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Entropy and the approach to the thermodynamic limit in three-dimensional simplicial gravity

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Abstract

We present numerical results supporting the existence of an exponential bound in the dynamical triangulation model of three-dimensional quantum gravity. Both the critical coupling and the number of nodes per unit volume show a slow power law approach to the infinite volume limit.

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Introduction

Much interest has been generated recently in lattice models for euclidean quantum gravity based on dynamical triangulations [1, 2, 3, 4, 5, 6, 7, 8]. The study of these models was prompted by the success of the same approach in the case of two dimensions, see for example [9]. The primary input to these models is the ansatz that the partition function describing the fluctuations of a continuum geometry can be approximated by performing a weighted sum over all simplicial manifolds or triangulations T.

$$Z = \sum_{T} \rho\left(T\right) \tag{1}$$

In all the work conducted so far the topology of the lattice has been restricted to the sphere S^d. The weight function $\rho\left(T\right)$ is taken to be of the form

$$\rho\left(T\right) = e^{-\kappa_d N_d + \kappa_0 N_0} \tag{2}$$

The coupling κ_d represents a bare lattice cosmological constant conjugate to the total volume (number of d-simplices N_d) whilst κ_0 plays the role of a bare Newton constant coupled to the total number of nodes N_0.

We can rewrite eqn. 1 by introducing the entropy function $\Omega_d\left(N_d, \kappa_0\right)$ which counts the number of triangulations with volume N_d weighted by the node term. This the primary object of interest in this note.

$$Z = \sum_{N_d} \Omega_d\left(N_d, \kappa_0\right) e^{-\kappa_d N_d} \tag{3}$$

For this partition sum to exist it is crucial that the entropy function Ω_d increase no faster than exponentially with volume. For two dimensions this is known [10] but the only evidence for this in higher dimensions has come from numerical simulation. The proof of a bound for three dimensions given by Boulatov [11] has been shown to be incorrect [12]. In four dimensions there is still some uncertainty in the status of this bound [13, 14, 8].

With this in mind we have conducted a high statistics study of the three dimensional model at $\kappa_0 = 0$, extending the simulations reported in [15] by an order of magnitude in lattice volume and with increased statistics. Whilst we observe a rather slow approach to the asymptotic, large volume limit, our results are entirely consistent with the existence of such a bound. This is our most important result. Furthermore, the measured mean node number per unit volume also shows strong finite volume effects. We will argue that the detailed nature of these provides a strong consistency check on our results for the bound.

If we write $\Omega_3\left(N_3\right)$ as

$$\Omega_3\left(N_3\right) = a e^{\kappa_3^c\left(N_3\right) N_3} \tag{4}$$

the effective critical cosmological constant κ_3^c is taken dependent on the volume and a bound implies that $\kappa_3^c \rightarrow \text{const} < \infty$ as $N_3 \rightarrow \infty$. In contrast for a model where the entropy grew more rapidly than exponentially κ_3^c would diverge in the thermodynamic limit.

To control the volume fluctuations we add a further term to the action of the form $\delta S = \gamma\left(N_3 - V\right)^2$. Lattices with $N_3 \sim V$ are distributed according to the correct Boltzmann weight up to correction terms of order $O\left(\frac{1}{\sqrt{\gamma V}}\right)$ where we use $\gamma = 0.005$ in all our runs. This error is much smaller than our statistical errors and can hence be neglected.

Likewise, as a first approximation, we can set κ_3^c equal to its value at the mean of the volume distribution V which allows us to compute the expectation value of the volume exactly since the resultant integral is now a simple gaussian. We obtain

$$\langle N_3\rangle = \frac{1}{2\gamma}\left(\kappa_3^3\left(V\right) - \kappa_3\right) + V \tag{5}$$

Equally, by measuring the mean volume $\langle N_3\rangle$ for a given input value of the coupling κ_3 we can estimate $\kappa_3^c\left(V\right)$ for a set of mean volumes V. The algorithm we use to generate a Monte Carlo sample of three dimensional lattices is described in [16]. We have simulated systems with volumes up to 128000 3-simplices and using up to 400000 MC sweeps (a sweep is defined as V *attempted* elementary updates of the triangulation where V is the average volume).

