# Multiple Ising Models Coupled to 2–d Gravity: a CSD Analysis*

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We simulate single and multiple Ising models coupled to 2–d gravity and we measure critical slowing down (CSD) with the standard methods. We find that the Swendsen-Wang and Wolff cluster algorithms do not eliminate CSD. We interpret the result as an effect of the mesh dynamics.

In recent years the introduction of cluster algorithms [1,2] has provided the means to beat CSD present in a variety of statistical models near criticality. Reviews of such techniques may be found in refs. [3]. In the context of a wider study of the critical behavior of single and multiple Ising models coupled to 2–d gravity [4,5], we have analyzed CSD for various update algorithms.

We are interested in the behavior of Ising spins attached to the vertices of a random triangulation. The triangulation is described by the adjacency matrix $C_{\rm ij}$, which is one if i and j are neighbors and vanishes otherwise. $C_{\rm ij}$ is the discrete analogue of the world-sheet metric $g_{\rm ij}$. We restrict ourselves to toroidal triangulations of minimum loop length three and of minimum coordination number $q = 3$. The partition function of this model is

$$Z_N = \sum_{T\in\mathcal{T}_N} \sum_{\sigma_{\rm i=\pm 1}} \exp\left(-\beta\,\mathcal{H}\right) \qquad (1)$$

$$\mathcal{H} = \sum_{\alpha=1}^{n_{\rm s}} \sum_{\rm i,j=1}^{N} C_{\rm ij}(T)\sigma_{\rm i}^{\alpha}\sigma_{\rm j}^{\alpha}, \qquad (2)$$

where $\alpha$ labels the spin species. The parameter $n_{\rm s}$ determines the number of Ising spins (species) attached to each vertex. We studied the model for $n_{\rm s} = 1$, 2 and 3.

To implement numerically the partition function (1) with a Monte Carlo simulation one has to update the Ising spins as well as the triangulation $C_{\rm ij}$. The former is achieved using one of the cluster algorithms, Swendsen-Wang (SW) or Wolff [1,2], the latter using the link-flip [6] with a standard Metropolis update. For comparison, we also simulated the model using the Metropolis update for the Ising spins. For triangulations of $N$ vertices a single sweep consists of attempted flips of $3N$ links followed by updates of the spins. A Wolff update consists of consecutive flips of FK [7] clusters that reverse the sign of at least 40% of the spins.

The observables that we measured were the energy density, the magnetization density, the average absolute value of the scalar curvature and, in some cases, the average and maximum cluster size. In order to detect and quantify the CSD of these observables, we used the following relation to estimate the integrated autocorrelation time $\tau_{\rm int}$.

$$\mathrm{Var}(O)_{\rm true} = 2\,\tau_{\rm int}\,\mathrm{Var}(O)_{\rm naive}; \qquad (3)$$

$\mathrm{Var}(O)_{\rm true}$ is the variance of the observable extracted with the *binning* method. The CSD exponent $z/d_H$ was obtained by fitting the data to the scaling law

$$\tau_{\rm int} \propto N^{z/d_H}, \qquad (4)$$

where the Hausdorff dimension of the surface is left explicit, since it is hard to measure the linear size of these models.

We simulated each model, $n_{\rm s} = 1$, 2 and 3, with all three update algorithms for four different volumes (512, 1024, 2048, 4096). In the $n_{\rm s} = 1$ case we also ran for $N = 8192$. We ran the $n_{\rm s} = 1$

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Table 1
Critical exponent for the Magnetization and the Energy from fits.

| | $z/d_H$ — Magnetization | | | $z/d_H$ — Energy | | |
|---|---|---|---|---|---|---|
| Model | Metropolis | SW | Wolff | Metropolis | SW | Wolff |
| $n=1$ | $.85 \pm .06$ | $.58 \pm .05$ | $.54 \pm .05$ | $.62 \pm .03$ | $.057 \pm .005$ | $.04 \pm .03$ |
| $n=2$ | $.95 \pm .05$ | $.62 \pm .06$ | $.58 \pm .09$ | $.35 \pm .1$ | $.08 \pm .02$ | $.17 \pm .08$ |
| $n=3$ | $.9 \pm .1$ | $.49 \pm .08$ | $.55 \pm .1$ | $.5 \pm .1$ | $.05 \pm .04$ | $.37 \pm .08$ |

model at the critical value of $\beta$ (known analytically); for the others, we chose $\beta$ by looking at the peak of the lattice susceptibility and the intersection of Binder's cumulants. We thermalized the configuration for $1 \times 10^5$ sweeps and measured the observables 3–$5 \times 10^5$ times. We took one measurement per sweep.

