The Phase Diagram of Fluid Random Surfaces with Extrinsic Curvature

Mark BOWICK, Paul CODDINGTON, Leping HAN,
Geoffrey HARRIS and Enzo MARINARI$^{(*)}$

Dept. of Physics and NPAC,
Syracuse University,
Syracuse, NY 13244, USA

bowick@suhep.phy.syr.edu paulc@sccs.syr.edu han@sccs.syr.edu

gharris@sccs.syr.edu marinari@roma2.infn.it

$(*)$: and Dipartimento di Fisica and INFN,
Università di Roma *Tor Vergata*
Viale della Ricerca Scientifica, 00173 Roma, Italy

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Abstract

We present the results of a large-scale simulation of Dynamically Triangulated Random Surfaces with extrinsic curvature embedded in three-dimensional flat space. We measure a variety of local observables and use a finite size scaling analysis to characterize as much as possible the regime of crossover from crumpled to smooth surfaces.

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1 Introduction

In this paper we use Monte Carlo simulations to investigate a theory of bosonic strings embedded in three target space dimensions with the addition of an extrinsic curvature term to the action. We present a complete high-statistics analysis of the behaviour of a set of relevant observables. Since computing correlation functions on dynamically triangulated surfaces is a difficult task, we have focused on elucidating the phase diagram by analyzing local observables in great detail.

String theory, in a number of guises, has been conjectured to describe the underlying fundamental physics of a wide variety of physical phenomena and models. These include the strong interaction at long distances, the three-dimensional Ising model and unified models incorporating gravity. In its simplest form, the bosonic string, it is a theory of free fluctuating surfaces. The functional integral for the Euclideanized bosonic string is just the partition function for an ensemble of random fluctuating fluid surfaces. Such surfaces are also ubiquitous in nature, being found for example in macro-emulsions and the lipid bilayers that form an important part of cell membranes [1]. These systems are fluid because their component 'molecules' are loosely bound. Their constituents are arranged so that the net surface tension (nearly) vanishes; thus these membranes are subject to large thermal fluctuations. In one important respect, however, these chemical/biological membranes differ fundamentally from the surfaces we discuss and simulate; they are self-avoiding. The worldsheets of the bosonic string, in contrast, generically self-intersect.

The bosonic string for surfaces embedded in 26 dimensional space has been studied extensively. Much progress in numerically simulating strings has been made through the use of Dynamically Triangulated Random Surfaces (DTRS)[4, 5]. For theories of surfaces embedded in $D \leq 1$, analysis of the continuum Liouville theory [2, 3] and of matrix models has led to consistent predictions for critical exponents and correlation functions. In the 'double scaling limit', in fact, exact nonperturbative solutions [6] have been found from these matrix models; the functional integral over surfaces has been essentially summed over all genera.

These analytical techniques have failed, for the most part, in probing the theory of random surfaces in the more interesting embedding regime $D > 1$. Indeed the formulas for critical exponents computed in Liouville theory give

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complex results when continued to the range $1 < D < 25$.[1] The matrix models describing $D > 1$ strings have so far been too complicated to solve.

Monte Carlo simulations for $D > 1$ [8, 9, 10, 11] indicate that these theories do not appear to describe the fluctuations of two dimensional *smooth* surfaces in the continuum limit. Extremely spiky, branched-polymer-like configurations with high (perhaps infinite) Hausdorff dimension dominate the functional integral[2].

Evidence for this pathology was obtained, for instance, when it was shown that the bare string tension, essentially the amount of work per unit area needed to perturb a boundary loop on these configurations, cannot vanish at the critical point [12]. (This implies that the renormalized string tension diverges in the continuum limit.) It has been speculated in ref. [13] that the proliferation of vortex configurations of the internal geometry (the 'Liouville mode') induces the degeneration of these surfaces in the embedding space. A related explanation that is often proffered is that a negative mass-squared particle[3], which comes on shell in the string theory for $D > 1$, creates instabilities which are made manifest by these singular configurations.

The tachyon, and apparently these related instabilities, can be eliminated in particular cases by introducing fermionic coordinates and supersymmetry on the worldsheet and implementing an appropriate projection of states. Presumably, the fermions effectively smooth out the surfaces. This would be consistent with what has been observed for one-dimensional geometries; the random walk of a spin one-half particle has Hausdorff dimension one and thus appears to be smooth [14]. Many authors have proposed an alternative modification of the string action [15, 16, 17, 18] via the addition of a term that directly suppresses extrinsic curvature[4]. We shall examine this class of theories in this paper.

To write down our action we introduce an explicit parametrization of a

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[1]More precisely, one encounters these instabilities in Liouville theory when the quantity $c - 24\Delta > 1$, where c denotes the central charge of the matter theory which describes the embedding of the surfaces and Δ is the conformal weight of the lowest weight state in this theory [7]. Since here we are considering flat space, $c = D$ and $\Delta = 0$.

[2]In the same way singular configurations dominate the Gaussian theory, which is essentially a theory of free random walks.

[3]In some contexts this is referred to as the 'tachyon'.

[4]In fact, integrating the fermions out of the Green-Schwarz superstring yields an action similar to the one we consider, but with the addition of a complex Wess-Zumino type term [19].

generic surface $\mathcal{M}$ in R^3 with coordinates (σ_1, σ_2) and the embedding $X^\mu(\sigma_i)$. μ runs from 1 to 3 (since we only study the case of a $3d$ embedding space). The induced metric (the pullback of the Euclidean R^3 metric via the embedding) is given by

$$h_{ij} = \partial_{\sigma_i} X^\mu \partial_{\sigma_j} X_\mu \ . \tag{1}$$

We will use Greek letters for the embedding space indices; they can be raised and lowered at will since our background space is flat. Associated with each point in $\mathcal{M}$ are tangent vectors $(t_i^\mu \in T\mathcal{M})$ and a normal vector $n^\mu \in T\mathcal{M}^\perp$. The extrinsic curvature matrix K_{ij} (the second fundamental form) can be defined by

$$\partial_i n^\mu = -K_{ij} t^{\mu j} \ . \tag{2}$$

The eigenvalues of this matrix are the inverses of the radii of curvature of $\mathcal{M}$. One usually describes the geometry of these surfaces in terms of the mean curvature [20, 21]

$$H = \frac{1}{2} h^{ij} K_{ij} \tag{3}$$

and the Gaussian curvature

$$K = \epsilon^{ik} \epsilon^{jl} K_{ij} K_{kl} \ . \tag{4}$$

One can show that the Gaussian curvature can be computed solely from the metric h_{ij}, while the mean curvature depends explicitly on the embedding X^μ.