Our results for $\kappa_3^c\left(V\right)$, computed this way, are shown in fig. 1 as a function of $\ln V$. The choice of the latter scale is particularly apt as the presence of a factorial growth in Ω_3 would be signaled by a logarithmic component to the effective $\kappa_3^c\left(V\right)$. As the plot indicates there is no evidence for this. Indeed, the best fit we could make corresponds to a *convergent* power law

$$\kappa_3^c\left(V\right) = \kappa_3^c\left(\infty\right) + aV^{-\delta} \tag{6}$$

If we fit all of our data we obtain best fit parameters $\kappa_3^c\left(\infty\right) = 2.087(5)$, $a = -3.29(8)$ and $\delta = 0.290(5)$ with a corresponding χ^2 per degree of freedom $\chi^2 = 1.3$ at 22% confidence (solid line shown). Leaving off the smallest lattice $V = 500$ yields a statistically consistent fit with an even better $\chi^2 = 1.1$ at 38% confidence. We have further tested the stability of this fit by dropping either the small volume data ($V = 500 - 2000$ inclusive), the large volume data ($V = 64000 - 128000$ inclusive) or intermediate volumes ($V = 8000 - 24000$). In each of these cases the fits were good and yielded fit parameters consistent with our quoted best fit to all the data. Furthermore, these numbers are consistent with the earlier study [15]. We are thus confident that this power law is empirically a very reasonable parameterisation of the approach to the thermodynamic limit. Certainly, our conclusions must be that the numerical data *strongly* favour the existence of a bound.

One might object that the formula used to compute κ_3^c is only approximate (we have neglected the variation of the critical coupling over the range of fluctuation of the volumes). This, in turn might yield finite volume corrections which are misleading. To check for this we have extracted κ_3^c directly from the measured distribution of 3-volumes $Q\left(N_3\right)$. To do this we computed a new histogram $P\left(N_3\right)$

$$P\left(N_3\right) = Q\left(N_3\right)e^{\kappa_3 N_3 + \gamma\left(N_3 - V\right)^2} \tag{7}$$

As an example we show in fig. 2 the logarithm of this quantity as a function of volume for $V = 64000$. The gradient of the straight line fit shown is an unbiased estimator of the critical coupling $\kappa_3^c(64000)$. The value of 1.9516(10) compares very favourably with the value $\kappa_3^c(64000) = 1.9522(12)$ obtained using eqn. 5. Indeed, this might have been anticipated since we might expect corrections to eqn. 5 to be of magnitude $O\left(V^{-(1+\delta)}\right)$ which even for the smallest volumes used in this study is again much smaller than our statistical errors.

In addition, we have measured the mean node number per unit volume. We will argue that the finite volume corrections to this quantity are essentially determined by the behaviour of $\kappa_3^c(V)$. This follows from the usual rule $\langle N_0 \rangle = \frac{\partial \ln Z}{\partial \kappa_0}$ with Z replaced by $\Omega_3(V, \kappa_0)$ for our quasi-microcanonical simulations.

$$\langle N_0/V \rangle = \frac{\partial \kappa_3^c(V, \kappa_0)}{\partial \kappa_0} \tag{8}$$

