1 Figure Captions

Fig. 1 The edge extrinsic-curvature specific heat $C(\lambda)$ as a function of λ. Multi-histogram with errors are shown for $N = 576$ (long and short-dashed lines) and N=1, 152 (solid lines). Four individual data points are also shown for N=2, 304 (solid circles). One sees that the specific heat peak has saturated - it is not growing with the systemsize N above 576.

Fig. 2 The specific heat $C(\beta)$ of the two-dimensional $O(3)$ non-linear sigma model as a function of β for lattice volumes N=16, 25, 64, 100, 900, 2, 500, 4, 900 and 10, 000. The peak saturates quickly for $N \geq 100$ and "β_c" does not increase with the volume.

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N	$F[K]$	$F[q-6]$
576	$5.71 \pm .08$	$8.39 \pm .04$
1152	$5.59 \pm .05$	$8.32 \pm .03$
2304	$5.70 \pm .10$	$8.37 \pm .06$

Table 2: The mean square fluctuations of the extrinsic Gaussian curvature K and the defect coordination number $q-6$, with errors, for different lattice sizes.

estimate of R, which allows us to estimate for the Hausdorff dimension at the pseudo-critical point λ the value $d_H = 4.35 \pm .3$. This is an intriguing result, since 4 is the extrinsic Hausdorff dimension of a class of branched polymers, as constructed, for instance, in references [39]. Such configurations are expected to dominate the string functional integral for large embedding dimension D.

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$|\lambda - \lambda_c|^{-\alpha}$, ν is the analogous exponent for the correlation length and d is the intrinsic Hausdorff dimension of the surface. In other words there may be a cusp singularity at λ_c as, for example, in the case of the superfluid (λ) transition in He^4 [30, 31], for which $\alpha = -0.0127 \pm 0.0026$. Since we do not have a measurement of νd, which may even be rather large, we have no reliable idea of the exponent α itself. Generally speaking one finds that second order transitions on fixed lattices become higher order on dynamical lattices, as for example in the case of the $2d$-Ising model [32, 33]. Since there seems to be a 2nd order crumpling transition for non-self-avoiding tethered (fixed-triangulation) surfaces [35, 36, 37, 38], it would be consistent for the transition to be higher than 2nd order when the model is coupled to gravity.

All told our work gives only weak evidence for a continuum crumpling transition. The strongest evidence in favour of such a transition, at present, is the scaling behaviour of the string tension and mass gap reported in [2]. This highlights the need for more extensive measurements on these important observables.

We have also measured the fluctuations of the extrinsic Gaussian curvature $|K|$, defined as the average magnitude of the deficit angle at each vertex as measured in the embedded space. Likewise, we have also computed the fluctuations of the mean defect coordination number $|q-6|$, which is proportional to the intrinsic Gaussian curvature. In reference [3] we had observed that fluctuations of these observables were quite large near the coupling λ_c but then drop quite dramatically for slightly higher λ. We find that on larger lattices the fluctuations of these observables at λ_c also do not grow with N; thus their behaviour does not provide unequivocal evidence of the presence of a phase transition. In Table 2 we give the mean-square fluctuations of both observables.

Finally we note the behaviour of the gyration radius at λ_c (which we will take as being 1.425 in the following). For large λ (> 2), the scaling of $R(N) \sim N^{\frac{2}{d_H}}$ with N gives a Hausdorff dimension close to 2 (as we expect for flat surfaces). In the crumpled region the Hausdorff dimension rapidly increases with diminishing λ. We had pointed out in [3] that finite size effects were relevant in the sector close to λ_c and that we could not estimate a reliable number from the lattice sizes analyzed. Here the largest lattice we simulated ($N = 2304$) does not give useful data, since the error in R is too large, but on the 1152 and 576 node lattices we get a fairly precise

clusters updated a volume exceeding 30 times the volume of the lattice [3]. For each point of the $N = 16,\ 64,\ 900,\ 2,\ 500$ and $4,\ 900$ lattices we used $20,\ 000$ measurements. We took a measurement every time the Wolff clusters updated a volume exceeding 3 times the volume of the lattice. It is very clear that the peak levels off quickly for $N \geq 100$ and that "β_c" is not increasing with the size of the lattice. Measurements of the asymptotic value of $C(\beta)$ have been reported in the past [23, 24]. The authors of [25, 26, 27] explain the peak as the excitation of an extra degree of freedom, the so-called σ-particle [28]. The would-be transition occurs when the mass of the σ-particle becomes comparable to the inverse correlation length of the $O(3)$ model. It may be that there is a similar interpretation of the observed peak of $C(\lambda)$.