We shall be concerned primarily with the Polyakov form of the string action [22], in which an additional *intrinsic metric* g_{ij} is introduced to describe the surface geometry. We discretize our model by triangulating surfaces. In this construction, each triangle is equilateral with area 1 in the intrinsic metric; the coordination number at each vertex determines the intrinsic curvature of the surface. The coordinates i label the vertices of the triangulation. Then the discrete analogue of the intrinsic metric is the adjacency matrix C_{ij} whose elements equal 1 if i and j label neighbouring nodes of the triangulation and vanish otherwise. Two-dimensional diffeomorphism invariance reduces to the permutation symmetry of the adjacency matrix at this discrete level. One of the keys, in fact, to the power of this construction is the

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preservation of this symmetry. Each vertex of the triangulation is embedded in R^3 via the mapping X_i^μ. Given the embedding X, we can also associate a unit normal vector $(n^\mu)_{\hat{k}}$ with each triangle on the surface (Roman indices with hats label the triangles). Note that all of the surface curvature of our triangulations is concentrated along the links and vertices. The surface is still flat in the direction tangent (but not transverse) to each link, so that the mean curvature has support on the links, while the Gaussian curvature is non-zero only at the vertices. The intrinsic curvature R_i at vertex i is given by the deficit angle determined solely by the triangulation

$$R_i = \pi \frac{(6 - q_i)}{q_i}, \qquad (5)$$

where q_i denotes the connectivity of the lattice at vertex i. The Gaussian curvature K on the other hand is expressed in terms of the deficit angle in the embedding space.

We shall study the theory defined by the action

$$S = S_{Gauss} + \lambda S_E = \sum_{i,j,\mu} C_{ij}(X_i^\mu - X_j^\mu)^2 + \lambda \sum_{\hat{k},\hat{l},\mu} C^{\hat{k}\hat{l}}(1 - n_{\hat{k}}^\mu \cdot n_{\hat{l}}^\mu) . \qquad (6)$$

Thus, for $\lambda > 0$, we have introduced a ferromagnetic interaction in the surface normals. The model defined by this action has been studied in [23, 24, 25, 26, 27, 28] and references therein.

From (2) and the definition of the induced metric, it follows that this is a discretization of the continuum action

$$S = \int \sqrt{|\det g|}(g^{ij}\partial_i X^\mu \partial_j X^\mu + \frac{\lambda}{2}g^{ij}h^{kl}K_{ik}K_{jl}) . \qquad (7)$$

Note that the second term in this action is manifestly positive, Weyl and reparametrization invariant and that λ is a dimensionless coupling. So, naively, it is not clear whether it is relevant or not. If it were relevant, one would then anticipate that (since it obeys all of the appropriate symmetries) it should be effectively generated in any string action, and that it should engender ordering of the normals. It should then lead to another RG fixed point at a finite value of λ, which would characterize a phase transition between the crumpled phase (observed when $\lambda = 0$) and a 'smooth(er)'

phase.[5] We proceed first to review previous work which has addressed the question of whether or not these theories exhibit a crumpling transition.

1.1 Previous Analytical Work

A renormalization group analysis [16, 31, 32, 17] indicates, however, that there should be no phase transition at finite coupling when such extrinsic curvature dependent operators are added to the action. The computations of refs. [16, 31, 32] use the action

$$S = \int d^2\sigma \left(\mu_0 \sqrt{\det h} + \frac{1}{\alpha} \sqrt{\det h} (h^{ij} K_{ij})^2 \right) , \qquad (8)$$

in the regime in which the string tension μ_0 is small (unlike the usual particle physics limit of string theory, which is characterized by large μ_0). After integrating out fluctuations of the embedding X^μ between momentum scales Λ and $\tilde{\Lambda}$, it is found that the renormalization of the extrinsic curvature coupling is given to one-loop order by

$$\beta(\alpha) \equiv \Lambda \frac{d\alpha}{d\Lambda} = -\frac{3}{4\pi}\alpha^2 , \qquad (9)$$

so that α is driven to infinity in the infra-red. This theory thus exhibits asymptotic freedom. Surfaces are smooth (the normals are correlated) below a persistence length [33]

$$\xi_p \sim \exp(\frac{4\pi}{3\alpha_{bare}}) \qquad (10)$$

and are disordered above this scale. Some intuition into this result can be gained by observing that this theory is similar to the $O(3)$ sigma model,

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[5]The extrinsic curvature term is also higher-derivative, indicating that the field theory described by this action is non-unitary. This fact alone does not imply that the associated string-scattering amplitudes do not satisfy unitarity. Polchinski and Yang [29] do, however, contend that in this case the string theory will not be unitary. Even if this were so, this model could still be an appropriate description of the statistical mechanics of fluctuating surfaces, although not one corresponding to a physical fundamental string theory. Braaten and Zachos [30] have also showed that the generic static classical solutions of a similar higher-derivative theory of rigid strings are unstable. This would also imply that these actions could only be the basis for effective, but not fundamental, quantum theories of strings.

which is asymptotically free [14]. The normals to M are the analogues of $O(3)$ vectors, though in this case they are constrained to be normal to a surface governed by the action (7).

