RECENT ADVANCES IN WALL-BOUNDED SHEAR FLOWS

Survey of contributions from the USA

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Abstract

Recent advances in the application of asymptotic methods to the analysis of turbulent wall-bounded shear flows in the limit of large Reynolds number are surveyed. In particular, we consider *(i)* the turbulent wake downstream of sharp and wedge-shaped trailing edges, *(ii)* turbulent boundary-layer flow under the influence of a large and adverse pressure gradient and the related turbulent separation problem, and *(iii)* three-dimensional turbulent boundary-layer flow. The above are discussed in the context of incompressible flow. The emphasis is on contributions from the USA; however, related work elsewhere is also discussed.

1. Introduction

It is convenient to begin the discussion on wall-bounded shear flows by summarizing the well-known structure of the two-dimensional attached turbulent boundary layer first. It is assumed here that all lengths and velocities are nondimensionalized by the characteristic length of the body and a representative external velocity; the Reynolds number Re is based on these two quantities and the kinematic viscosity. The boundary-layer edge velocity and wall shear stress are denoted by U_e and τ_w, respectively, and the friction velocity is given by $u_\tau = \sqrt{\tau_w}$. It is well-known (Fendell, 1972;

Mellor, 1972; Yajnik, 1970) that the flow is two-tiered with the streamwise velocity u and Reynolds stress $-\overline{u'v'}$ expanding in the outer layer according to

$$u = U_e\left[1 + u_*\frac{\partial F_1}{\partial \eta} + \cdots\right], \qquad -\overline{u'v'} = u_\tau^2\sigma + \cdots; \qquad \eta = \frac{y}{\Delta_o}, \tag{1}$$

where the parameter u_* is formally of $O(1/\log Re)$ and is defined (Fendell, 1972) according to $u_* = u_\tau/U_e$ for two-dimensional flow. The normal distance from the wall is y and, for the attached flow considered here, the outer-layer thickness $\Delta_o \sim O(1/\log Re)$.

In the wall layer of thickness $O(Re^{-1}u_*^{-1})$, where to leading order the total shear stress is constant, the streamwise velocity and Reynolds stress expand as

$$u = u_\tau U^+(y^+), \qquad -\overline{u'v'} = u_\tau^2\sigma^+(y^+); \qquad y^+ = Reu_\tau y. \tag{2}$$

It is well-accepted that in the overlap region between the two layers,

$$\frac{\partial F_1}{\partial \eta} \sim \frac{1}{\kappa}\log \eta + C_o \quad\ \eta \to 0; \quad\ U^+(y^+) \sim \frac{1}{\kappa}\log y^+ + C_i \quad\ y^+ \to \infty, \tag{3}$$

where κ is the von Kármán constant; this yields the match condition given by

$$\frac{1}{u_*} = \frac{1}{\kappa}\log(ReU_e u_*\Delta_o) + C_i - C_o. \tag{4}$$

2. Wake Flow

We first consider the nature of the turbulent wake flow downstream of a sharp trailing edge. The flow past a finite-length flat plate aligned with the external flow provides a simple configuration for analysis. Following Bogucz and Walker (1988), it is convenient to define a small parameter $\epsilon \sim O(1/\log Re)$ with the value of u_* at the trailing edge. The sudden change in the wall boundary condition at the trailing edge of the flat plate is expected to produce a flow structure in which the oncoming outer layer splits into the near inner and outer wakes. Consider the inner wake first where Bogucz and Walker (1988) define a normal scaled variable z according to $z = y/\Delta_n(x)$, where x denotes the streamwise distance with the origin at the trailing edge; thus Δ_n is a measure of the growth of the near inner wake. Noting that $U_e = 1$, the asymptotic form of the outer-layer velocity in terms of z is

$$u \sim 1 + \epsilon\left[\frac{1}{\kappa}\log\left(\frac{\Delta_n}{\Delta_o}\right) + \frac{1}{\kappa}\log z + C_o\right] \qquad z \to \infty, \tag{5}$$

suggesting that the expansion of u in the inner wake is given by

$$u = 1 + \epsilon \left[U_o(x) + \frac{\partial f}{\partial z} \right] + \cdots. \qquad (6)$$

The expansion above is a slight departure from that used by Bogucz and Walker (1988), but, to the order considered here, the results are identical (see also Neish and Smith (1988)). Retaining the expansion for the Reynolds stress in 1, the governing equation is then given by

$$\frac{\partial \sigma}{\partial z} + \frac{\Delta_n'}{\epsilon} z \frac{\partial^2 f}{\partial z^2} - \frac{\Delta_n U_o'}{\epsilon} = \frac{\Delta_n}{\epsilon} \frac{\partial^2 f}{\partial z \partial x}. \qquad (7)$$

