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Derivation of the Beam and Warming Algorithm for Compressible Navier-Stokes Equations

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Abstract

The Beam and Warming algorithm for solving the compressible Navier-Stokes Equations is derived here for a target audience not familiar with Computational Fluid Dynamics (CFD). Starting with the differential form of the equations in conservative form, the algorithm is derived with particular emphasis on the various simplifications which are necessary for an efficient implementation. In order to highlight the important aspects of the algorithm, only the two-dimensional equations are considered with simple central differencing and no artificial dissipation.

1 Introduction

The Navier-Stokes equations in conservative form are given by

$$
\begin{aligned}
\partial_t \rho + \partial_j (\rho u_j) &= 0 , \\
\partial_t (\rho u_i) + \partial_j [\rho u_j u_i + p\delta_{ij} - \tau_{ij}] &= \rho F_i , \\
\partial_t(\rho E) + \partial_j [\rho u_j H - u_i \tau_{ij} - k \partial_j T] &= \rho u_i F_i + \dot{q} ,
\end{aligned} \tag{1}
$$

where

$$\tau_{ij} = \mu [\partial_j u_i + \partial_i u_j] + \lambda \delta_{ij} \partial_k u_k , \tag{2}$$

ρ: density

u_i: velocity component in i

p: pressure

F_i: external force in i

E: total internal energy $= e + \frac{u_i u_i}{2}$, where e is the specific internal energy

H: total enthalpy $= h + \frac{u_i u_i}{2}$, where h is the specific enthalpy

T: temperature

k: thermal conductivity

μ: absolute viscosity

λ: second coefficient of viscosity ($= -\frac{2}{3}\mu$ upon invoking Stokes assumption)

A perfect gas is assumed, in which case,

$$e = E - \frac{u_i u_i}{2} = C_v T \Rightarrow T = \frac{1}{C_v} \left[E - \frac{u_i u_i}{2} \right] . \tag{3}$$

Furthermore, since $p = \rho R T$, then

$$p = \rho R T = \rho \frac{R}{C_v} \left[E - \frac{u_i u_i}{2} \right] = (\gamma - 1) \rho \left[E - \frac{u_i u_i}{2} \right] . \tag{4}$$

To simplify the subsequent development of the Beam and Warming Algorithm, consider a two-dimensional flow in cartesian coordinates. Expanding the Navier-Stokes equations gives

$$\begin{aligned}
\rho_t + \quad & (\rho u)_x + (\rho v)_y = 0 \ , \\
(\rho u)_t + \quad & (\rho u^2)_x + (\rho uv)_y + p_x - [(2\mu + \lambda)u_x + \lambda v_y]_x - [\mu\,(u_y + v_x)]_y = \rho F_x \ , \\
(\rho v)_t + \quad & (\rho uv)_x + (\rho v^2)_y + p_y - [\mu\,(u_y + v_x)]_x - [(2\mu + \lambda)v_y + \lambda u_x]_y = \rho F_y \ , \\
(\rho E)_t + \quad & (\rho uH)_x + (\rho vH)_y - [2\mu uu_x + \mu v\,(u_y + v_x) + \lambda u\,(u_x + v_y) + kT_x]_x \\
- \quad & [2\mu vv_y + \mu u\,(u_y + v_x) + \lambda v\,(u_x + v_y) + kT_y]_y = \rho uF_x + \rho vF_y + \dot{q} \ .
\end{aligned}$$