Without the extrinsic curvature term, (8) is the Nambu-Goto action, while (7), which we use in our simulations, is based on the action quantized by Polyakov. Classically (when the equations of motion for the Polyakov action are solved and substituted back into the action), the two actions are equivalent. It has also been demonstrated [34] that the two quantizations are equivalent in the critical dimension $D = 26$. In lower dimensions (note that the Nambu-Goto action clearly does not make sense for $D < 2$), it is not so clear that quantizations 'based' on the two actions are indeed the same. The work of Polchinski and Strominger [35] suggests that there are alternative quantizations. Distler ([36]) has also questioned the equivalence of these quantizations in $D = 3$. Indeed, even if the two quantizations are equivalent, it does not automatically follow that the two theories are still the same once an extrinsic curvature dependent term has been added.

In fact, Polyakov in [17] uses a hybrid form of the action (8) and still obtains the same result for the beta function. He introduces an intrinsic metric g_{ij}, chooses the conformal gauge $g_{ij} = \rho \delta_{ij}$ and considers

$$S = \frac{1}{2\alpha} \int d^2\sigma (\mu_o \rho + \rho^{-1} (\partial^2 X^\mu)(\partial^2 X^\mu) + \lambda^{ij} (\partial_i X^\mu \partial_j X^\mu - \rho \delta_{ij})) . \quad (11)$$

Classically, the Lagrange multiplier λ^{ij} constrains the intrinsic metric to equal the induced metric (this equality is not enforced by the classical equations of motion for the original Polyakov action). This constraint should be relaxed quantum mechanically if, as Polyakov [14] argues, the condensate of this Lagrange multiplier assumes a value of the order of the momentum cut-off. If this dynamical assumption is correct, then one can essentially derive the equivalence of this Nambu-Goto like and the original Polyakov quantizations. In the large D (embedding dimension) limit, saddle point calculations [37] show that λ indeed does acquire a large expectation value, and that for small values of the string tension μ_o, the coupling α is asymptotically free, as the RG calculations suggest.

There are, however, a couple of caveats and suggestions in the analytic literature that do allow for the existence of a crumpling transition for fluid surfaces. Polyakov remarks that if, in the infrared region, fluctuations of the

internal geometry (ρ) are suppressed relative to fluctuations of the extrinsic metric, then the beta function is proportional to α and hence the continuum limit of the theory exhibits non-trivial scaling behaviour; this presumably cannot be the case in the large D limit. Another RG calculation, performed by Yang [38] using the Polchinski-Strominger action [35] with an extrinsic curvature dependent term, indicates that the two-loop correction (which is proportional to α^3) might be large enough to yield a zero of the beta function and thus a non-trivial IR fixed point. The Polchinski-Strominger action is based on the assumption that the Liouville mode ρ effectively decouples (its mass is much greater than the momentum scale set by the string tension); it is not clear why this assumption should hold for the model that we simulate. Finally, note that these computations are perturbative (in $1/D$ or α). It is possible that non-perturbative effects could drive a crumpling transition.

1.2 Previous Numerical Evidence

Monte Carlo simulations of the action (7) on dynamically triangulated random surfaces (DTRS) were first performed by Catterall [23] and shortly thereafter by Baillie, Johnston and Williams [24, 25] and Catterall, Kogut and Renken [28]. They simulated triangulations with the topology of the sphere, and measured the specific heat

$$C(\lambda) \equiv \frac{\lambda^2}{N}(< S_E^2 > - < S_E >^2) \tag{12}$$

on surfaces with up to $N = 144$ nodes (and $N = 288$ nodes in the last reference). They found a peak in the specific heat; the peak size appeared to grow with N. A similar model that can be vectorized rather straightforwardly was also considered; the set of planar ϕ^3 graphs was simulated [28, 39]. Each vertex of these ϕ^3 graphs was embedded in R^3 and the action (7) was used; graphs of up to 1000 nodes were simulated (these would be dual to 500 node triangulations). It was found that the specific heat peak grew with N, albeit slowly, as

$$C_{max} = AN^\omega + B , \tag{13}$$

with $\omega = 0.185(50)$. Further work by Ambjørn, Irbäck, Jurkiewicz, Petersson and Varsted [26, 27], using dynamical triangulations with the topology of

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the torus and lattices with up to $N = 576$ nodes, indicated that the rate of increase of the peak height severely diminishes with increasing N. The data strongly suggests that in fact the specific heat peak height does **not** diverge as $N \to \infty$. These authors also measured the bare string tension and mass gap, by embedding the torus in a background toroidal space spanned by a loop, and measuring the dependence of the free energy on the loop size. They found that these measurements (when taken for λ values near the peak position) are consistent with the appropriate scaling relations (with vanishing bare string tension and mass gap) that should characterize a phase transition to smooth surfaces. This measurement, although it constitutes the best evidence there is so far for a real phase transition at $\lambda = \lambda_c$, is still quite an indirect way of measuring correlation functions. As we will discuss, these scaling relations could contradict other observed phenomena such as the absence of diverging correlation times and increasing finite size effects at the putative critical point.

Thus it appears that numerical evidence could allow for the existence of a crumpling transition (most probably of higher order), while analytical calculations generally indicate that no such transition should occur.

In [40] the peak was measured in a DTRS simulation that incorporated self-avoidance and the extrinsic curvature term S_E, with a solid-wall potential substituted for the Gaussian term in the action. The results for the specific heat turned out to be very similar to those found in the simulations we have just discussed, for example, in [27]. The specific heat peak is, in this context, considered to be a lattice artifact, because the peak height levels off with large N (of order 500). These simulations included a crude block-spin measurement that suggests that the renormalization group flow of λ is consistent with the analytical result of asymptotic freedom.