Our data for this quantity are shown in fig. 3. From eqn.8 the finite volume corrections to this quantity should be similar to those of the critical coupling $\kappa_3^c(V)$. Specifically, if we attempt a power law fit to the data shown in fig. 3 we should find a power statistically consistent with that governing the approach to infinite volumes of the critical coupling. Initially we have fitted the data in two ways

$$\langle N_0/V \rangle = b + cV^{-d} \tag{9}$$

In the first the parameter $b = \frac{\partial \kappa_3^c(\infty)}{\partial \kappa_0}$ is set to zero and we fit for only two parameters $c = \frac{\partial a}{\partial \kappa_0}$ and d (using the notation introduced in eqn.6). These fits appear to be rather poor – even leaving off all the data for lattices with volume $V \leq 32000$ the resultant fit $c = 0.572(8)$, $d = -0.296(1)$ has a $\chi^2/\text{dof} = 7.0$. The problem is the rather steep rise at small volumes which is inconsistent with the long tail. Notice, though that the power d is rather close to that obtained from the critical coupling (which we have called δ). This may be taken as an argument in favour of the fit – the poor χ^2 can be interpreted as resulting from the presence of rather large subleading corrections at most of the lattice volumes we could reach. As an alternative we have fitted with the b-parameter left free. In this case acceptable fits are obtained by taking data from lattices $V \geq 8000$ – our best fit yields $b = 0.0045(1)$, $c = 1.14(2)$, $d = -0.380(3)$ at a χ^2 per degree of freedom of $\chi^2 = 1.6$. Fits to subsets of the large volume data yield consistent results. The errors quoted are simply those of the fit – they are almost certainly underestimates of the true errors due to the presence of subleading corrections. Notice that whilst the fits are 'better' in the sense of their χ^2 values, the exponent d is now rather larger than δ. Of course, as we have remarked, since it is hard to estimate the absolute errors in these parameters they may still be consistent.

We can attempt to model the small volume behaviour more closely by allowing for the presence of power law corrections to the leading exponential behaviour of $\Omega_3(V)$. The presence of such corrections would lead to an additional term in eqn. 9 of the form $e \ln V/V$ (we set the constant $b = 0$). The data for $V \geq 8000$ can now be fitted as

$e = 4.4(4)$, $c = 0.452(8)$ and $d = 0.278(2)$ with a $\chi^2 = 1.7$. This fit has the merit of yielding an estimate for d close to the δ power, with a reasonable goodness-of-fit. Both this fit and the $b = 0$ fit imply that in the thermodynamic limit the lattices have an infinite node coordination number.

Finally, we show in fig. 4, a plot of the mean intrinsic size of the ensemble of simplicial graphs versus their volume. This quantity is just the average geodesic distance (in units where the edge lengths are all unity) between two randomly picked sites. The solid line is an empirical fit of the form

$$L_3 = e + f\,(\ln V)^g \tag{10}$$

Clearly, the behaviour is close to logarithmic (as appears also to be the case in four dimensions [7]), the exponent $g = 1.047(3)$ from fitting all the data ($\chi^2 = 1.7$ per degree of freedom). This is indicative of the extremely compact nature of the typical simplicial manifolds dominating the partition function at this node coupling. It is natural to associate this with the very small (possibly zero) value of the b-parameter discussed in the last section.

An alternative way to parametrise the data (essentially small deviations from a simple logarithm) might be to add a correction term of the form $\ln\ln V$.

$$L_3 = e + f\ln V + g\ln\ln V \tag{11}$$

This gives a competitive fit with $e = -1.45(4)$, $f = 1.438(4)$ and $g = -0.55(3)$ with $\chi^2 = 1.6$. One might be tempted to favour this fit on the grounds that it avoids the problem of a power close to but distinct from unity. However, there are very many other ways to fit the data which are *a priori* equally acceptable. The situation must remain ambiguous without further theoretical insight.

To summarise this brief note we have obtained numerical results consistent with the existence of an exponential bound in a dynamical triangulation model of three dimensional quantum gravity. One rather robust way to characterise the approach to the thermodynamic limit is via a small power. Furthermore, we have argued that these finite volume corrections are the same for both the critical coupling and mean node coordination. Our numerical results support this picture. We have also attempted to address the question of whether the coordination number diverges in the thermodynamic limit i.e whether the parameter b is indeed zero or merely small.