In table 1 and figures 1, 2 we report our estimates of the dynamic exponent $z/d_H$. We deduce the following:

1. There is considerable CSD in all cases. Nonetheless, the use of a cluster algorithm significantly improves the situation, reducing the autocorrelation time and the dynamic exponent.

2. The magnetization is the observable that suffers the most from CSD. This behavior differs from the case of a flat Ising model, where the energy and magnetization have comparable CSD.

3. The two cluster algorithms, SW and Wolff, perform very similarly within the statistical accuracy of our data.

4. There seems to be little difference between one and two species coupled to 2–d gravity. In [5] we found that the numerical behavior of these two models is very similar. Actually, the presence of logarithmic corrections to scaling in the $n_s = 2$ model indicates that the scaling ansatz (4) might cease to be valid. This situation is similar to the case of the 2–d 4–state Potts model. Here, Li and Sokal [8] have shown that measurements of $z$, obtained using the ansatz (4), violate rigorous bounds. They suggest that these measurements of $z$ are not correct because the fits to $\tau$ fail to take into account logarithmic corrections. In the $n_s = 3$ model the situation is even worse since there is no theoretical prediction of the form of corrections to scaling. It is likely that the numbers quoted for the $n_s = 2$ and 3 models differ considerably from their asymptotic values.

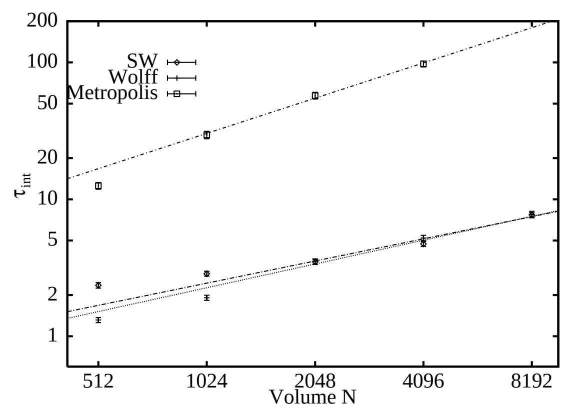


Figure 1. Comparison of the integrated autocorrelation times for the magnetization in the $n = 1$ model. The dashed lines are log-log regression fits.

The presence of CSD in these models is a consequence of the fact that we update the mesh with a local algorithm. In a related study [9], it was found that pure percolation clusters built on a dynamical mesh suffer from critical slowing down as well. In that case CSD must be attributed to the

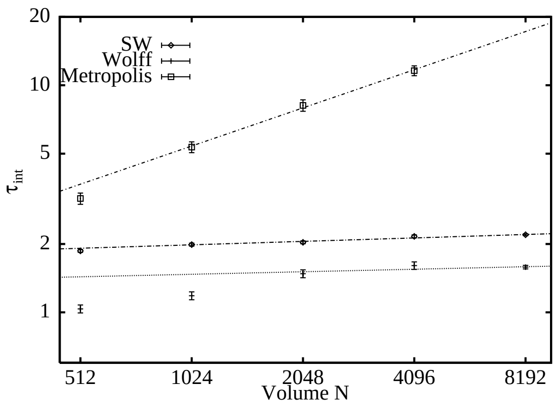


Figure 2. Comparison of the integrated autocorrelation times for the energy in the $n = 1$ model.

mesh dynamics alone, since the percolation clusters do not have a dynamical behavior of their own. It has been shown that the dynamical triangulation algorithm generates a distribution of baby universes [10]; these are regions of the mesh that are connected to the rest of the surface by narrow bottlenecks. The presence of these bottlenecks inhibits cluster growth in and out of baby universes. Since this structure of baby universes is slow to decorrelate under the local link-flip updates, we expect that the mean percolation cluster size will be afflicted by CSD.

In our case, we build FK clusters, but the effect of the bottlenecks on their formation is similar. The maximum and mean size of the FK clusters are directly related to the magnetic observables [11,12]. The mean cluster size $\mathcal{S}_{\rm FK}$ is defined as $\langle s\rangle_{\rm Wolff}$ and $\langle s^2\rangle_{\rm SW}/\langle s\rangle_{\rm SW}$, $s$ being the number of sites of the clusters one builds in the update process. For $\beta \leq \beta_c$, $\mathcal{S}_{\rm FK}$ is equivalent to the susceptibility $\chi = \beta/N\ \langle M^2\rangle$. It is therefore straightforward to interpret the CSD we found for the magnetic observables as one consequence of the CSD in the dynamics of the local mesh update. Note that the above arguments do not apply directly to the energy density. This is consistent with our observation of little or no CSD for the energy with SW or Wolff updates.

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