Rearrange the above equations according to

$$\begin{aligned}
\rho_t + (\rho u)_x + (\rho v)_y \quad &= \quad 0 \ , \\
(\rho u)_t + \left(\rho u^2 + p\right)_x + (\rho uv)_y \quad &- \quad [(2\mu + \lambda)u_x]_x - [\lambda v_y]_x \\
&- \quad [\mu v_x]_y - [\mu u_y]_y = \rho F_x \ , \\
(\rho v)_t + (\rho uv)_x + \left(\rho v^2 + p\right)_y \quad &- \quad [\mu v_x]_x - [\mu u_y]_x \\
&- \quad [\lambda u_x]_y - [(2\mu + \lambda)v_y]_y = \rho F_y \ , \\
(\rho E)_t + (\rho uH)_x + (\rho vH)_y \quad &- \quad [2\mu uu_x + \mu vv_x + \lambda uu_x + kT_x]_x \\
&- \quad [\mu vu_y + \lambda uv_y]_x \\
&- \quad [\mu uv_x + \lambda vu_x]_y \\
&- \quad [2\mu vv_y + \mu uu_y + \lambda vv_y + kT_y]_y \\
&= \quad \rho uF_x + \rho vF_y + \dot{q} \ . \qquad (5)
\end{aligned}$$

Next, define the following vector functions:

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix}; \qquad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uH \end{bmatrix}; \qquad \mathbf{G} = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vH \end{bmatrix};$$

$$\mathbf{U}_I = \begin{bmatrix} 0 \\ (2\mu + \lambda)u_x \\ \mu v_x \\ 2\mu uu_x + \mu vv_x + \lambda uu_x + kT_x \end{bmatrix}; \qquad \mathbf{U}_E = \begin{bmatrix} 0 \\ \lambda v_y \\ \mu u_y \\ \mu vu_y + \lambda uv_y \end{bmatrix};$$

$$\mathbf{V}_I = \begin{bmatrix} 0 \\ \mu u_y \\ (2\mu + \lambda)v_y \\ 2\mu vv_y + \mu uu_y + \lambda vv_y + kT_y \end{bmatrix}; \qquad \mathbf{V}_E = \begin{bmatrix} 0 \\ \mu v_x \\ \lambda u_x \\ \mu uv_x + \lambda vu_x \end{bmatrix};$$

and, finally,

$$\mathbf{H} = \begin{bmatrix} 0 \\ \rho F_x \\ \rho F_y \\ \rho uF_x + \rho vF_y + \dot{q} \end{bmatrix}. \tag{6}$$

With the above definitions, the Navier-Stokes equations may be written as

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \frac{\partial \mathbf{U}_I}{\partial x} + \frac{\partial \mathbf{U}_E}{\partial x} + \frac{\partial \mathbf{V}_I}{\partial y} + \frac{\partial \mathbf{V}_E}{\partial y} + \mathbf{H} \tag{7}$$

Note: $\frac{\partial \mathbf{U}_I}{\partial x}, \frac{\partial \mathbf{V}_I}{\partial y}$ do not contain cross-derivatives, and $\frac{\partial \mathbf{U}_E}{\partial x}, \frac{\partial \mathbf{V}_E}{\partial y}$ do contain cross-derivatives.

From equation (6), it follows that

$$\begin{aligned} \mathbf{F} &= \mathbf{F}(\mathbf{q}) , & \mathbf{G} &= \mathbf{G}(\mathbf{q}) , \\ \mathbf{U}_I &= \mathbf{U}_I(\mathbf{q}, \mathbf{q}_x) , & \mathbf{U}_E &= \mathbf{U}_E(\mathbf{q}, \mathbf{q}_y) , \\ \mathbf{V}_I &= \mathbf{V}_I(\mathbf{q}, \mathbf{q}_y) , & \mathbf{V}_E &= \mathbf{V}_E(\mathbf{q}, \mathbf{q}_x) . \end{aligned}$$

2 Second-order Temporal Discretization

Let the equations be discretized at $t = (n+1)\Delta t$. A Taylor series expansion about $t + \Delta t$ (or $(n+1)\Delta t$) yields

$$\mathbf{q}^n = \mathbf{q}^{n+1} - \Delta t \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n+1} + \frac{\Delta t^2}{2} \left.\frac{\partial^2 \mathbf{q}}{\partial t^2}\right|_{n+1} + O\left(\Delta t^3\right) , \tag{8a}$$

$$\mathbf{q}^{n-1} = \mathbf{q}^{n+1} - 2\Delta t \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n+1} + 2\Delta t^2 \left.\frac{\partial^2 \mathbf{q}}{\partial t^2}\right|_{n+1} + O(\Delta t)^3 . \tag{8b}$$