We have argued that this is a delicate question and depends very strongly on how one parametrises the subleading corrections to the leading finite volume behaviour. It is clear that a simple power fit with b zero does not model the data. However, it is hard to distinguish a non-zero b-fit from one with additional terms involving $\ln V/V$. The latter have some motivation as arising from possible power corrections to the exponential behaviour of the entropy function Ω_3. Without any theoretical guidance it is impossible to be sure of this question.

Finally, we show data for the scaling of the mean intrinsic extent with volume which suggests a very large (possibly infinite) fractal dimension for the typical simplicial manifolds studied.

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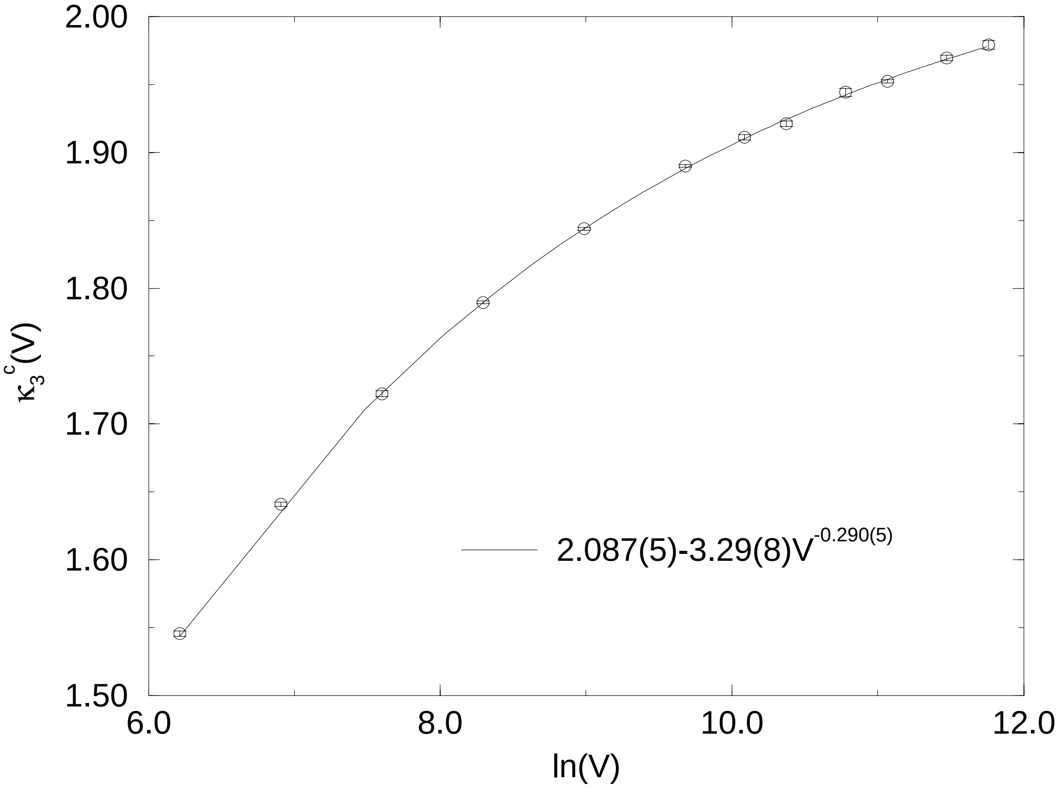

