Strings with Extrinsic Curvature: An Analysis of the Crossover Regime*

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We present the results of a set of Monte Carlo simulations of Dynamically Triangulated Random Surfaces embedded in three dimensions with an extrinsic curvature dependent action. We analyze several observables in the crossover regime and discuss whether or not our observations are indicative of the presence of a phase transition.

1. Introduction

In this work, we investigate a theory of fluid, fluctuating random surfaces embedded in three dimensions. Theories of fluctuating surfaces (string theories) have been conjectured to describe a wide variety of physical phenomena and models, including the strong interaction at large distances, the 3d Ising model, and unified theories incorporating gravity. Lipid bilayers and microemulsions are also examples of fluctuating surfaces [?]. Such biological and chemical membranes exhibit self-avoidance, which we do not take into account in our simulations. Models of fluctuating random surfaces can in fact be solved exactly when the surfaces are embedded in dimensions $D \leq 1$; these solutions break down though when continued to the more physical regime $D > 1$. This may be related to the observation that simulations of fluctuating surfaces in $D > 1$ using a discretization of the standard Polyakov string action are dominated by crumpled, spiky configurations.

Our lattice model is constructed by triangulating each surface. Each node of the triangulation is embedded in R^3 by the functions X_i^μ; i labels the ith node and μ runs from 1 to 3. The triangulation is characterized by the adjacency matrix C_{ij}, whose elements equal 1 if i and j label neighboring nodes of the triangulation and vanish otherwise. The triangles are assumed to be equilateral (as measured by the intrinsic metric of each surface); the connectivity at each node determines the intrinsic curvature. We also associate a normal vector $(n^\mu)_{\hat{k}}$ with each triangle (indices with hats label the triangles). We shall study the theory defined by the action [?, ?, ?, ?, ?, ?, ?]

$$S = S_G + \lambda S_E = \sum_{i,j,\mu} C_{ij}(X_i^\mu - X_j^\mu)^2 + \lambda \sum_{\hat{k},\hat{l},\mu} C^{\hat{k}\hat{l}}(1 - n_{\hat{k}}^\mu \cdot n_{\hat{l}}^\mu) \ . \quad (1)$$

For $\lambda = 0$ this is simply a discretization of the Polyakov string action. The final term, which depends on the discretized extrinsic curvature, introduces a ferromagnetic interaction between surface normals, which one might hope would cause smoother surfaces to dominate the partition function. We would like to know if there is a smooth phase and a phase transition (at some finite λ_c) between this phase and the crumpled phase observed at $\lambda = 0$. If this were so, an interesting continuum limit of this lattice model could perhaps be constructed at this phase transition point, yielding a new continuum string theory.

2. The Simulation

We have considered triangulations with the topology of the torus, to minimize finite size effects. The above action was used, with the BRST invariant measure utilized also by Baillie, Johnston and Williams [?], so that

$$Z = \sum_{G \in T(1)} \int \prod_{\mu,i} dX_i^\mu \prod_i q_i^{\frac{D}{2}} \exp(-S_G - \lambda S_E) \ , \quad (2)$$

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where $D = 3$, q_i is the connectivity of the ith vertex and $T(1)$ refers to the set of triangulations of genus 1. We used the standard Metropolis algorithm to update our configurations. To sweep through the space of triangulations we performed flips on randomly chosen links. Flips were automatically rejected if they yielded a degenerate triangulation. After a set of $3M$ flips was performed, $3M$ randomly selected embedding coordinates were updated via random shifts from a flat distribution. Most of the Monte Carlo simulations were performed on HP-9000 (720 and 750 series) workstations; we also collected some data by simulating lattices on each of the 32 nodes of a CM-5.

We ran on lattices ranging in size from $N = 36$ to 576 (N signifies the number of vertices) with 4 to 7 different values of λ for each N. Most of the data was this data was taken in the region $\lambda \in (1.325, 1.475)$. For small N, the runs consisted of 3×10^6 sweeps, while we performed longer runs (of up to 27×10^6 sweeps for $N = 576$) for larger lattices, because the auto-correlation times for our simulations were very large. (The correlation time for the radius of gyration was greater than 10^6 sweeps for $N = 576$!) To determine our observables as a function of λ we used a histogram reconstruction procedure. We patched different histograms by weighting them with the associated statistical indetermination (which was estimated by a jack-knife binned procedure). Various consistency checks indicate that this procedure is very reliable.

3. Observables

We measured the edge action S_E and the associated specific heat $C(\lambda) \equiv \frac{\lambda^2}{N}(< S_E^2 > - < S_E >^2)$. In Fig. 1 we plot the specific heat curve (constructed via the histogram procedure) and we tabulate its maximum and peak position for various lattice sizes in Table 1.

We see that the specific heat peak grows vigorously with N for small lattices, but that this growth quickly levels off for larger N. These observations agree fairly well with previous work [?, ?, ?]. For the larger lattices it appears that the peak position shifts very slowly towards higher values of λ, though this increase is not statistically significant. The shape of the peak does not change dramatically with N; it narrows perhaps a bit between $N = 144$ and $N = 576$.

Figure 1. $C(\lambda)$. Dotted lines: N=144. Dashed lines: N=288. Solid lines: N=576.

N	$C(\lambda)^{\max}$	λ_c
36	3.484(8)	1.425(35)
72	4.571(15)	1.410(15)
144	5.37(14)	1.395(30)
288	5.55(7)	1.410(25)
576	5.81(17)	1.425(30)

Table 1
The maximum of the specific heat and its position, with errors, for different lattice sizes.