Simulations using other discretizations for the extrinsic curvature dependent term have yielded somewhat different results [23, 24]. The specific heat peak, measured in simulations employing what is referred to as the 'area discretization', is rather feeble and levels off for small values of N (by $N = 72$) (the authors interpret this as being indicative of perhaps a 'third' order transition). Actions based on these various discretizations have been simulated for fixed, triangular meshes. These systems model tethered or crystalline membranes, in which the constituent molecules are tightly bound together. In the tethered case, the specific heat peak obtained from simulations of the edge action (7) grows vigorously as a function of N for very large (128×128)

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lattices [41]. This is strong evidence for the existence of a second order transition which, in this case, is in accord with the analytic results – these calculations are reviewed by Nelson [42] and David [1, 43] and involve mean field and large D computations which suggest that the β function is linear at leading order, with a zero for finite α, i.e. a UV fixed point. When the alternate area discretization is used in the tethered case, the specific heat peak again stops growing. Recent work has demonstrated that this other discretization is pathological in the tethered case; the class of 'corrugated' surfaces, which are singular in one direction and smooth in the other, then dominates the path integral [41].

Thus, given the muddle of somewhat contradictory evidence, it is unclear whether or not a crumpling transition exists for fluid surfaces. We have pursued this question by taking high statistics measurements of the specific heat peak and by measuring other observables describing the geometry of these surfaces, since observables with different quantum numbers can give quite different information. For example, in the Ising model the magnetization behaves quite unlike the internal energy (which is invariant under the standard Z_2 transformation).

To analyze and interpret this data, we have applied insights gained from work on better understood systems, primarily spin models and lattice gauge theories. Issues of the equivalence of the Nambu-Goto and Polyakov quantizations have also motivated us to compare the intrinsic and induced geometry of the surfaces that we simulate.

1.3   The Plan of the Paper

We hope that in this section we have introduced the problem in sufficient detail. In Section 2 we define the quantities we have decided to measure, and explain why they are physically interesting. Next, in Section 3, we present the details of our numerical simulations. In Section 4 we describe and discuss our results concerning the phase diagram of the theory and we devote Section 5 to the discussion of correlation times. In Section 6 we propose various interpretations of these results and in Section 7 we comment on future possible developments.

2 Observables

To minimize finite size effects, we have considered triangulations with the topology of the torus. The action (7) was used, with the BRST invariant measure utilized also by Baillie, Johnston, and Williams ([24]), so that

$$Z = \sum_{G \in T(1)} \int \prod_{\mu,i} dX_i^\mu \prod_i q_i^{\frac{d}{2}} \exp(-S_{Gauss} - \lambda S_E) \ , \qquad (14)$$

where $d = 3$, q_i is the connectivity of the ith vertex and $T(1)$ refers to the set of triangulations of genus 1. The authors of [26, 27] do not include this connectivity dependent term in their measure. The long-distance physics of the simulations is presumably insensitive to the presence of this term. Because we have chosen a different measure, though, our quantitative results cannot be precisely compared with theirs.

We measured a variety of quantities that characterize the extrinsic and intrinsic geometry of these surfaces. These observables include:

1. The edge curvature S_E and the associated specific heat $C(\lambda)$, which is a sensitive indicator of the presence of a phase transition.

2. The squared radius of gyration R_G;

$$R_G \equiv \frac{1}{N} \sum_{i,\mu} (X_i^\mu - X_{com}^\mu)^2 \ , \qquad (15)$$

   where the com subscript refers to the center of mass of the surface. By measuring the N dependence of the gyration radius, we can extract a value for the extrinsic Hausdorff dimension, which is given by

$$R_G \sim N^\nu \sim N^{\frac{2}{d_{extr}}} \quad . \qquad (16)$$

3. The magnitude of the extrinsic Gaussian curvature. We measure a discretization of $\int |K| \sqrt{|h|}$, with

$$|\mathcal{K}| = \frac{1}{N} \sum_i |2\pi - \sum_{\hat{j}} \phi_i^{\hat{j}}| \quad . \qquad (17)$$

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Here $\phi_i^{\hat{j}}$ denotes the angle subtended by the $\hat{j}$th triangle at the ith vertex. This quantity, therefore, measures the magnitude of the deficit angle in the embedding space averaged over all vertices. We also record the mean square fluctuation of $\mid \mathcal{K} \mid$, denoted by $F[\mid \mathcal{K} \mid]$.

4. The corresponding intrinsic quantity $\mid \mathcal{R} \mid$, given by

$$\mid \mathcal{R} \mid = \frac{\pi}{3N} \sum_i \mid 6 - q_i \mid , \tag{18}$$

and its fluctuations. When the intrinsic and extrinsic metrics are equal, the intrinsic and extrinsic deficit angles are identical and $K = R/2$.

5. To study the correlation between intrinsic and extrinsic geometry, we also measure the quantity which we refer to as $\mathcal{K} * \mathcal{R}$:

$$\mathcal{K} * \mathcal{R} \equiv \frac{\int K R}{\sqrt{\int K^2 \int R^2}} = \frac{\sum_i (2\pi - \sum_{\hat{j}} \phi_i^{\hat{j}})(6 - q_i)}{\sqrt{\sum_i (2\pi - \sum_{\hat{j}} \phi_i^{\hat{j}})^2 \sum_i (6 - q_i)^2}} . \tag{19}$$

This quantity is 1 when the metrics are equal, 0 if they are un- correlated, and negative when these curvatures are anti-correlated.