Subtracting equation 8(b) from four times equation 8(a) yields

$$4\mathbf{q}^n - \mathbf{q}^{n-1} = 3\mathbf{q}^{n+1} - 2\Delta t \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n+1} + O\left(\Delta t^3\right) ,$$

or, upon rearranging the above,

$$\mathbf{q}^{n+1} - \mathbf{q}^n = \frac{2}{3}\Delta t \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n+1} + \frac{1}{3}\left[\mathbf{q}^n - \mathbf{q}^{n-1}\right] + O\left(\Delta t^3\right) . \tag{9}$$

Denote

$$\Delta \mathbf{q}^n = \mathbf{q}^{n+1} - \mathbf{q}^n ,$$

$$\frac{\partial}{\partial t}\left(\Delta \mathbf{q}^n\right) = \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n+1} - \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n} .$$

Upon substituting the above into equation (9),

$$\Delta \mathbf{q}^n = \frac{2}{3}\Delta t \frac{\partial}{\partial t}\Delta \mathbf{q}^n + \frac{2}{3}\Delta t \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n} + \frac{1}{3}\Delta \mathbf{q}^{n-1} + O\left(\Delta t^3\right) . \tag{10}$$

Note: (1) Equation (9) indicates that $\left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n+1}$ is second order accurate in t. (2) To achieve the above second-order accuracy, the time steps Δt must be uniform.

A general form of the temporal discretization which encompasses equation (10) is given by

$$
\begin{aligned}
\Delta \mathbf{q}^n &= \frac{\theta \Delta t}{1+\alpha} \frac{\partial}{\partial t} (\Delta \mathbf{q}^n) + \frac{\Delta t}{1+\alpha} \left. \frac{\partial \mathbf{q}}{\partial t} \right|_n + \frac{\alpha}{1+\alpha} \Delta \mathbf{q}^{n-1} \\
&+ O\left(\left(\theta - \frac{1}{2} - \alpha \right) \Delta t^2, \, \Delta t^3 \right) \; . \qquad (11)
\end{aligned}
$$

The choice $\theta = 1$, $\alpha = 1/2$ recovers the second-order accurate equation (10).

We now work with the general form equation (11). Upon substitution of equation (7) into equation (11) yields

$$
\begin{aligned}
\Delta \mathbf{q}^n &= \frac{\theta \Delta t}{1+\alpha} \left[\frac{\partial}{\partial x} \left\{ -\Delta \mathbf{F}^n + \Delta \mathbf{U}_I^n + \Delta \mathbf{U}_E^n \right\} \right. \\
&\qquad + \left. \frac{\partial}{\partial y} \left\{ -\Delta \mathbf{G}^n + \Delta \mathbf{V}_I^n + \Delta \mathbf{V}_E^n \right\} + \Delta \mathbf{H}^n \right] \\
&+ \frac{\Delta t}{1+\alpha} \left[\frac{\partial}{\partial x} \left\{ -\mathbf{F}^n + \mathbf{U}_I^n + \mathbf{U}_E^n \right\} + \frac{\partial}{\partial y} \left\{ -\mathbf{G}^n + \mathbf{V}_I^n + \mathbf{V}_E^n \right\} + \mathbf{H}^n \right] \\
&+ \frac{\alpha}{1+\alpha} \Delta \mathbf{q}^{n-1} + O\left(\left(\theta - \frac{1}{2} - \alpha \right) \Delta t^2, \, \Delta t^3 \right), \qquad (12)
\end{aligned}
$$

where the following notation is adopted:

$$
\begin{aligned}
\mathbf{F}^n &= \mathbf{F}(\mathbf{q}^n) \; , \\
\Delta \mathbf{F}^n &= \mathbf{F}\left(\mathbf{q}^{n+1} \right) - \mathbf{F}(\mathbf{q}^n) \; . \qquad (13)
\end{aligned}
$$

Similarly, the same notation is used for the other vector functions.