Self-similarity is invoked (see also Alber (1980)) which requires the coefficients in the second and third terms to be constants. Assuming $\Delta_n'/\epsilon = \kappa$, integration gives

$$\Delta_n = \epsilon \kappa (x - x_o), \qquad (8)$$

where Bogucz and Walker (1988) show $x_o \sim O(Re^{-1}\epsilon^{-1})$. To leading order, 8 shows that the near inner wake grows linearly, and, from 5,

$$U_o \sim \frac{1}{\kappa} \log \left[\frac{\epsilon}{\Delta_o} \kappa x \right] \qquad x > 0, \qquad (9)$$

indicating that the centerline velocity increases logarithmically with distance from the trailing edge. With the result in 8, the similarity equation becomes

$$\sigma' + \kappa z f'' = 1, \qquad (10)$$

subject to

$$f(0) = f''(0) = \sigma(0) = 0, \qquad f' \sim \frac{1}{\kappa} \log z + C_o, \quad \sigma \sim 1 \quad z \to \infty, \qquad (11)$$

where the symmetry condition and a match with the upstream boundary layer is used to determine the boundary conditions. The validity of the expansion in 6 as $x \to 0$ may be determined from 9; it follows from the match condition in 4 that the expansion is valid only for $x \gg O(Re^{-1}\epsilon^{-1})$ and thus a local region of extent $O(Re^{-1}\epsilon^{-1}) \times O(Re^{-1}\epsilon^{-1})$ at the trailing edge is required to smooth out the nonuniformity in the solution and no further simplifications are possible in the governing equations in this region.

Next consider the outer wake where the streamwise velocity expands regularly and Bogucz and Walker (1988) show that

$$u = 1 + \epsilon \left[F_1' + x \frac{\epsilon}{\Delta_o} (-\eta F_1'') + \cdots \right]. \qquad (12)$$

Note that as $\eta \to 0$, the defect function $F_1' \sim (1/\kappa)\log\eta + C_o$ thus matching with the logarithmic behavior in the inner wake solution in 10. Since the inner wake thickness Δ_n increases linearly with x, the logarithmic overlap region between the outer and inner wake begins to lift up; see also discussion below. The normal velocity is shown to be given by

$$v = -\epsilon^2 \left[1 + F_1 - \eta F_1'\right], \tag{13}$$

which results in a nonuniformity because in the upstream boundary layer, $v = \epsilon^2(\eta F_1' - F_1)$ and therefore $v = \epsilon^2$ as $\eta \to \infty$ since $\eta F_1' \to 0, F_1 \to -1$ as $\eta \to \infty$; the latter follows from the definition of the outer boundary-layer scale (Fendell, 1972) $\Delta_o = \delta^*/u_*$, where δ^* is the displacement thickness. However, in the wake, $v = 0$ as $\eta \to \infty$ to the order considered here. This nonuniformity is resolved in the usual way by consideration of weak-interaction whereby a second-order $O(\epsilon^2)$ correction to the external flow field is obtained followed by the introduction of perturbation quantities for the velocities and pressure of $O(\epsilon^2)$ in the outer layer. Bogucz and Walker (1988) are thus able to demonstrate that the nonuniformity in the normal velocity may be resolved within the scale of the boundary layer in a square region $O(\epsilon) \times O(\epsilon)$ centered at the trailing edge.

The structure of the flow discussed thus far is consistent with the results obtained by Alber (1980), Neish and Smith (1988) and Bogucz and Walker (1988); however, it is in the treatment of the inner wake that differences arise and is due to the different turbulence models adopted. Alber (1980) and Bogucz and Walker (1988) adopt a simple model for the eddy viscosity ε^* for the flat-plate outer-layer flow defined by

$$\varepsilon^* = u_* \Delta_o \varepsilon; \qquad \varepsilon = \begin{cases} \alpha & \eta \geq \frac{\alpha}{\kappa} \\ \\ \kappa\eta & \eta < \frac{\alpha}{\kappa} \end{cases}, \tag{14}$$

