise} \end{cases} \tag{6}$$

This defines a metric on the lattice by fixing all links to have length one. Thus all triangles (cells) in the triangulation are equilateral and of fixed area. The Euler characteristic follows from Euler's relation $\chi = V - E + F$, for V vertices,

E edges (links) and F faces (triangles). Local c
the deficit angle

$$R_i = \frac{\pi}{3}\frac{6 - q_i}{q_i}$$

where q_i is the coordination number of vertex

$$q_i = \sum_j C_{ij}.$$

To simulate the integral over metrics the adja
to fluctuate so that the coordination number
degree of freedom. The local environment of
This considerably complicates the study of suc
point of view but also makes them more intere
Dynamically Triangulated Random Surfaces D
update C_{ij} is a flip on a fundamental parallelo

common edge. The discrete version of the part
grals over metrics by sums over admissible tria
in the form

$$Z[G,\mu] = \sum_{h=0}^{\infty} e^{\frac{2-2h}{4G}} \sum_{n=0}^{\infty} e$$

where $Z_{h,n}$ is the number of distinct triangula
h. $Z_{h,n}$ is a discrete version of $\Omega_h(A)$, since
fixed area elementary triangles. The combinato
is related to the quantum field theory problem
distinct Feynman diagrams of a matrix Φ^3 field

constructs the dual of each triangulation. It may then be shown that the original partition function (9) is related to the solution of the matrix model defined by the integral

$$\zeta(g,N) = \int d^{N^2}\Phi \ \exp\{-N\mathrm{Tr}(\frac{\Phi^2}{2} - \frac{g}{3}\Phi^3)\} \tag{10}$$

over NxN-Hermitian matrices Φ. The exact relation is

$$Z[G,\mu] = \log\zeta(g,N) \quad , \tag{11}$$

where one must identify

$$N = e^{\frac{1}{4G}} \quad \text{and} \quad g = e^{-\mu}. \tag{12}$$

It is necessary to take Φ to be a matrix to generate topologically non-trivial triangulations. In fact it is easy to see that N appears weighted as N^χ for a

Feynman diagram of Euler characteristic χ in t
in a double power series in g and N [7].

In the continuum it has been shown [8–10]

$$\Omega_h(A) \sim e^{\mu_c A} A^{\gamma_h}$$

where the string susceptibility γ_h is

$$\gamma_h = -\frac{1}{2} + \frac{5}{2}h.$$

This result can be generalized to include particu
surface. These are the so-called minimal confor
integers p and q. A key parameter of these mod
measures the response of the free energy to loc
and is roughly a measure of the number of effe
model. For a (p,q) model c is given by

$$c = 1 - \frac{6(p-q)^2}{pq}$$

Note that c is less than one. Since a single sca
of (p,q) conformal matter coupled to $2d$-gravi
than one target-space dimension. The result (1
more generally

$$\gamma_h = 2 - \frac{(1-h)}{12}\Big\{25 - c + \sqrt{(}$$

It turns out that pure gravity corresponds to p
(15) that $c = 0$ as expected. Near the critica
equivalently critical coupling g_c) we see that

$$\int_0^\infty dA e^{-\mu A}\Omega_h(A) \sim (\mu$$

and the mean surface area

$$\langle A \rangle = -\frac{\partial \log Z}{\partial \mu}$$

diverges as $\frac{1}{\mu-\mu_c}$. The string susceptibility γ_h is clearly the critical exponent for the specific heat. Diverging surface area is an indication of criticality. Near μ_c one may thus construct a continuum limit with associated critical exponents that are universal in the sense that they do not depend on the fine details of the lattice. The linearity of γ_h in the genus h implies that $Z[G,\mu]$ is actually a function of only one scaling variable

$$x = (\mu - \mu_c)\ \exp\left\{\frac{1}{4G}\left(1 - \sqrt{\frac{1-c}{25-c}}\ \right)\right\}. \tag{19}$$

In the Fall of 1989 it was discovered that the complete partition function $Z = Z(x)$ may be determined by taking the so-called double-scaling limit in which $\mu \to \mu_c$ and $N \to \infty$ with $x = (\mu_c - \mu)N^{2m/2m+1}$ held fixed [12–15]. To reach the double-scaling limit for a fixed m requires fine tuning the parameters of a degree $2m$ polynomial potential in the matrix model. The integer m is called the order of multicriticality. The critical behavior at the m^{th} multicritical point is governed by a universal scaling of the density of eigenvalues of the matrix model at the edge of its support [16]. The order of multicriticality is related to the particular conformal matter being coupled by $p = 2$ and $q = 2m - 1$. The specific heat $f(x) = -\partial^2 \ln Z/\partial\mu^2$ is given in this limit by an ordinary nonlinear differential equation of Painlevé I type. For $m = 2$ (pure gravity), for example, it is

$$f^2(x) + \frac{1}{3}f''(x) = x. \tag{20}$$

The string susceptibility determining the behavior of f around the critical point $f \sim (\mu_c - \mu)^{-\gamma_h}$, is given by

$$\gamma_h = -\frac{2}{p+q-1} = -\frac{1}{m}. \tag{21}$$

More general (p,q) models are described by introducing multi-matrix models.