6. We measure, finally, the average maximum coordination number of the surface vertices, $\max_i q_i$.

3 The Numerical Simulation

In our simulations we have used the standard Metropolis algorithm to update the embedding fields X_i^{μ}. To sweep through the space of triangulations we performed flips (see reference [10]) on randomly chosen links. Flips were automatically rejected if they yielded a degenerate triangulation; i.e. one in which a particular vertex has fewer than three neighbors, or in which a vertex is labeled as its own neighbor, or where more than one link connects two vertices. (It has been proven in ref. [10, 11] that the entire space of graphs of a given topology can be spanned by only performing these flips.) After a set of $3N$ flips was performed, $3N$ randomly selected embedding coordinates were updated via random shifts from a flat distribution

N=36			1.375	1.400	1.425	1.475		λ
			3	3	3	3		$\times 10^6$ sweeps
N = 72			1.375	1.400	1.425	1.475		λ
			3	3	3	3		$\times 10^6$ sweeps
N=144	.8	1.25	1.35	1.40	1.45	1.50	2.0	λ
	3	3	3	3	3	3	3	$\times 10^6$ sweeps
N=288	.8		1.375	1.40	1.425	1.475	2.0	λ
	14.4		21.0	15.0	16.2	13.5	14.4	$\times 10^6$ sweeps
N= 576	.8	1.325	1.375	1.40	1.425	1.475	2.0	λ
	12.0	27.0	27.0	27.0	27.0	27.0	9.6	$\times 10^6$ sweeps

Table 1: A record of the number of sweeps performed at each different λ value for different lattice sizes.

$$X^\mu \to X^\mu + \delta X^\mu \ . \tag{20}$$

The mean magnitude of these shifts

$$< \delta X^\mu \delta X^\mu > \tag{21}$$

was chosen so that the acceptance rate for updates of the X^μ was roughly 50 percent. Most of the Monte Carlo simulations were performed on HP-9000 (720 and 750 series) workstations; we also collected some data by simulating lattices on each of the 32 nodes of a CM-5. Our code was in Fortran, with a Fibonacci random number generator.

In Table (1) we summarize our runs. Note that we have performed quite long runs on the larger lattice sizes. We will discuss in Section (5) why we believe runs of this length are just sufficient to yield accurate values of the observables for the largest lattice size ($N = 576$).

In all of our figures the different dots will be printed with their associated statistical error (sometimes too small to be visible). The statistical error is computed by means of a standard binning procedure. We will explicitly discuss the cases in which our estimator for the statistical error is not asymptotic.

The lines in these figures are from a histogram reconstruction (see for example [44, 45]). We patch different histograms [46, 47, 48] by weighting them with the associated statistical indetermination (which we estimate by

a jack-knife binned procedure); this procedure seems to be very effective and reliable. All of the reconstruction curve sets (3: dotted, dashed and continuous for 3 surface sizes on each figure) consist of 3 curves (which sometimes appear as a single one). The middle curve is the histogram reconstruction and the upper and the lower ones bound the data within the errors obtained by the procedure we have just described.

For $N = 144$ we have patched the four histograms originating from $\lambda = 1.35$, 1.40, 1.45, 1.50. For $N = 288$ we have used $\lambda = 1.375$, 1.40, 1.425 and 1.475. For $N = 576$, we chose $\lambda = 1.375$, 1.40 and 1.425.

We have only drawn the reconstructed, patched curves (with their reliable errors) in the regions where we trust them. For example, close to the pseudo-critical region we can trust a peak pattern only when we can reconstruct the peak by starting from both sides of the transition (without multi-histogram patching). So we have always used single histogram reconstructions to check these criteria, before constructing the final, multi-histogram data.

4   The Phase Diagram

We have measured, as stated previously, a large number of local observables. We will see that a mixed picture emerges from these measurements. For example the observables related to the dynamical triangulations exhibit a characteristic pattern, to be discussed in detail below.

We start by showing, in Fig.1, the edge curvature S_E as a function of λ. The crossover region is around $\lambda \simeq 1.4$. For small values of λ, the surface is crumpled (see the latter part of this section). In this region, finite size effects are already negligible for our lattice sizes, and our 3 data points are on top of each other. We can see weak finite size effects by comparing the continuous lines in the transient region. The $N = 144$ dotted line is far from the ones of the two larger lattices, which lie, on the contrary, on top of each other. Finite size effects are larger in the large λ phase. One would expect, close to a phase transition with a diverging correlation length, an increase in finite size effects which we do not observe here. The lattice should feel the presence of the zero mass excitation, and the finite size corrections should be larger than everywhere else (in the case of periodic boundary conditions they would obey a power-law, rather than decaying exponentially with size). This is surely not firm evidence against the presence of a phase transition,

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N	$C(\lambda)^{\mathrm{max}}$	λ_c
36	3.484(8)	1.425(35)
72	4.571(15)	1.410(15)
144	5.37(14)	1.395(30)
288	5.55(7)	1.410(25)
576	5.81(17)	1.425(30)

Table 2: The maximum of the specific heat and its position, with errors, for different lattice sizes.

but it does show that the putative critical behaviour is atypical.

The errors in the 'flat phase' ($\lambda = 2.0$) are not under control. Our estimators do not plateau under repeated iterations of the binning procedure. In this regime, correlation times are large, as we will discuss in next section. This *caveat* holds for this figure and for all the quantities we have measured.

In Fig. 2 we show the related specific heat $C(\lambda)$ in the same λ region. In Fig. 3 we enlarge the pseudo-critical λ region, in order to show the reconstructed peak of the specific heat. As already noted our reconstruction procedure is quite reliable here.

In Table 2 we give the maximum of the edge curvature specific heat and its location for different lattice sizes.

We see that the specific heat peak grows vigorously with N for small lattices, but that this growth levels off for larger N. From the data for the three largest lattices, we can extract a specific heat exponent $\omega = .06 \pm .05$, with ω defined as in equation (13), and the constant B set to zero. If we estimate an effective exponent from the $N = 144$ and 288 lattices, we get $.05 \pm .06$, and from the two largest lattice sizes we get $.07 \pm .06$; this demonstrates that we do not see, within our statistical precision, any sign of a non-pure-power, non-asymptotic behavior. Note that, if the constraint that B vanishes is relaxed, our data is not accurate enough to yield a meaningful fit to equation (13). A very small (asymptotically finite) correlation length is sufficient to produce such a small effect on our quite small lattice sizes. These results appear to be consistent with those of the Copenhagen group [27], and they are not so far from the ones of the Urbana group [28, 39].

The critical value of λ shifts very slowly to higher values for increasing N for the largest lattices, although the increase is not statistically significant.