where α is the constant outer-layer Clauser (1956) eddy viscosity and assumed to be $\alpha = 0.016$ (Mellor and Gibson, 1966). The model for the outer layer over the flat plate must merge smoothly with the near-wake model downstream, and, for $\eta \to 0$, the eddy viscosity in terms of the inner-wake variables is given by $\varepsilon^* = \epsilon\Delta_n\kappa z$. Thus, $\sigma = \kappa z f''$ which is consistent with the boundary conditions as $z \to \infty$ in 11. Alber (1980) assumes that the linear variation in the eddy viscosity continues all the way to the centerline and, in this case, it follows from 10 that $f' \sim a_o + \kappa^{-1} z$ as $z \to 0$ thus violating the symmetry condition. On the other hand, Bogucz and Walker (1988) argue that this behavior is inconsistent and propose a model for the inner wake given by

$$\varepsilon^* = \epsilon\Delta_n\varepsilon_w; \qquad \varepsilon_w = \begin{cases} \kappa z & z \geq \hat{\alpha} \\ \kappa\hat{\alpha} & z < \hat{\alpha} \end{cases}, \tag{15}$$

where $\hat{\alpha}$ is a constant and Bogucz and Walker (1988) recommend a value $\hat{\alpha} = 0.6$ based on comparison of calculated similarity profiles with experimental data. The model in 15 assures the satisfaction of the symmetry condition at the centerline. Since $\Delta_n \sim \epsilon\kappa x$, it is clear that the extent of linear variation in eddy viscosity decreases with downstream distance thus diminishing the extent of the logarithmic variation in the wake velocity until at $x \sim (\Delta_o \alpha)/(\epsilon\kappa^2\hat{\alpha})$, it is completely extinguished. Unfortunately, the near-wake model does not asymptote to the model appropriate for the far wake thus calling for an additional region patching the near and far wakes.

In constrast to the modeling above, Neish and Smith (1988) use the Cebeci-Smith (1974) mixing-length model for the upstream surface layer. The continuation of this model in the inner wake yields $\sigma = \kappa^2 z^2 |f''| f''$ and this form is assumed throughout the inner wake. Although the asymptotic condition as $z \to \infty$ is satisfied, it may be inferred from 10 that $f' \sim a_o + 2\kappa^{-1} z^{1/2}$ as $z \to 0$. However, this irregular behavior is adjusted through what is termed the 'cuspidal layer' that lies between the inner wake and the centerline and is of $O(Re^{-2/3}\epsilon^{-1/3})$ thickness. Upon defining

$$u = 1 + \epsilon(U_o + a_o) + Re^{-1/3}\epsilon^{1/3}x^{-1/3}\hat{u}(\hat{y}); \qquad y = Re^{-2/3}\epsilon^{-1/3}x^{1/3}\hat{y}, \quad (16)$$

the governing equation in the cuspidal layer is given by

$$\frac{1}{\kappa} = \frac{\partial}{\partial \hat{y}}\left[\frac{\partial \hat{u}}{\partial \hat{y}} + \kappa^2 \hat{y}^2 (\frac{\partial \hat{u}}{\partial \hat{y}})^2\right], \qquad \frac{\partial \hat{u}}{\partial \hat{y}} = 0 \quad \hat{y} = 0, \qquad \hat{u} \sim \frac{2}{\kappa^{3/2}}\hat{y}^{1/2} \quad \hat{y} \to \infty, \quad (17)$$

where the last condition arises from matching the inner limit of the near inner wake solution.

Unlike the studies discussed earlier on the wake flow past a flat plate, Melnik and Grossman (1982) consider the nonuniformity at the wedge-shaped trailing edge of an airfoil profile due to a singularity in the outer external flow. The airfoil profile is assumed to be thin, symmetric and nonlifting and is characterized by a thickness ratio t that is assumed small. In order to focus on the salient features of the flow, they limit their attention to incompressible flow.

Consider a velocity expansion according to

$$\begin{aligned} u(x,y) &= [1 + tu_{01} + \cdots] + \epsilon^2[u_{20} + tu_{21} + \cdots], \\ v(x,y) &= [tv_{01} + \cdots] + \epsilon^2[v_{20} + tv_{21} + \cdots]. \end{aligned} \quad (18)$$

The normal velocities $v_{01}(x,0)$ and $v_{20}(x,0)$ represent the effective transpiration velocities due to the profile shape and the viscous displacement, respectively. These boundary conditions determine the external perturbation field and, in particular, Melnik and Grossman (1982) show that

$$u_{01}(x,0) = \frac{\theta_{te}}{\pi}(\log|x| + a_1) \qquad |x| \to 0, \tag{19}$$

where the origin in x is assumed to be at the trailing edge; $t\theta_{te}$ is the trailing wedge angle and a_1 is a constant dependent on the profile shape. This logarithmic singularity in turn causes stronger singularities in v_{21} and u_{21}, thus rendering the expansion in 18 inappropriate near the trailing edge. Moreover, the pressure drop across the boundary layer is shown to be of $O(\epsilon^3 t x^{-2})$ which becomes increasingly important as $|x| \to 0$. Their subsequent analysis is concerned with the elimination of the above inviscid singularity by considering viscid-inviscid interaction. To this end, two methodologies are employed: *(i)* Interactive boundary-layer theory (IBLT), and *(ii)* Strong-Interaction theory (SI).