Note that the original matrix integral for pure gravity is unbounded from below at the critical point $g_c = e^{-\mu_c}$ since it corresponds to a cubic potential. There seems to be no escape from this pathology. Pure gravity is still not non-perturbatively well-defined by the matrix model. Models with matter

corresponding to m odd are well-defined, how
able lesson. It may be necessary to add certain
to render the model non-perturbatively sensib
defined model may be obtained by introducing
in the one-dimensional string [17]. The target
anticommuting $(\theta, \bar{\theta})$ dimension. The total ce
Grassmannian dimension cancelling the $d = 1$ c

Suppose now that we wish to describe mo
sponding to surfaces embedded in a target sp
than one. The surface is given by $x^\mu(\xi_1, \xi_2)(\mu$
viously have $c > 1$. An immediate problem i
According to the continuum results the string
$1 < c < 25$. This suggests that the model has a
understanding of this instability is gained by ex
the discrete versions of these models with the
either by the Nambu-Goto action

$$S_{NG} = \int d^2\xi \sqrt{h}$$

where h is the determinant of the induced metr
simply the area of the surface in the induced m

$$S_P = \int d^2\xi \sqrt{g} \nabla x^\mu \nabla$$

Analytical and computational investigations cl
uum limit of these models is dominated by su
branched tree of tubes of diameter of order
called branched polymer configurations and are
dimensional.

The origin of these spikes is clear in the Na
infinitesimally thin long tube has vanishing area
by the area action. The large entropy for such
nates the statistical mechanics of these surfaces
shown to be in the same universality class.

A bending rigidity may be added to the action to suppress branched polymer configurations [18–20]. Consider the extrinsic curvature matrix (Gauss' second fundamental form) K_{ij} given by

$$K^{\mu}_{ij} = D_i D_j x^{\mu}, \tag{24}$$

where D_i is the covariant derivative along the surface. This is the only additional term relevant under rescaling $x \rightarrow \lambda x$ that may be added to the string action and so will eventually be generated by radiative corrections in any case. In three dimensions the trace of K is the mean curvature $H = 1/r_1 + 1/r_2$, where r_i are the principal radii of curvature of the surface. The extrinsic curvature action is

$$S_{EC} = \kappa \int d^2\xi \sqrt{g} (\mathrm{Tr} K)^2. \tag{25}$$

Its discrete form may be written as

$$S_{EC} = \kappa \sum_{<ij>} (1 - \hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j) \quad , \tag{26}$$

where i and j represent triangles that share a common edge and $\hat{\mathbf{n}}_i$ is the unit normal to triangle i. S_{EC} clearly suppresses local fluctuations in the mean curvature of the surface. But the key question is whether there is long-range order in the normals to the surface. The bending rigidity is, in fact, a running coupling — it depends on the scale at which it is measured. A perturbative calculation in the inverse coupling κ^{-1} reveals that strings with bending rigidity are asymptotically free in the same sense as Quantum Chromodynamics. Fluctuations screen the theory and soften the effective bending rigidity as the length scale increases. The momentum p dependence of κ is found to be [18]

$$\kappa^{-1}(p) = \frac{\kappa_0^{-1}}{1 - \frac{d}{2}\frac{\kappa_0^{-1}}{2\pi}\log\frac{\Lambda}{p}} \quad , \tag{27}$$

where Λ is the cutoff or inverse lattice spaci

of the target space. At large length scales κ

suppression of fluctuations in the alignment c

two-point function decays exponentially

$$< \hat{\mathbf{n}}(\xi_1, \xi_2) \cdot \hat{\mathbf{n}}(0) >=$$

with persistence length ξ_p. Thus the surface is at length scales r exceeding ξ_p. This conclusi the study of liquid membranes as well. A typica found in nature (which can also be manufactu bilayer. It consists of two layers of amphiphilic heads and long hydrophobic hydrocarbon tails selves in thin extended sheets. Within the bilay free to diffuse, so that the in-plane elastic con Another candidate liquid membrane is a mono an oil-water interface in a microemulsion. In fa three-dimensional phases is a candidate syste We see here a beautiful interplay between stri the statistical mechanics of fluctuating liquid membranes.

In the last few years such systems have been extensively explored via numerical simulations on a wide range of computers, including parallel machines [21–24]. There are some novel but not fully understood results. The full action which is simulated is given by a quadratic interaction term plus the extrinsic curvature term

$$S = \sum_{<i,j>} (x_i^\mu - x_j^\mu)^2 + \kappa \sum_{[i,j]} (1 - \hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j) \tag{29}$$

where the first sum is over nearest neighbors and the second over adjacent triangles. For $\kappa < \kappa_c \simeq 1.5$ one sees the expected crumpled surface (see fig. 7). The radius of gyration of these surfaces grows only logarithmically with their area corresponding to infinite Haussdorf dimension d_H defined by

$$R_G^2 \simeq A^{\frac{2}{d_H}} \quad , \tag{30}$$

where R_G is the radius of gyration. For $\kappa > \kappa_c$ the surfaces become extended and considerably smoother with d_H approaching two, which would be the value one would get for a flat surface (see fig. 8). The nature of the cross-over at κ_c is still uncertain. It may be that the system is undergoing a true thermodynamic

phase transition. If it is of second order then t
at the critical coupling would be an interesting
a real extended $2d$ surface rather than a branch
dimensional character. In this case it must be
vary with scale (there is a fixed point of the be
coupling κ_c. At this point there is said to be a
most exciting possibility from the string point
that we have successfully regularized and defi
string with more than one embedding dimensio
The challenge would then be to understand the
string theory at the crumpling transition and t

It may also be that the observed cross-ove
and that the persistence length is simply reach
that is simulated on the computer. In this case
is always crumpled on sufficiently large distan
bility for a liquid membrane but would still le
regularization of a string in $d > 1$ dimensions
large-scale simulations in three and four embed
of the above possibilities is in fact correct [25].

Finally it is of great interest to extend th
angulated surfaces to manifolds of higher di
and four dimensional manifolds. One can the
Einstein-Hilbert quantum gravity and seek criti
perturbative definition of a perturbatively no
theory. This would be a very exciting develop
seems to indicate that there are indeed phase t

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