In addition the shape of the specific heat (for example the width) is

basically unchanged as we go to larger lattices. From Figs. 2 and 3 we do not infer evidence of criticality.

In Fig. 4a we show the radius of gyration of the surface, R_G, as defined in (15). Here obviously the volume scaling is non-trivial: larger surfaces have larger radius. The histogram reconstruction already ceases to work for quite low values of λ for the larger lattice. This effect could be related to the interesting finite size scaling behavior of this quantity, which we illustrate in better detail in Fig. 4b. Here we plot

$$\nu(N) \equiv \frac{\log \frac{R(N)}{R(\frac{N}{2})}}{\log(2)} \ . \tag{22}$$

This is an effective inverse Hausdorff dimension, which is a function of λ. In the large λ limit $\nu \to 1$ and $d_{extr} \to 2$, as expected for flat surfaces. In the low λ limit d_{extr} becomes very large. In the pseudo-critical region ν is a linear function of λ. Curiously enough, the latter curve yields a Hausdorff dimension of 4, a value characteristic of branched polymers, near the location of the specific heat peak. This value is not particularly reliable though because of finite-size effects and also because it changes rapidly in this region. In ref. [27] a value compatible with ours $(D_H(\lambda_c) > 3.4)$ is quoted for the critical theory. We stress however (and also here we are in complete agreement with [27]) that the Hausdorff dimension in the pseudo-critical region depends heavily and quite unusually on N.

In both the high and low λ regions finite size effects are quite small (compatible with zero to one standard deviation). In the pseudo-critical region, on the contrary, finite size effects are large. This effect cannot be explained by the shift in λ which one gets from the shift of the peak of the specific heat, which is far too small. This behaviour is very different from that which we discussed for S_E; it seems to indicate the possibility of some sort of critical behavior close to $\lambda = 1.4$.

In Fig. 5 we plot the expectation values of the magnitude of the extrinsic Gaussian curvature $\mid \mathcal{K} \mid$. If the induced metric is equal to the intrinsic metric, then $\mid \mathcal{K} \mid = \frac{|\mathcal{R}|}{2}$.

This plot is not substantially different from that of S_E. We note that finite size effects are somewhat larger in this case than for the edge action, but they follow the same pattern (exhibiting a big increase in the flat phase).

The plot of the fluctuations of the extrinsic Gaussian curvature, $F[\mathcal{K}]$,

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which we present in Fig. 6, shows something very new. A very sharp crossover, with perhaps a peak developing for large N, dominates the pseudo-critical behavior. Fluctuations do not seem to depend on λ in the crumpled phase, while they drop dramatically, in a very small λ interval, in the flat region. Here again, finite size effects are sizeable in the pseudo-critical region. The position of the crossover does not depend sensitively on N, while the detailed shape at λ_c seems to change slightly with N.

It is difficult to give a precise interpretation of a plot like this, but, as we said, the crossover is very clear here.

In Fig. 7 we give the intrinsic curvature $\mathcal{R}$ and in Fig. 8 its fluctuations. Both plots are very similar to the related, extrinsic curvature, $\mathcal{K}$ plots. $|\mathcal{R}|$ drops off rapidly, just as $|\mathcal{K}|$ does. Through the peak region, though, $|\mathcal{K}|$ decreases by about a factor of 5 while $|\mathcal{R}|$ diminishes to only about .6 of its value on the left-hand side of the peak. Since the action explicitly suppresses mean curvature, and the mean and extrinsic Gaussian curvature are closely related (for instance, $H^2 > \frac{K}{2}$), we would expect that for large λ extrinsic fluctuations would be suppressed much more than fluctuations of intrinsic geometry.

In Fig. 9 we plot the intrinsic-extrinsic curvature correlation. The plot of $\mathcal{K} * \mathcal{R}$ indicates that intrinsic and extrinsic geometry are strongly correlated for small λ, but as one passes through the peak region they become decorrelated. This is not particularly surprising, given that the action directly suppresses only extrinsic fluctuations. Note that RG calculations based on the Nambu-Goto action plus an extrinsic curvature term (with no dependence on an intrinsic metric) perturb about a background that is both intrinsically and extrinsically flat. Given the observed decorrelation between intrinisic and extrinsic geometry, we would not anticipate that this background appears in the low- temperature limit of the model which we simulate.

In Fig. 10a we plot the expectation value of the maximum coordination number, which has non-trivial scaling behavior. In Fig. 10b we give its scaling exponent, defined analogously to the exponent we have exhibited for the gyration radius. In the pseudo-critical region q_{max} scales (for our 3 lattice sizes) as a power, with an exponent close to 0.1; we do not know if this scaling is meaningful.

5   Correlation Times

We will discuss here correlation times for different observables. As we already pointed out correlation times become very large in the large λ region. In agreement with ref. [27] (see their Fig. 1) we do not see any increase of the correlation times close to the pseudo-critical point.

We will not present precise estimates of correlation times (exponential or integrated) – they are too large to get precise estimates. We will limit ourselves to a discussion of a few figures, which give quite a clear idea of what is happening. The comparison with Fig. 1 of ref. [27] cannot be very direct, since our action is different and because their dynamics may be more effective than ours. Still, the comparison is quite puzzling, since we estimate and exhibit correlation times which are much (orders of magnitude) larger than the ones of [27]. Applying customary methods to estimate τ_{int} can lead to an underestimate of correlation times if more than one time scale is present (that does surely happen with our data if we integrate it on a window of reasonable size).

In Fig. 11a we plot S_E for $N = 144$, $\lambda = 1.4$, and in Fig. 11b the gyration radius for these values (with a different time scale). Clearly, the correlation time is at least of order $40,000$ sweeps in the first case and $100,000$ sweeps in the second one. In Figs. 12a and b, we plot the same quantities for $\lambda = 1.5$. Here correlation times are larger, of order $50,000$ steps for S_E and larger than $150,000$ steps for R_G. In Figs. 13 we draw the same plot on the largest lattice we study ($N = 576$) for $\lambda = 1.4$. Here we can see dramatic correlations, with times of at least $100,000$ steps for S_E and of at least $1,000,000$ steps for R_G.