As is customary in IBLT, the external inviscid and boundary-layer equations are coupled to produce solutions for the pressure and displacement thickness, but the normal momentum equation is ignored. The essential idea that Melnik and Grossman (1982) pursue is to seek a solution for the displacement-thickness induced velocity that exactly cancels the logarithmic singularity in the profile induced velocity in 19. Thus, both components of the perturbation streamwise velocity are required to be of $O(t)$ and their analysis indicates that this may only be accomplished within a local streamwise length scale that is of $O(\epsilon^2)$. Melnik and Grossman (1982) are able to show that the logarithmic singularity in the external flow may be eliminated; however, they find that the normal pressure-gradient effect is significant and, since it cannot be taken into account within the framework of IBLT, the theory remains incomplete in correctly describing the wedge trailing-edge flow.

This situation is rectified within the SI theory. Within this framework, the structure of the flow consists of three layers of streamwise extent $O(\epsilon)$ near the trailing edge. In the main deck of normal thickness $O(\epsilon)$, the governing equations include the normal momentum equation but the shear stress does not enter to leading order. Below the main deck is a 'blending' layer of thickness $O(\epsilon^2)$, and, finally, the innermost wall layer of thickness $O(Re^{-1}\epsilon^{-1})$. Also, a square region $O(Re^{-1}\epsilon^{-1}) \times O(Re^{-1}\epsilon^{-1})$ at the trailing edge is required and serves the same function as discussed for the flat plate flow above. Since the shear stress does not enter to leading order, the solution in the main deck may be obtained without recourse to any turbulence modeling. Melnik and Grossman (1982) construct an analytical solution that matches the upstream oncoming flow.

3. Turbulent Separation

We know consider the challenging problem of determining the structure of a turbulent boundary layer undergoing incipient separation. The theories put forth in the studies discussed here differ from one another considerably attesting to the difficulty of analyzing such flows. We first discuss the theory developed by Melnik (1989) that results from an asymptotic analysis in terms of a two-parameter expansion, viz. $\alpha, \epsilon \to 0$ but where $\epsilon\alpha^{-1/2} \sim O(1)$. The unusual aspect of this study is that the small parameter α is identified with the 'constant' in the algebraic Clauser (1956) model for the outer region of the turbulent boundary layer. At the outset, the Clauser eddy viscosity model along with the mixing-length formulation closer to the wall is adopted. The primary objective in Melnik's (1989) approach is to determine a solution that is uniformly valid in the attached as well as farther downstream on approach to separation. To this end, Melnik (1989) introduces an additional region which is termed the 'equilibrium layer' of thickness $O(\alpha^{3/2})$ between the classical outer and wall layers. The flow in the new outer layer is no longer considered to be representable as a sum of the freestream velocity and small defect; rather the flow is governed by the full nonlinear equations with turbulence modeled by the Clauser eddy viscosity given by $\nu_T = U_e\delta^*\alpha$ throughout the extent of the new outer layer. Thus, the outer solution is smooth as the wall is approached and, in general, results in a nonzero slip velocity. Note that Melnik's (1989) approach of introducing an additional layer was also used previously by Sychev (1987); however, in this case, a defect form for the velocity was considered to be valid in the outer layer with an intermediate layer of thickness $O(\epsilon^2)$.