Figure 1: Critical coupling vs volume

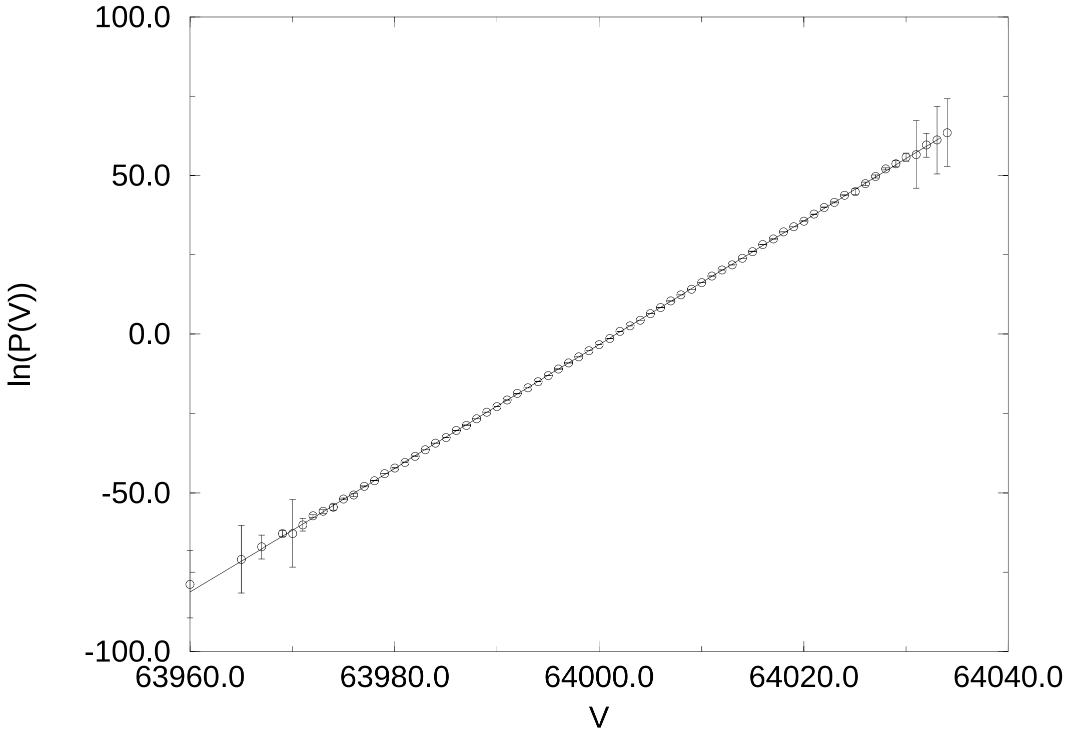

Figure 2: Modified distribution of 3-volumes