We are currently histogramming our data to examine as well the behaviour of the complex zeroes of the partition function when λ is allowed to become complex. For $SU(2)$ lattice gauge theory, which also exhibits a specific heat peak without an associated phase transition, it has been shown that there are complex zeroes which are near the real axis but do not converge to it in the infinite-volume limit [17, 18]. High-temperature expansions also indicate that the $O(3)$ model susceptibility has a complex singularity near the real axis [29]. Our search has been so far inconclusive. We used the single histogramming technique [17, 18] for $N = 36,\ 72,\ 144,\ 288,$ and 572. We checked our code by reproducing the results of [18] for $N = 64$. The modulus of the partition function becomes quite small near the real axis and we were not able to see the complex zeroes due to statistical fluctuations. For $N = 36$ and 144, for example, the complex zeroes have $\mathrm{Im}\ \lambda > 0.3$ and 0.1 respectively. Higher statistics and an improvement of the method [20] would help to probe deeper into the complex λ plane and study the scaling of the complex zeroes with the volume. Using single histogramming and the simulations of the two dimensional $O(3)$ model mentioned in the previous paragraph, we were not able to see the complex zeroes of the $O(3)$ partition function reported in [29] either.

It is still possible that there is a true continuous phase transition, the crumpling transition, occurring at λ_c. Assuming a continuous transition, a standard finite-size-scaling argument only tells us that $\omega = \frac{\alpha}{\nu d} < 0$, where α is the exponent governing the divergence of the specific heat, $C(\lambda) \sim$

[3] For the $N = 16,\ 64,\ 900$ and $4,900$ lattices the integrated autocorrelation times were between 1 and 2 Wolff updatings of the entire lattice

N	$C^{(max)}$	λ_c
36	$3.484 \pm .008$	$1.425 \pm .035$
72	$4.571 \pm .015$	$1.410 \pm .015$
144	$5.37 \pm .08$	$1.395 \pm .017$
288	$5.55 \pm .05$	$1.410 \pm .015$
576	$5.81 \pm .06$	$1.425 \pm .010$
1152	$5.69 \pm .04$	$1.425 \pm .010$
2304	$5.75 \pm .10$	$1.425 \pm .010$

Table 1: The maximum of the specific heat and its position, with errors, for different lattice sizes.

true asymptotic behaviour. These results also invalidate the interpretation raised in the introduction [4 3]. A one-loop renormalization group calculation shows that the persistence length ξ grows with bending rigidity λ as $\exp(\frac{3}{4\pi}\lambda)$. Equating ξ with the spatial extent of the lattice $N^{1/d_{in}}$, where d_{in} is the intrinsic Hausdorff dimension of the lattice, one sees that they become comparable at a coupling $\lambda_c \sim \frac{3}{4\pi d_{in}} \ln(N)$. In the continuum limit $N \to \infty$, λ_c diverges. Since, for reasonable values of d_{in}, we do not see the increase in λ_c with N predicted by the above relationship we can state with some confidence that the origin of the observed specific heat peak is not explained by the persistence length becoming comparable to the extent of the lattice.

In [3] we also discussed other, more subtle, possibilities that could account for the observed behaviour of $C(\lambda)$ without invoking a phase transition. One was based on the analogy between the present model and the $O(3)$ sigma-model in $2d$ [21]. This model is also asymptotically free and consequently disordered at all non-zero temperatures. Yet numerical simulations show a distinct peak in the specific heat which grows for small lattices and then saturates, just as we find in the model of a rigid string treated here. This may be seen in Fig. 2 where we have plotted the specific heat $C(\beta)$ for the two dimensional $O(3)$ model. The simulations were done on square lattices of volume $N=16$, 25, 64, 100, 900, 2, 500, 4, 900 and 10, 000 using the Wolff algorithm[22]. For each point of the $N=25$, 100 and 10, 000 lattices we used 100, 000 measurements. We took a measurement every time the Wolff

a set of $3N$ flips are performed, $3N$ randomly selected embedding coordinates are updated by random shifts from a flat distribution.

The observable of most direct physical interest is the edge extrinsic-curvature specific heat

$$C(\lambda) = \frac{\lambda^2}{N}(< S_E^2 > - < S_E >^2) \ . \qquad (2)$$

This exhibits a peak at a coupling λ_c which depends on the exact discrete form of the action chosen [11, 12, 13, 14, 15, 16, 2, 3]. In [3] we found that the maximum value of the specific heat grows with the system size as $C_{max} = AN^{\omega}$, with $\omega = 0.06 \pm 0.05$. In our new series of simulations on lattices with 1152 nodes we ran 54 million sweeps at $\lambda = 1.425$, 21 million sweeps at $\lambda = 1.430$ and 18 million sweeps at $\lambda = 1.435$ [1]. On the data from these three points we use multi-histogram reconstruction [17, 18, 19]. This works well in that three different reconstructions give coherent results. On lattices of 2304 nodes we have poorer statistics. We ran 17 million sweeps at $\lambda = 1.425$ plus approximately 5 million sweeps at $\lambda = 1.40$, 1.42 and 1.43 as a consistency check. On the 2304 lattice histogramming does not work well. This is to be expected since the statistics are not good enough for such a large lattice. Still we have checked that our measurements at $\lambda = 1.425$ give consistent results, that the error estimate is reliable and that we are, with good accuracy, at the peak of the specific heat. In all these simulations required the equivalent of approximately one year of CPU time on an HP 9000 (720 series) workstation.