In the thinner equilibrium layer below the outer layer, Melnik (1989) expands the velocity in a defect form about the nonzero slip velocity according to

$$u = u_s[1 + \alpha^{1/2}\gamma_s U(Y)]; \qquad y = \alpha^{3/2}\gamma_s b(x)Y, \tag{20}$$

where $\alpha^{1/2}\gamma_s u_s = u_\tau$. The expansion in 20 ensures that the patch point in the eddy viscosity lies within the equilibrium layer, and the factor $b(x)$ is introduced to make it coincide with $Y = 1$. The Reynolds stress is expanded as

$$-\overline{u'v'} = \alpha u_s^2\gamma_s^2 G(Y), \qquad G \to 1 \quad Y \to 0, \tag{21}$$

where the limiting form for G is obtained from a match with the constant shear stress wall layer. With the above expansions, the governing equation is given by

$$G = 1 + \alpha^{1/2}\Lambda(x)Y + O(\alpha), \tag{22}$$

where

$$G = \kappa^2 Y^2 \left|\frac{\partial U}{\partial Y}\right| \frac{\partial U}{\partial Y} \quad Y \leq 1, \qquad \Lambda(x) = \frac{b}{2\gamma_s u_s^2} \frac{d}{dx}\left[u_s^2 - U_e^2\right]. \tag{23}$$

An equation for $Y \geq 1$ is also obtained but is not needed for the discussion here. Since $\alpha \rightarrow 0$, the above equation indicates that the equilibrium layer may be characterized as a constant shear stress layer. However, Melnik (1989) argues that as separation is approached, $u_s \rightarrow 0$ and at some downstream location $\Lambda \sim O(\alpha^{-1/2})$. It is maintained that by assuming $\lambda = \alpha^{1/2}\Lambda \sim O(1)$, a solution that is uniformly valid in the attached as well as separating boundary layer may be obtained. The leading-order solutions in these regions are then determined by taking the limits $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$, respectively. As λ increases from zero, the equilibrium layer is transformed from a constant to a linearly-varying shear stress layer. Integration of 22 with $\lambda = \alpha^{1/2}\Lambda \sim O(1)$ gives

$$\begin{aligned} U(Y) = \frac{1}{\kappa}\Bigg[&\log\frac{Y}{\sqrt{1+\lambda}} + 2\left(\sqrt{1+\lambda Y} - 1\right) - \\ &2\log\left(\frac{\sqrt{1+\lambda Y}+1}{2}\right) - D(\lambda)\Bigg], \qquad Y \leq 1, \end{aligned} \tag{24}$$

where $D(\lambda)$ is a known, although cumbersome, function of λ. The limiting solution as $Y \rightarrow 0$ with $\lambda \sim O(1)$ is given by $\kappa^{-1}[\log Y - D(\lambda)]$ and is matched to the log-law velocity in the thin wall layer which results in an implicit match condition for the scaled friction velocity γ_s.

It is of interest to determine the limiting behavior of the solution in the attached ($\lambda \rightarrow 0$) and separating ($\lambda \rightarrow \infty$) regions. Melnik (1989) shows that

$$U(Y) \sim \frac{1}{\kappa}[\log Y + 1] + O(\alpha), \qquad \lambda \rightarrow 0, \tag{25}$$

$$U(Y) \sim \frac{2\sqrt{\lambda}}{\kappa}\left[Y^{1/2} - \frac{3}{4}\right] + O(\lambda^{-1/2}), \qquad \lambda \rightarrow \infty. \tag{26}$$

Note that the half-power law variation in the latter is not surprising since this behavior follows from the assumption of a linearly-varying shear stress layer and a mixing-length formulation for the eddy viscosity. A solution for $Y \geq 1$ is also obtained and matched to the outer-layer solution. Melnik (1989) shows that the appropriate boundary conditions that arise for the outer solution are

$$\overline{u}(x, \overline{y} = 0) = u_s, \quad \left.\frac{\partial \overline{u}}{\partial \overline{y}}\right|_{\overline{y}=0} = \frac{u_s^2\gamma_s^2}{U_e\overline{\delta}^*}, \quad y = \alpha\overline{y}, \quad \delta^* = \alpha\overline{\delta}^*. \tag{27}$$

The boundary conditions in 27 thus establishes an iterative procedure whereby the wall shear stress may be calculated by considering the numerical solution of the outer layer only. Finally, Melnik (1989) considers the limit $u_s \to 0$ to analyze the separation singularity in the outer layer and determines a Goldstein-like singularity in u_s, and establishes $x \sim O(\alpha^{1/2})$ as the streamwise scale of turbulent separation.