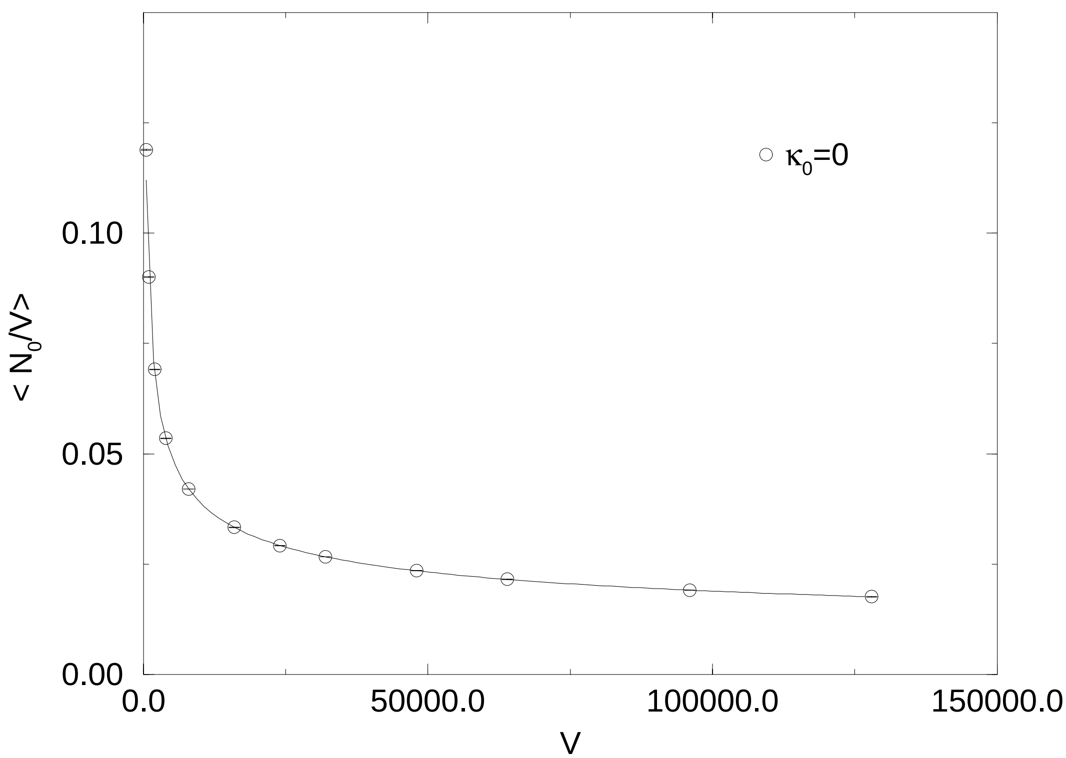

Figure 3: Number of nodes per unit volume

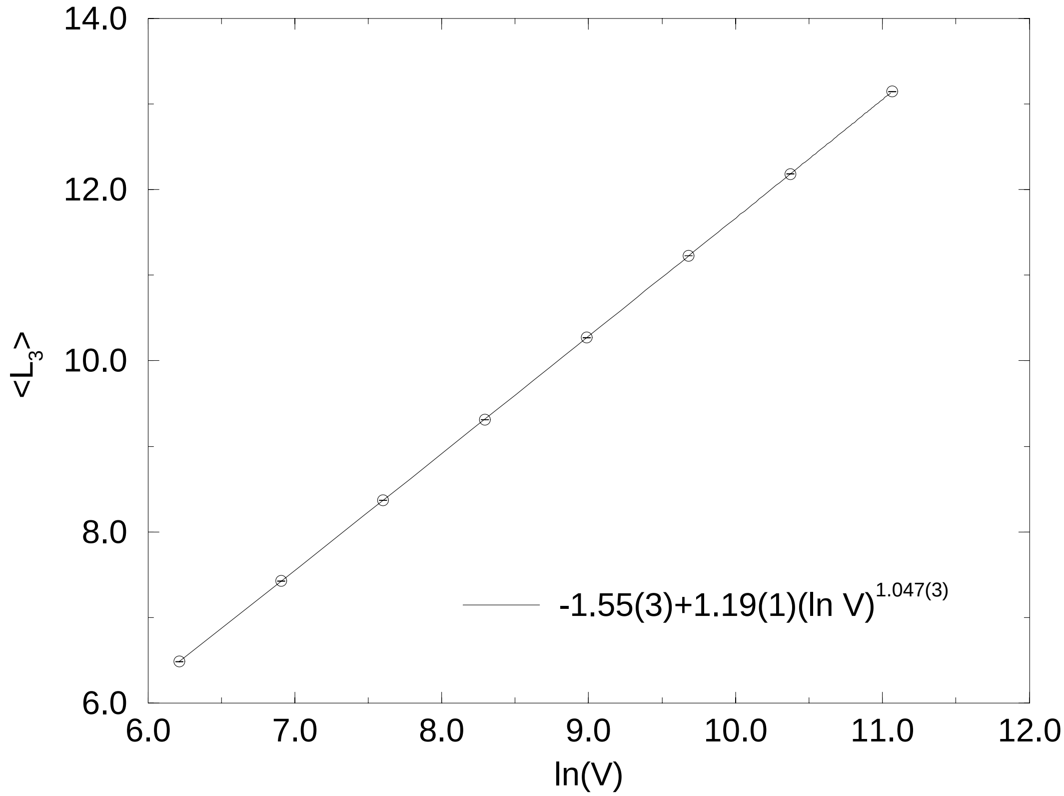

Figure 4: Mean intrinsic extent