Note: Equation (12) is a nonlinear equation in $\Delta \mathbf{q}^n$. Once an appropriate spatial differencing scheme is employed, the equations could be solved by using a Newton-Raphson method; however, this is an expensive iterative procedure.

Three simplifications to equation (12) are now made to allow for an efficient solution procedure.

Step 1:

From equation (12), it follows that $\Delta\mathbf{q}^n \sim O(\Delta t)$. Also,

$$[\Delta\mathbf{q}^n]_x = \left[\mathbf{q}^{n+1} - \mathbf{q}^n\right]_x = \mathbf{q}_x^{n+1} - \mathbf{q}_x^n = [\Delta\mathbf{q}_x]^n \sim O(\Delta t) \ . \tag{14}$$

Similarly, for the y-derivative. Let

$$\begin{aligned} \Delta\mathbf{q}_x^n &= [\Delta\mathbf{q}^n]_x = [\Delta\mathbf{q}_x]^n \ , \\ \Delta\mathbf{q}_y^n &= [\Delta\mathbf{q}^n]_y = [\Delta\mathbf{q}_y]^n \ , \end{aligned}$$

and expand the vector functions $\mathbf{F}$ and $\mathbf{G}$ about $\mathbf{q}^n$. Thus,

$$\mathbf{F}^{n+1} = \mathbf{F}^n + \Delta\mathbf{q}^n \left.\frac{\partial\mathbf{F}}{\partial\mathbf{q}}\right|_n + O\left(\Delta\mathbf{q}^n\right)^2 \ ,$$

$$\mathbf{G}^{n+1} = \mathbf{G}^n + \Delta\mathbf{q}^n \left.\frac{\partial\mathbf{G}}{\partial\mathbf{q}}\right|_n + O\left(\Delta\mathbf{q}^n\right)^2 \ .$$

Using the notation in equation (13) and $\Delta\mathbf{q}^n \sim O(\Delta t)$,

$$\Delta\mathbf{F}^n = \mathbf{A}^n\Delta\mathbf{q}^n + O\left(\Delta t^2\right) \ , \tag{15a}$$

$$\Delta\mathbf{G}^n = \mathbf{B}^n\Delta\mathbf{q}^n + O\left(\Delta t^2\right) \ , \tag{15b}$$

where

$$\mathbf{A}^n = \left.\frac{\partial\mathbf{F}}{\partial\mathbf{q}}\right|_n \ ; \quad \mathbf{B}^n = \left.\frac{\partial\mathbf{G}}{\partial\mathbf{q}}\right|_n \ . \tag{16}$$

Similarly, the functions $\mathbf{U}_I$ and $\mathbf{V}_I$ are expanded according to

$$\begin{aligned} \mathbf{U}_I^{n+1} &= \mathbf{U}_I^n + \Delta\mathbf{q}^n \left.\frac{\partial\mathbf{U}_I}{\partial\mathbf{q}}\right|_n + \Delta\mathbf{q}_x^n \left.\frac{\partial\mathbf{U}_I}{\partial\mathbf{q}_x}\right|_n + O\left(\Delta t^2\right) \ , \\ \mathbf{V}_I^{n+1} &= \mathbf{V}_I^n + \Delta\mathbf{q}^n \left.\frac{\partial\mathbf{V}_I}{\partial\mathbf{q}}\right|_n + \Delta\mathbf{q}_y^n \left.\frac{\partial\mathbf{V}_I}{\partial\mathbf{q}_y}\right|_n + O\left(\Delta t^2\right) \ , \end{aligned} \tag{17}$$

or,

$$\begin{aligned} \Delta\mathbf{U}_I^n &= \mathbf{P}\Delta\mathbf{q}^n + \mathbf{R}\Delta\mathbf{q}_x^n + O\left(\Delta t^2\right) \ , \\ \Delta\mathbf{V}_I^n &= \mathbf{Q}\Delta\mathbf{q}^n + \mathbf{S}\Delta\mathbf{q}_y^n + O\left(\Delta t^2\right) \ , \end{aligned} \tag{18}$$