Durbin and Belcher (1992) also propose a three-layered structure for what is termed an adverse pressure gradient (APG) boundary layer appropriate for flow just upstream of separation on a streamwise length scale of $L_p^{-1} = -U_e^{-1} U_e'$. The expansions are carried out in terms of a small parameter denoted here by ϵ_p and defined by

$$\epsilon_p = \frac{1}{(ReU_eL_p)^{1/3}}. \tag{28}$$

The outer layer is of $O(L_p\epsilon_p)$ where, in agreement with Melnik (1989), the velocity defect is not considered small and the full nonlinear equations are used. In contrast, however, the layer adjacent to the wall is postulated to be only $O(\epsilon_p)$ smaller than the outer layer and involves a balance of shear stress and pressure-gradient terms and is thus a layer of linearly-varying shear stress. In this inner layer, the asymptotic behavior of the streamwise velocity is assumed to be given by

$$\frac{u}{U_e} \sim \epsilon_p A_u \hat{y}^{1/2} \qquad \hat{y} \to \infty; \qquad y = L_p \epsilon_p^2 \hat{y}, \tag{29}$$

which is shown not to match the $O(1)$ velocity in the outer layer necessitating the inclusion of an intermediate layer of thickness $O(L_p\epsilon_p^{4/3})$. This layer is also a region of linearly-varying shear stress where the velocity is of $O(\epsilon_p^{2/3})$.

The structure of the flow, in particular the implicit formula for the evaluation of the skin friction which involves higher-order terms, is highly dependent on the turbulence model adopted. A k–ε model is used in the innermost layer, a constant Clauser eddy-viscosity in the outer layer, and an interpolation between these models in the intermediate layer. Durbin and Belcher (1992) do not address the issue of asymptotically matching the oncoming attached boundary layer to their proposed structure and associated questions on the streamwise length scale over which this may be accomplished and the manner in which the logarithmic variation in the streamwise velocity is suppressed.

The above analyses assumed that the outer layer is governed by the full nonlinear equations. It is perhaps instructional to review Neish and Smith's (1992) analysis in order to gain an insight into how an upstream small defect outer layer may be transformed into a nonlinear region farther downstream. Let the external velocity approach a stagnation point at $x = 1$ according to

$$U_e \sim \hat{\lambda}(1-x) \qquad x \to 1^-. \tag{30}$$

With the expansions for the outer layer given in 1, the governing equations are given by (Fendell, 1972)

$$\frac{\partial \sigma}{\partial \eta} + \frac{1}{u_*}\frac{d\Delta_o}{dx}\eta\frac{\partial^2 F_1}{\partial \eta^2} + \frac{\Delta_o}{U_e u_*}\frac{dU_e}{dx}\left[\eta\frac{\partial^2 F_1}{\partial \eta^2} - 2\frac{\partial F_1}{\partial \eta}\right] = \frac{\Delta_o}{u_*}\frac{\partial^2 F_1}{\partial x \partial \eta}. \tag{31}$$

Integration of the above equation across the boundary-layer thickness and using $\sigma \sim 1$, $F_1 \sim 0$ as $\eta \to 0$ and $\sigma \sim 0$, $\eta F_1' \sim 0$, $F_1 \sim F_{1\infty}$ as $\eta \to \infty$ along with the first of 3, it may be shown that $-(U_e^3 \Delta_o F_{1\infty})' = u_* U_e^3$. Integration in x along with the use of 30 yields

$$-\Delta_o F_{1\infty} \sim \frac{E u_*}{\hat{\lambda}^3(1-x)^3} \qquad x \to 1^-, \tag{32}$$

where E is a constant and represents the contribution farther upstream where $U_e \sim O(1)$. From the first of 1, it follows that $\delta_* = -F_{1\infty}\Delta_o u_*$ and if a Clauser eddy-viscosity model is adopted for the outer layer, then $\sigma = -\alpha F_{1\infty} F_1''$. Along with the result in 32, it follows from 31 that a turbulence-inertial balance is possible if

$$\Delta_o \sim u_*(1-x)^{-1}; \quad F_1 \sim (1-x)^{-2} \qquad x \to 1^-. \tag{33}$$

Note that the latter indicates a strong increase in the defect function. Therefore, the expansion for the velocity in 3 ceases to be valid when $(1-x) \sim O(u_*^{1/2})$. Along with the first of 33, this implies that a square region of size $O(u_*^{1/2}) \times O(u_*^{1/2})$ comes into play in which the external velocity is of $O(u_*^{1/2})$. Noting that $u_* \sim O(\epsilon)$, one may formally define $((x-1), y) = (\epsilon^{1/2}\overline{X}, \epsilon^{1/2}\overline{Y})$ and $(u, v, p) = (\epsilon^{1/2}\overline{U}, \epsilon^{1/2}\overline{V}, \epsilon\overline{P})$. Therefore, the governing equations in the new region are not only nonlinear but also include the normal momentum equation. Upon requiring that $\overline{V} = 0$ at $\overline{Y} = 0$ results in a slip velocity $\overline{U}(\overline{X}, 0) = \overline{U}_s$.