In Fig. 14 we plot, for the same time history and on the same scale, both S_E and R_G. This figure shows a clear anticorrelation: larger surfaces are flatter and have smaller curvature (this is apparent in the region close to the 2000th step).

6   Interpretation of Results

The results of this paper show that the transition from crumpled to flat surfaces with increasing λ is quite complex. Guided by the present data and the very interesting results of ref. [27] we present in this section our current interpretation of the situation.

On lattices with up to 576 vertices we can clearly see a sharp crossover, but the absence of a diverging specific heat, of diverging correlation times and of strong finite size effects suggests that we are not observing a usual second order phase transition. On the other hand, quantities like the intrinsic curvature, or the mass measurements of [27] show that something non-trivial is happening.

Let us review the crux of our observations again. This model of crumpled surfaces appears to exhibit sharp crossover behavior in the region around $\lambda = 1.4$. The sharp growth in the gyration radius and the suppression of curvature fluctuations indicate that the normals acquire long-range correlations, up to the size of the systems we examine. Presumably the zero string tension measurement of [27] also shows that the disordered regime differs from the regime in which the surfaces are ordered (up to scale of the lattices that are simulated) by only a small shift in λ. This evidence might indicate the presence of a phase transition at this point. If so, it is very likely to be of order higher than 2 (or, rather implausibly, it could be second order with an extremely low negative specific heat exponent; our lattices are much too small for us to extrapolate with confidence the value of the specific heat exponent as $N \to \infty$).

If the transition were higher order, the peak should exhibit a cusp, but we would need far more accurate data to detect this. The existence of this phase transition would then suggest the existence of a new continuum string theory, though many other issues would have to be resolved (e.g. unitarity) to determine if such a theory is physically desirable.

There are other possible interpretations of our data. We need to consider the influence of finite-size effects, since the surfaces which we simulate are quite small, even smaller than one might naively assume because they are not intrinsically smooth. For instance, random surfaces characteristic of $D = 0$ gravity have a Hausdorff dimension of roughly $d_{intr} = 2.8$ [49, 50]; it has been predicted that surfaces embedded in 1 dimension have Hausdorff dimension $2 + \sqrt{2}$ [49]. Thus, for instance, if the surfaces in our simulations had an intrinsic dimension of 3, they would have a linear size of fewer than 9 lattice spacings[6].

Perhaps the simplest alternative explanation for the presence of this peak

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[6]Of course, our lattices are too small, by one or two orders of magnitude, to really exhibit a convincing fractal structure.

is suggested by the arguments of Kroll and Gompper [40]. They argue that the peak occurs when the persistence length of the system approaches the size of the lattice ($\xi_p \sim N^{\frac{1}{d}}$)[7]. For larger λ, fluctuations on a larger scale become more important, but when this scale is greater than the lattice size, these fluctuations are suppressed. Thus one might surmise that the specific heat will drop for large λ. (It clearly goes to zero for small λ because of the presence of the prefactor λ^2; the lattice implements a ultraviolet cutoff that freezes out very short-range fluctuations.)

The one-loop renormalization group calculation (10) predicts that the persistence length grows as $\xi_p \sim \exp(C\lambda)$; C is inversely proportional to the leading coefficient of the beta function. We would expect that the peak position should shift to the right with increasing N in this scenario as

$$\lambda_{peak}(N') - \lambda_{peak}(N) = \frac{\ln(\frac{N'}{N})}{d_{intr}C}. \tag{23}$$

Quite a large value of C is needed to explain the rapid crossover; roughly values of $C \sim 10, d_{intr} \sim 3$ are more or less consistent with the magnitude of the peak shift and crossover width. The RG calculations using different forms of the action yield $C = \frac{4\pi}{3}$ (see equation 10), but this may not apply to the action we simulate.

This reasoning also indicates that the peak should widen as the lattice size increases; we do not observe this at all. It seems plausible though that these arguments, based only on the leading term of the high lambda expansion, are too naive.

An alternative scenario, which builds on the ideas in the above paragraph, is suggested by the tantalizing similarities between the results of our fluid surface simulations and what has been observed for the $d = 4$ $SU(2)$ Lattice Gauge Theory [44] and for the $d = 2$ $O(3)$ model. Let us discuss the case of the $O(3)$ model.

The $O(3)$ model, which is asymptotically free, exhibits a specific heat peak near $\beta = 1.4$. This peak was first measured via Monte Carlo simulations by Colot [52]. It can also be obtained by differentiating the energy data measured by Shenker and Tobochnik [53, 54]. The origin of this peak is understood [54, 55]; it is due to the fluctuations of the sigma particle, a

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[7]For couplings above this point, our simulations would simply be measuring finite size effects.

low-mass bound state of the massless $O(3)$ pions. The sigma induces short-range order, and contributes to the specific heat as a degree of freedom only at high temperatures (when the correlation length in the system becomes smaller than its inverse mass). The peak thus occurs at the beginning of the crossover regime, when the correlation length is several lattice spacings.

According to the low temperature expansion, the correlation length grows as $\xi \sim \exp(2\pi\beta)/\beta$. Thus one would expect a fairly rapid crossover in the $O(3)$ model; the correlation length should increase by roughly a factor of 9 when β is shifted by about $.35^8$. Such a crossover is indeed observed, though it is not so apparent that it is as dramatic as the crossover behaviour observed for fluid surfaces.[9]

Recent simulations of the $O(3)$ model [51] indicate that the specific heat peak grows significantly when the lattice size L is increased from 5 to 15, and that virtually no growth in peak height is evident as L is increased further up to 100. Also, the peak position shifts to the right as L grows, and then appears to stabilize for large L. This is more or less what we observe in our simulations of fluid surfaces, on lattices of small size. We point out these similarities largely to emphasize that there does exist an asymptotically free theory (with low mass excitations) which exhibits crossover behavior qualitatively similar to that observed in our simulations.