where

$$\mathbf{P} = \left.\frac{\partial \mathbf{U}_I}{\partial \mathbf{q}}\right|_n \quad ; \quad \mathbf{Q} = \left.\frac{\partial \mathbf{V}_I}{\partial \mathbf{q}}\right|_n \ ,$$

$$\mathbf{R} = \left.\frac{\partial \mathbf{U}_I}{\partial \mathbf{q}_x}\right|_n \quad ; \quad \mathbf{S} = \left.\frac{\partial \mathbf{V}_I}{\partial \mathbf{q}_y}\right|_n . \tag{19}$$

Substituting equations (15)–(19) into equation (12) gives

$$\begin{aligned}
\Delta \mathbf{q}^n \;=\; & \frac{\theta \Delta t}{1+\alpha}\left[\frac{\partial}{\partial x}\left\{-\mathbf{A}^n \Delta \mathbf{q}^n + \mathbf{P}\Delta \mathbf{q}^n + \mathbf{R}\Delta \mathbf{q}_x^n + \Delta \mathbf{U}_E^n\right\}\right. \\
+ & \left.\frac{\partial}{\partial y}\left\{-\mathbf{B}^n \Delta \mathbf{q}^n + \mathbf{Q}\Delta \mathbf{q}^n + \mathbf{S}\Delta \mathbf{q}_y^n + \Delta \mathbf{V}_E^n\right\} + \Delta \mathbf{H}^n\right] \\
+ & \frac{\Delta t}{1+\alpha}\left[\frac{\partial}{\partial x}\left\{-\mathbf{F}^n + \mathbf{U}_I^n + \mathbf{U}_E^n\right\} + \frac{\partial}{\partial y}\left\{-\mathbf{G}^n + \mathbf{V}_I^n + \mathbf{V}_E^n\right\} + \mathbf{H}^n\right] \\
+ & \frac{\alpha}{1+\alpha}\Delta \mathbf{q}^{n-1} + O\left(\left(\theta - \frac{1}{2} - \alpha\right)\Delta t^2,\, \Delta t^3\right) .
\end{aligned} \tag{20}$$

Note: $\Delta \mathbf{U}_E^n$ and $\Delta \mathbf{V}_E^n$ are left unchanged and are not linearized like the other functions for the following reason:

Suppose indeed that $\Delta \mathbf{U}_E^n$ and $\Delta \mathbf{V}_E^n$ were linearlized in the same fashion as in equation (18). Then, upon transferring the first term, i.e., $\frac{\theta \Delta t}{1+\alpha}[\quad]$, from the right-hand side to the left-hand side, an implicit system of equations for the unknown $\Delta \mathbf{q}^n$ would be set up. Furthermore, suppose a simple second-order central difference scheme is employed to discretize the system of equations. Then, for all terms other than $\frac{\partial}{\partial x}[\Delta \mathbf{U}_E]^n$ and $\frac{\partial}{\partial y}[\Delta \mathbf{V}_E]^n$, a five-point stencil would result; however, since the two terms above involve cross-derivatives, a nine-point stencil is created. (In three dimensions, it would be a nineteen-point stencil). The solution procedure would then be quite expensive. In order to circumvent this difficulty, the terms $\Delta \mathbf{U}_E^n$ and $\Delta \mathbf{V}_E^n$ are treated in the following manner:

Step 2:

A Taylor series expansion about t gives

$$\mathbf{U}_E^{n+1} = \mathbf{U}_E^n + \Delta t \left.\frac{\partial \mathbf{U}_E}{\partial t}\right|_n + O\left(\Delta t^2\right) ,$$

$$\mathbf{U}_E^{n-1} = \mathbf{U}_E^n - \Delta t \left.\frac{\partial \mathbf{U}_E}{\partial t}\right|_n + O\left(\Delta t^2\right) \ .$$