Next consider the upstream defect layer where

$$u = U_e\left[1 + u_*\left(\frac{1}{\kappa}\log\eta + C_o\right)\right] \qquad \eta \to 0. \tag{34}$$

Since $F_1 \to (1-x)^{-2}$ as $x \to 1^-$, then the x-dependence must be reflected in C_o in a similar fashion, i.e. $C_o \sim (1-x)^{-2}$. Formally defining $\overline{C}_o = u_* C_o$

and expressing the quantities in 34 in terms of the new variables, it is seen that to leading order,

$$\overline{U} = \overline{U}_e(1 + \overline{C}_o) \qquad \overline{Y} \to 0, \tag{35}$$

which is the required slip velocity; the logarithmic behavior is relegated to a higher-order effect. This argument proposed by Neish and Smith (1992) provides a possible scenario whereby an oncoming small defect outer layer transforms to a nonlinear region eradicating the logarithmic behavior in the process. It may perhaps be of interest to investigate the structure of the flow for other limiting forms of the external velocity.

4. Three-Dimensional Boundary Layers

The three-dimensional boundary layer is best described in a streamline coordinate system. At the edge of the boundary layer, the external velocity U_e is aligned with the streamwise coordinate x_1. The cross-stream and normal coordinates, x_2 and x_3, respectively, complete the orthogonal system. In general, the external streamline is curved which creates a cross-stream pressure gradient and, under its influence, a cross-stream velocity component u_2 develops within the boundary layer. Consequently, the velocity vector rotates away from its direction at the boundary-layer edge. Both components of the velocity must satisfy the no-slip condition at the wall and, therefore, the cross-stream velocity attains its maximum within the boundary layer. The velocity skew angle is denoted by θ, and its value at the wall, θ_w, is the wall skew angle.

Goldberg and Reshotko (1984) were the first to conduct an asymptotic analysis of the three-dimensional turbulent boundary layer by extending the two-dimensional theory of Mellor (1972). The wall-layer quantities were expanded in powers of ϵ and thus pressure-gradient effects, which are formally of $O(Re^{-1}\epsilon^{-3})$, were not taken into account. It was concluded that the wall layer is a region of constant shear stress to all orders. Moreover, it was postulated that the maximum in the cross-stream velocity lies in the outer layer and an empirical formula to represent the characteristic 'bulge' in the cross-stream velocity profile was proposed.

Subsequently, Degani *et al.* (1992; 1993) also analyzed the three-dimensional turbulent boundary layer and found that the structure of the streamwise velocity u_1 is similar to that in two-dimensional flow. In the outer layer,

$$u_1 = U_e\left[1 + u_*\frac{\partial F_1}{\partial \eta} + \cdots\right]; \qquad \frac{\partial F_1}{\partial \eta} \sim \frac{1}{\kappa}\log\eta + C_o \quad \eta \to 0, \tag{36}$$

and, in the wall layer,

$$u_1 = U_e u_* U^+, \qquad U^+ \sim \frac{1}{\kappa} \log y^+ + C_i \qquad y^+ \to \infty, \tag{37}$$

where the scaled friction velocity u_* is now given by

$$u_* = \frac{u_\tau \cos\theta_w}{U_e}. \tag{38}$$

The resulting match condition is identical to that in 4 but with u_* given by 38. On the other hand, the structure of the cross-stream velocity was found to be more involved and an important result of the analysis is that, for attached flow, the wall skew angle scales according to

$$\tan\theta_w = \theta_* u_*, \qquad u_* \sim O(\frac{1}{\log Re}), \quad \theta_* \sim O(1), \tag{39}$$

which indicates that $\theta_w \sim O(1/\log Re)$; θ_* is the scaled wall skew angle that is typically obtained as part of the boundary-layer solution. Results similar to those in 4, 38 and 39 were obtained by Spalart (1989) who considered a model problem in which the external velocity remains constant in magnitude but rotates at a uniform frequency. The cross-stream velocity in the wall layer is expanded according to

$$u_2 = u_\tau \sin\theta_w U^+ = U_e u_*^2 \theta_* U^+. \tag{40}$$

In order to match the wall-layer velocity in 40, an expansion to two orders is necessary in the outer layer where u_2 is given by

$$u_2 = U_e u_* \theta_* \left[\frac{\partial G_1}{\partial \eta} + u_* \frac{\partial G_2}{\partial \eta} + \cdots \right], \tag{41}$$

with

$$\frac{\partial G_1}{\partial \eta} \sim 1, \quad \frac{\partial G_2}{\partial \eta} \sim \frac{1}{\kappa} \log \eta + C_o \qquad \eta \to 0. \tag{42}$$