The specific heat peak for $N = 576$, 1152 and 2304 is shown in Fig. 1. In Table 1 we give our results for the maximum of the specific heat and the associated coupling λ_c as a function of N [2].

Clearly the maximum of the specific heat curve C_{max} is effectively constant for surfaces with 576 or more nodes. The (pseudo)-critical coupling λ_c is also constant for $N = 576$ and above. With the present data we can definitely exclude the presence of a divergence in the specific heat. The growth of the specific heat peak observed on small lattices [11, 12, 13] does not reflect

[1] Since the autocorrelation time τ is of order 400,000 sweeps on the 1152 lattice these runs have at least 45τ measurements.

[2] We have reanalyzed the data presented in reference [3], using a different method of weighting relative errors when combining histograms. Thus, some of the errors quoted here are smaller than the respective uncertainties in reference [3].

The theory of $2d$ fluid random surfaces embedded in R^3, with an extrinsic curvature term (bending rigidity) in the action, has received considerable analytical and numerical attention in the last decade [1, 2, 3]. In [3] we presented the results of a large-scale Monte Carlo simulation of a dynamically-triangulated torus in R^3 with up to 576 nodes, corresponding to 1152 triangles. Although we observed a rapid crossover from a *crumpled* regime for $\lambda < \lambda_c$ to a *smooth* regime for $\lambda > \lambda_c$, where λ is the extrinsic curvature coupling constant and $\lambda_c \approx 1.425$, it was not at all clear whether a true continuous thermodynamic phase transition separated the two regimes. In fact several alternative interpretations of the data were discussed in [3]. Perhaps the simplest possibility, advocated in [4], is that the persistence length ξ describing the exponential decay of the normal-normal two-point function in the crumpled (disordered) regime simply reaches the finite size of the system at λ_c. In this case the observed smooth regime would be a finite-size artifact with the true continuum theory really being crumpled for all couplings λ, in accordance with perturbative analytical results [5, 6, 7, 8]. Since ξ grows exponentially with λ, according to the one-loop beta-function, this interpretation would imply that λ_c diverges logarithmically with system size N. To resolve this issue and to gain further insight into the model it was clearly desirable to extend the numerical simulations to larger lattice sizes and to clarify the influence of finite-size effects. In this short letter we present an extension of our previous work to include toroidal lattices 1152 and 2304 nodes.

As in [3] we study the theory defined by the action

$$S = S_{Gauss} + \lambda S_E = \sum_{i,j,\mu} C_{ij} (X_i^\mu - X_j^\mu)^2 + \lambda \sum_{\hat{k}\hat{l}} (1 - n_{\hat{k}}^\mu \cdot n_{\hat{l}}^\mu) \ , \qquad (1)$$

where C_{ij} is the adjacency matrix, X_i^μ is the position in R^3 of node i ($i = 1, \ldots, N$) and $n_{\hat{k}}^\mu$ is the normal vector to a triangle $\hat{k}$ in the cellular decomposition of a lattice discretization of a torus. The discretization S_E of a continuum extrinsic curvature term takes support on the edges (links) of the lattice and is known as the (discrete) edge extrinsic-curvature. The simulation consists of a standard Metropolis algorithm for the updating of the nodes X_i^μ and a DIRS-algorithm [9, 10] to sweep through the space of triangulations. The basic flip move is attempted on randomly chosen links. After

Fluid Random Surfaces with Extrinsic Curvature: II

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Abstract

We present the results of an extension of our previous work on large-scale simulations of dynamically triangulated toroidal random surfaces embedded in R^3 with extrinsic curvature. We find that the extrinsic-curvature specific heat peak ceases to grow on lattices with more than 576 nodes and that the location of the peak, κ_c, also stabilizes. The evidence for a true crumpling transition is still weak. If we assume it exists we can say that the finite-size scaling exponent $\frac{\beta}{\nu d}$ is very close to zero or negative. On the other hand our new data does rule out the observed peak as being a finite-size artifact of the persistence length becoming comparable to the extent of the lattice.

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