Hence, for a uniform time step,

$$\Delta \mathbf{U}_E^n = \Delta \mathbf{U}_E^{n-1} + O\left(\Delta t^2\right) \ ,$$

and, similarly,

$$\Delta \mathbf{V}_E^n = \Delta \mathbf{V}_E^{n-1} + O\left(\Delta t^2\right) \ . \tag{21}$$

In effect, the cross-derivative terms are handled "explicitly." With the simplification in equation (21), equation (20) now takes the following form:

$$\begin{aligned}
\Delta \mathbf{q}^n \ &- \frac{\theta \Delta t}{1+\alpha}\left[\frac{\partial}{\partial x}\left\{-\mathbf{A}^n \Delta \mathbf{q}^n + \mathbf{P}\Delta \mathbf{q}^n + \mathbf{R}\Delta \mathbf{q}_x^n\right\}\right. \\
&\left. + \frac{\partial}{\partial y}\left\{-\mathbf{B}^n \Delta \mathbf{q}^n + \mathbf{Q}\Delta \mathbf{q}^n + \mathbf{S}\Delta \mathbf{q}_y^n\right\}\right] \\
&= \ \mathbf{H}^{*n} + \ O\left(\left(\theta - \frac{1}{2} - \alpha\right)\Delta t^2 , \, \Delta t^3\right) \ ,
\end{aligned} \tag{22a}$$

where

$$\begin{aligned}
\mathbf{H}^{*n} = &\frac{\theta \Delta t}{1+\alpha}\left[\frac{\partial}{\partial x}\left(\Delta \mathbf{U}_E^{n-1}\right) + \frac{\partial}{\partial y}\left(\Delta \mathbf{V}_E^{n-1}\right) + \Delta \mathbf{H}^n\right] \\
&+ \frac{\Delta t}{1+\alpha}\mathbf{H}^n + \frac{\alpha}{1+\alpha}\Delta \mathbf{q}^{n-1} \ .
\end{aligned} \tag{22b}$$

Note: The forcing function $\mathbf{H}^{*n}$ of the temporally discretized equations is not the same as the physical forcing function $\mathbf{H}^n$.

An expansion of equation (22a) gives

$$\begin{aligned}
\Delta \mathbf{q}^n \quad - \quad & \frac{\theta \Delta t}{1+\alpha}\Big[- \mathbf{A}_x^n \Delta \mathbf{q}^n - \mathbf{A}^n \Delta \mathbf{q}_x + \mathbf{P}_x^n \Delta \mathbf{q}^n + \mathbf{P}^n \Delta \mathbf{q}_x \\
+ \quad & \mathbf{R}_x^n \Delta \mathbf{q}_x^n + \mathbf{R}^n \Delta \mathbf{q}_{xx}^n - \mathbf{B}_y^n \Delta \mathbf{q}^n - \mathbf{B}^n \Delta \mathbf{q}_y^n \\
+ \quad & \mathbf{Q}_y^n \Delta \mathbf{q}^n + \mathbf{Q}^n \Delta \mathbf{q}_y + \mathbf{S}_y^n \Delta \mathbf{q}_y^n + \mathbf{S}^n \Delta \mathbf{q}_{yy}^n \Big] = \mathbf{H}^{*n} \ .
\end{aligned}$$

Denote

$\delta_y,\, \delta_x$: generic first-order derivative

$\delta_{yy},\, \delta_{xx}$: generic second-order derivative

With the above notation, the discretized equation may be written compactly according to

$$\Big[\mathbf{I} + \frac{\theta\Delta t}{1+\alpha}\Big[(\mathbf{A}_x^n - \mathbf{P}_x^n) + (\mathbf{A}^n - \mathbf{P}^n - \mathbf{R}_x^n)\,\delta_x - \mathbf{R}^n\delta_{xx} + \left(\mathbf{B}_y^n - \mathbf{Q}_y^n\right) + \left(\mathbf{B}^n - \mathbf{Q}^n - \mathbf{S}_y^n\right)\delta_y - \mathbf{S}^n\delta_{yy}\Big]\Big]\Delta\mathbf{q}^n = \mathbf{H}^{*n}\ . \quad (23)$$

If three-point differences are used, the above discretization would lead to a block pentadiagonal system of equations.