The analogy is perhaps deeper, though, since the fluid surface action (with extrinsic curvature) in certain guises looks like a sigma model action. So, perhaps it would not be so surprising from this point of view to find a sigma particle in these theories perhaps associated with $(\hat{n}^2 - 1)$, in which $\hat{n}$ denotes the unit normal to our surfaces.

Another additional possibility is that fluctuations of the intrinsic geometry (the Liouville mode) are responsible for short-range order and contribute to the specific heat peak.

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[8] In fact, the presence of the sigma significantly modifies this low-temperature expansion result [55] in this intermediate regime, but does not qualitatively destroy the rapidity of the crossover. Indeed, despite heroic efforts, it has been impossible to extend computationally beyond this regime and precisely verify the asymptotic low-temperature relation for the correlation length [56, 57].

[9] To quantitatively compare the width of the crossover regimes for these two models it would be necessary to measure a correlation length (perhaps extracted from the normal-normal correlation function) in these random surface simulations.

7 Further Work

There remains much to be done to clarify whether or not a crumpling transition occurs for a finite value of the extrinsic curvature coupling λ. It would be interesting (and probably a fair amount of work) to apply Wilson renormalization group techniques to the actual action (7) which we simulate, to determine the leading coefficient of the beta function. Additionally, perhaps a calculation of $1/D$ corrections to the large D computations already performed could unearth evidence of a sigma-type excitation in these theories (the effects of the sigma appear as $1/N$ corrections in the $O(N)$ model).

We also are histogramming our data to examine the behavior of complex zeroes (in complex λ space) of the partition function of our simulations [58]. It has been shown (in the case of $SU(2)$ lattice gauge theory) that such zeroes, when they are near but do not approach the real axis in the infinite volume limit, occur in theories which exhibit specific heat peaks with no associated phase transition [44]. High temperature expansions also indicate that the $O(3)$ model susceptibility has a complex singularity near the real axis [59]– presumably this corresponds to a zero of the partition function and is a manifestation of the sigma.

Of course, simulations on large lattices, with better statistics, should also help us evaluate whether a crumpling transition exists. We are testing algorithms, such as simulated tempering [60], in order to evade the long auto-correlation times that have characterized our simulations so far.

Even if no such transition exists for finite λ, one could still attempt to study a continuum theory in the strong coupling limit, as is done for QCD, for instance. To do so, we would like to examine global quantities, such as masses extracted from normal-normal correlation functions, rather than just the local quantities (energy, e.g.) that we have measured. Measuring these correlations requires a definition of distance on these triangulated lattices; the most successful definition of the metric is based on the propagation of massive particles (via inversion of the Laplacian) on these lattices [61].

8 Conclusion

We have thus explored the phase diagram of fluid random surfaces with extrinsic curvature, but unfortunately we have been unable to determine if

our model undergoes a phase (crumpling) transition at finite coupling. We have observed dramatic crossover behavior for particular observables in our Monte Carlo simulations, but on the other hand, the correlation times and certain finite-size effects do not behave as one would expect in the presence of a phase transition. The behavior of other lattice models also indicates that it is possible that we are observing the effects of finite-mass excitations on small lattices, rather than a phase transition. We hope that future work will clarify this murky state of affairs, to determine if there indeed exists a crumpling transition for fluid surfaces.

Acknowledgments

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9 Figure Captions

Fig. 1 The edge curvature S_E as a function of λ. As in all other pictures, filled circles and a dotted line correspond to $N = 144$, crosses and a dashed line indicate $N = 288$ and empty squares and a solid line represent $N = 576$.

Fig. 2 The edge curvature specific heat, $C(\lambda)$.

Fig. 3 As in Fig. 2, but with the multi-histogram reconstruction in the transient region.

Fig. 4a The gyration radius R_G defined in (15), plotted as in Fig. 1.

Fig. 4b The effective inverse Hausdorff dimension ν as a function of λ, as defined in (16). The filled dots and the dashed curve are from a fit to the $N = 288$ and $N = 144$ data, while the empty dots and solid curve represent the fit to $N = 576$ and $N = 288$.

Fig. 5 The extrinsic Gaussian curvature $\mid \mathcal{K} \mid$ defined in (17), plotted as in Fig. 1.

Fig. 6 The fluctuations of $\mid \mathcal{K} \mid$.

Fig. 7 The intrinsic curvature $\mid \mathcal{R} \mid$ defined in (18), plotted as in Fig.1.

Fig. 8 The fluctuations of $\mid \mathcal{R} \mid$.

Fig. 9 The intrinsic extrinsic curvature correlation, as defined in (19), plotted as in Fig.1.

Fig. 10a The average maximum coordination number of the surface vertices, $\max_i q_i$, plotted as in Fig.1.

Fig. 10b The scaling exponent of $\max_i q_i$, plotted as in Fig.1.

Fig. 11a S_E as a function of Monte Carlo time $(80,000$ steps) for $N = 144$, $\lambda = 1.4$.

Fig. 11b R as a function of Monte Carlo time $(300,000$ steps) for $N = 144$, $\lambda = 1.4$.

Fig. 12a S_E as a function of Monte Carlo time $(80,000$ steps) for $N = 144$, $\lambda = 1.5$.

Fig. 12b R as a function of Monte Carlo time $(300,000$ steps) for $N = 144$, $\lambda = 1.5$.

Fig. 13a S_E as a function of Monte Carlo time $(300,000$ steps) for $N = 576$, $\lambda = 1.4$.

Fig. 13b R as a function of Monte Carlo time $(3,000,000$ steps) for $N = 576$, $\lambda = 1.4$.

Fig. 14 S_E and R from the same Monte Carlo run, $N = 576$, $\lambda = 1.325$, $20,000$ steps.