The above expansions and asymptotic forms for the cross-stream velocity lead naturally to the characteristic profile shape. Specifically, in the outer layer close to the boundary-layer edge, the cross-stream velocity u_2 is dominated by the contribution from the leading-order term, $\partial G_1/\partial\eta$, which continuously increases in magnitude with decreasing distance from the wall. As the overlap region between the outer and wall layers is approached, the second-order term, $\partial G_2/\partial\eta$, begins to make an increasingly negative contribution until the cross-stream velocity reaches its maximum value. A further decrease in distance from the wall reduces the sum of the $O(u_*)$

and $O(u_*^2)$ terms to $O(u_*^2)$ in the wall layer in much the same fashion as the sum of the $O(1)$ and $O(u_*)$ terms reduce the streamwise velocity in the outer layer to $O(u_*)$ in the wall layer.

It thus follows from the above argument that the maximum in the cross-stream velocity is neither in the outer nor in the wall layers but within the overlap region between the two layers. A precise length scale of the location of the maximum from the wall may be obtained by assuming that the cross-stream shear stress may be related to the normal cross-stream velocity gradient through an eddy viscosity; it then follows that the maximum in u_2 lies at the point where the cross-stream shear stress is zero. With this, Degani *et al.* (1992; 1993) are able to show that

$$y_{max} \sim O\left(\frac{\Delta_o u_*}{-\log u_*}\right), \tag{43}$$

where y_{max} denotes the normal distance from the wall to the location of the maximum in u_2. Furthermore, the maximum value of the cross-stream velocity is given by

$$\left.\frac{u_2}{U_e \tan\theta_w}\right|_{max} \sim 1 + \frac{1}{\kappa} u_* \log u_* + \cdots \qquad y \to y_{max}. \tag{44}$$

Pressure-gradient effects, which are formally of $O(Re^{-1}u_*^{-3})$ in the wall layer, are expected to have a greater effect on the cross-stream flow which is $O(u_*)$ smaller than the streamwise flow. Denoting the skew angle of the total shear stress (i.e. the sum of laminar and Reynolds stresses) by θ_s, it emerges that

$$\tan\theta_s = \tan\theta_w \left[1 + \frac{\beta_n}{\theta_*}\frac{1}{Re_{\delta^*} u_*^2} y^+ + \cdots\right]. \tag{45}$$

Here $Re_{\delta^*} = ReU_e\delta^*$ and β_n is a cross-stream pressure-gradient parameter defined by

$$\beta_n = -\frac{\delta^*}{u_\tau^2 \cos\theta_w} K_2 U_e^2, \tag{46}$$

where K_2 denotes the curvature of the external streamline. Furthermore, the velocity skew angle is shown to be given by

$$\tan\theta_v = \frac{u_2}{u_1} = \tan\theta_w \left[1 + \frac{\beta_n}{\theta_*}\frac{1}{Re_{\delta^*} u_*^2} f(y^+) + \cdots\right], \tag{47}$$

where $f(y^+)$ is an unknown function that may be determined upon assuming a turbulence model. If a linear eddy viscosity model is assumed, the function $f(y^+) \sim y^+/\log y^+$ as $y^+ \to \infty$; then 46 and 47 indicate that the total shear stress skews more rapidly than the velocity. Degani *et al.* (1992; 1993) argue that although the pressure-gradient effects are of higher order and tend to zero in the limit of infinite Reynolds number, they are not negligible at the large but finite Reynolds numbers encountered in practice.

5. Conclusions

In this survey, recent analyses and summary of the salient results obtained from the application of asymptotic methods to turbulent wall-bounded shear flows are highlighted and provide valuable insight into a fundamental understanding of such flows. It is evident that asymptotic methods enable the extraction of a fairly comprehensive set of results from the governing equations, and, in many instances, without incorporating a specific turbulence model *a priori*. This is exemplified by the analyses discussed here for the near-wake flow past an aligned flat plate and the three-dimensional boundary layer. The structure of turbulent separation is not clearly defined yet, but recent progress and interest in analyzing this flow is encouraging. It is imperative to continue this advance and also address the challenging problem of three-dimensional turbulent separation due to its considerable practical significance.

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