Step 3: Approximate Factorization

Denote the operators $\mathbf{L}_x$ and $\mathbf{L}_y$ according to

$$\mathbf{L}_x = (\mathbf{A}_x^n - \mathbf{P}_x^n) + (\mathbf{A}^n - \mathbf{P}^n - \mathbf{R}_x^n)\,\delta_x - \mathbf{R}^n\delta_{xx}$$

$$\mathbf{L}_y = \left(\mathbf{B}_y^n - \mathbf{Q}_y^n\right) + \left(\mathbf{B}^n - \mathbf{Q}^n - \mathbf{S}_y^n\right)\delta_y - \mathbf{S}^n\delta_{yy}$$

Then, equation (23) may be written as

$$\left\{\mathbf{I} + \frac{\theta\Delta t}{1+\alpha}\left(\mathbf{L}_x + \mathbf{L}_y\right)\right\}\Delta\mathbf{q}^n = \mathbf{H}^{*n} + O\left(\left(\theta - \frac{1}{2} - \alpha\right)\Delta t^2,\, \Delta t^3\right)\ . \quad (24)$$

But,

$$\left\{\mathbf{I} + \frac{\theta\Delta t}{1+\alpha}\mathbf{L}_x\right\}\left\{I + \frac{\theta\Delta t}{1+\alpha}\mathbf{L}_y\right\} = \left\{\mathbf{I} + \frac{\theta\Delta t}{1+\alpha}\left(\mathbf{L}_x + \mathbf{L}_y\right)\right\} + O\left(\Delta t^2\right)\ .$$

Since $\Delta\mathbf{q}^n \sim O(\Delta t)$, equation (24) may be written as

$$\left\{\mathbf{I} + \frac{\theta\Delta t}{1+\alpha}\mathbf{L}_x\right\}\left\{I + \frac{\theta\Delta t}{1+\alpha}\mathbf{L}_y\right\}\Delta\mathbf{q}^n = \mathbf{H}^{*n} + O\left(\left(\theta - \frac{1}{2} - \alpha\right)\Delta t^2,\, \Delta t^3\right)\ . \quad (25)$$

Equations (24) and (25) have the same formal temporary accuracy.

The factorization in equation (25) permits an efficient solution procedure consisting of a set of two block-tridiagonal system which is solved very efficiently by the Block Thomas algorithm. Specifically, solve

$$\left\{\mathbf{I} + \frac{\theta \Delta t}{1+\alpha}\mathbf{L}_x\right\}\Delta\mathbf{q}^* = \mathbf{H}^{*n} \ ,$$

followed by

$$\left\{\mathbf{I} + \frac{\theta \Delta t}{1+\alpha}\mathbf{L}_y\right\}\Delta\mathbf{q}^n = \Delta\mathbf{q}^* \ .$$

3 Summary

- Cross-derivative terms are expensive to handle implicitly.
- If possible, it is best to handle them "explicitly" by moving them over to the right-hand side as demonstrated by the Beam and Warming Algorithm.
- A linearized (frozen coefficient assumption) stability analysis has shown the above algorithm to be stable for $\alpha = 1/2$. However, stable calculations for nonlinear equations is not guaranteed. This is usually a problem when nonorthogonal grids are used where the cross-derivative terms may not be small. Furthermore, stability requirements are far more stringent in three dimensions.
- The calculations at the first time step cannot be $O\left(\Delta t^2\right)$.
- Typically, the introduction of artificial dissipation is necessary to stabilize the calculations and suppress non-physical oscillations in the solution. The discussion of these models is beyond the scope of this article.

4 References

1. Beam, R. M. and Warming, R. F. (1978) "An Implicit Factored Scheme for the Compressible Navier-Stokes Equations," *AIAA Journal*, Vol. 16, No. 4, pp. 393-402.