We measured the magnitude of the extrinsic Gaussian curvature, $\int \mid K \mid \sqrt{\mid h \mid}$ (h is the induced metric and K is the determinant of the extrinsic curvature matrix), given by

$$\mid \mathcal{K} \mid = \frac{1}{N} \sum_i \mid 2\pi - \sum_{\hat{j}} \phi_i^{\hat{j}} \mid \quad . \tag{3}$$

Here $\phi_i^{\hat{j}}$ denotes the angle subtended by the $\hat{j}$th triangle at the ith vertex. This quantity, plotted in Fig. 2, measures the magnitude of the deficit angle in the embedding space averaged over all vertices. Note that the mean Gaussian curvature decreases rapidly in the neighborhood of $\lambda = 1.4$, indicating that a sharp crossover is occuring in this system. From this plot we can see that finite

Figure 2. $|\mathcal{K}|$, plotted as in Figure 1.

size effects increase with λ; they do not appear to peak in the region about $\lambda = 1.4$ as one might expect for a typical phase transition. We also measured various other observables which characterize both the intrinsic and extrinsic geometry of these surfaces (and the correlation between intrinsic and extrinsic geometry). These measurements are discussed in a longer write-up of this work [?]. They all exhibit sharp crossover behavior in the region near $\lambda = 1.4$. We found that the auto-correlation times of these observables grew rapidly as λ increased, but we did not note any maximum in these times in the region about $\lambda = 1.4$.

4. Interpretation

This model of crumpled surfaces appears to exhibit sharp crossover behavior in the region around $\lambda = 1.4$. The sharp change in the magnitude of the Gaussian curvature, the radius of gyration and other observables indicates that the normals acquire long-range correlations, up to the size of the systems we examine. The zero string tension measurement of [?] also shows that the disordered regime differs from the regime in which the surfaces are ordered (up to scale of the lattices that are simulated) by only a small shift in λ. This evidence might indicate the presence of a phase transition at this point. Since the peak growth rapidly diminishes for large N, such a phase transition would likely be higher than second order. Still, the apparent absence of diverging correlation times and increasing finite size effects in the peak region leads us to question whether we are actually observing a typical phase transition.

There are indeed other possible interpretations of our data. Note that the surfaces which we simulate are quite small. For instance, if the surfaces in our simulations had an intrinsic dimension of 2.87 (characteristic of $D = 0$ gravity), they would have roughly a linear size of fewer than 10 lattice spacings.

Perhaps the simplest alternative explanation for the presence of this peak is suggested by the arguments of Kroll and Gompper [?]. They argue that the peak occurs when the persistence length of the system approaches the size of the lattice ($\xi_p \sim N^{\frac{1}{d}}$); d denotes the intrinsic Hausdorff dimension. Fluctuations on a larger scale become more important. When this scale is greater than the lattice size these fluctuations are suppressed. Thus one might surmise that the specific heat will drop for large λ. The one-loop renormalization group calculation [?] predicts that the persistence length grows as $\xi_p \sim \exp(C\lambda)$; C is inversely proportional to the leading coefficient of the beta function. We would expect that the peak position should shift to the right with increasing N in this scenario as

$$\lambda_{peak}(N') - \lambda_{peak}(N) = \frac{\ln(\frac{N'}{N})}{dC}. \qquad (4)$$

This reasoning also indicates that the peak should widen as the lattice size increases; we do not observe this at all.

An alternative scenario, which builds on the ideas in the above paragraph, is suggested by the tantalizing similarities between the results of our fluid surface simulations and what has been observed for the $d = 4$ $SU(2)$ Lattice Gauge Theory [?] and for the $d = 2$ $O(3)$ model.

The $O(3)$ model, which is thought to be asymptotically free, exhibits a specific heat peak near $\beta = 1.4$ (first measured via Monte Carlo simulations by Colot [?]). The origin of this peak is understood [?]; it is due to the fluctuations of the sigma particle, a low-mass bound state of the massless $O(3)$ pions. The sigma induces

short-range order and contributes to the specific heat as a degree of freedom only at high temperatures (when the correlation length in the system becomes smaller than its inverse mass). The peak thus occurs at the beginning of the crossover regime, when the correlation length is several lattice spacings.

According to the low temperature expansion, the correlation length grows as $\xi \sim \exp(2\pi\beta)/\beta$. Thus one would expect a fairly rapid crossover in the $O(3)$ model; the correlation length should increase by roughly a factor of 9 when β is shifted by about 0.35. Such a crossover is indeed observed, though it is not so apparent that it is as dramatic as the crossover behaviour observed for fluid surfaces.

Recent simulations of the $O(3)$ model [?] indicate that the specific heat peak grows significantly when the lattice size L is increased from 5 to 15, and that virtually no growth in peak height is evident as L is increased further up to 100. Furthermore, the peak position shifts to the right as L grows and then appears to stabilize for large L. This is more or less what we observe in our simulations of fluid surfaces, on lattices of small size. We point out these similarities largely to emphasize that there does exist an asymptotically free theory (with low mass excitations) which exhibits crossover behavior qualitatively similar to that observed in our simulations. The analogy is perhaps deeper, though, since the fluid surface action (with extrinsic curvature) in certain guises looks like a sigma model action. It would not therefore be so surprising from this point of view to find a sigma particle in these theories, perhaps associated with $(\hat{n}^2 - 1)$, in which $\hat{n}$ denotes the unit normal to our surfaces.

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