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Scaling and the Fractal Geometry of Two-Dimensional Quantum Gravity

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Abstract

We use a scaling ansatz to examine geodesic correlation functions in spin systems coupled to two-dimensional gravity. The numerical data support the scaling assumption and indicate that the quantum geometry develops a non-perturbative length scale. The existence of this length scale allows us to extract a fractal dimension, which in the case of pure gravity is in agreement with other recent calculations. We discuss the influence of the back-reaction of the matter on the fractal dimension.

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1 Introduction

Remarkable strides have been made in recent years in our understanding of the properties of two-dimensional quantum gravity [1]. Calculations carried out within the framework of conformal field theory have yielded the gravitational dressing of *integrated* matter field operators, the correlation functions on the sphere, and the torus partition function. On the other hand the matrix models have provided us with powerful calculational tool that enables us to compute the above mentioned quantities and also enables us to perform the nonperturbative sum over topologies.

Many important geometrical quantities of physical interest are not as yet, however, well understood analytically. One such quantity is the Hausdorff dimension of the two dimensional surfaces corresponding to matter coupled to two dimensional gravity. In some sense one could think of the Hausdorff dimension as an order parameter characterising the different phases of the two dimensional surfaces. If there exists a power law relation between two reparametrisation invariant quantities with the dimension of length and volume, this provides a well-defined fractal dimension. But as there is no natural notion of a length scale in these theories one has to be introduced by hand, at least in the continuum formulation. In the discretized approach this length scale is provided by the short distance cut-off corresponding to the finite elementary link length.

Recently a transfer matrix formalism utilizing matrix model amplitudes has been developed that predicts the Hausdorff dimension $d_H = 4$ for pure $2d$ gravity [15]. This approach has not yet been extended to the case of unitary minimal models coupled to gravity. On the other hand the analysis of the diffusion equation for a random walk on the ensemble of $2d$ manifolds determined by the Liouville action yields a prediction for the Hausdorff dimension which agrees with the transfer matrix approach for pure gravity. It may also be extended to include the coupling of conformal matter of central charge $c \leq 1$ [7].

These analytic predictions for the Hausdorff dimension rely on the validity of certain scaling assumptions. It also appears that there are several potentially inequivalent definitions of an appropriate fractal dimensionality. It seems very worthwhile therefore to explore these issues numerically. Earlier numerical work addressing this question has been remarkably inconclusive [4, 2, 3]. Indeed for a while it was claimed that there was no well-defined Hausdorff dimension in the case of pure gravity [2]. In contrast clear numerical evidence for a fractal scaling of gravity coupled to $c = -2$ matter was found in [5].

In this letter we establish that this scaling behavior extends to pure gravity as well as the Ising and 3-state Potts models coupled to gravity. The key technique is a careful finite size scaling analysis of appropriate correlation functions. For pure gravity we find $d_H \approx 3.85$ in qualitative agreement with [15, 7, 6]. For the Ising and 3-state Potts models the values of d_H that we obtain do not show conclusive evidence of the back reaction of matter on the fractal structure.

This paper is organized as follows. In section 2 we describe the application of finite size scaling to loop-loop correlation functions. In section 3 we outline our numerical procedures and results. In section 4 we present the existing theoretical predictions for the Hausdorff dimension. Section 5 is a discussion of our conclusions.

2 Scaling

Finite size scaling is a well-established technique for the critical behavior of conventional statistical mechanical models [8]. In numerical studies of quantum gravity it has traditionally only been employed in a rather limited context - typically by extracting a power law scaling for integrated matter field operators at the critical point [9].

In general, the scaling ansatz asserts that if we have some observable $O(x, y)$ a function of two variables x and y, then close to criticality it will depend on only one *scaling* combination $\mu = y/x^q$ up to an overall scale set by x^p

$$O(x, y) \sim x^p f(y/x^q) \tag{1}$$

The powers p and q are related to the critical exponents of the model. We will use this as an ansatz for analysing geodesic correlators defined on dynamical triangulations. The latter are sampled via the usual Monte Carlo procedure.

The fundamental objects in two-dimensional gravity are loop-loop correlators. To define these consider two marked loops of length l and l' on a triangulation. If we consider matter coupled systems (a generic field σ living on the vertices) these loops will be dressed with fixed boundary spin configurations S and S'. If we define a geodesic distance r between the loops on the graph as the minimal number of links that must be traversed to go from l to l', we can define a correlation function $G_{l,l',S,S'}(r)$ as simply the fraction of all graphs satisfying these constraints.

$$G_{l,l',S,S'}(r) = \frac{\sum_\sigma \sum_{T(N),l,l',S,S'} e^{-S(\sigma,T)}}{\sum_\sigma \sum_{T(N)} e^{-S(\sigma,T)}} \tag{2}$$

We are working in a microcanonical ensemble in which we include only triangulations $T(N)$ with N triangles. This is convenient computationally and the effect of restricting to fixed volume can be exploited in the finite size scaling analysis. The configurations are distributed with the usual Boltzmann weight depending on the action $S(\sigma, T)$ of the spin model.

In practice we further consider the degenerate case when the loop sizes l and l' shrink to zero and we talk of the point-point correlator. The boundary spin configurations are then simply specified by describing the state of the spin on the marked point. By exploiting the symmetry of the spin models we can further reduce the possible correlators to two distinct types which we denote $f_1(r, N)$ and $f_2(r, N)$. The correlator f_1 then counts the number of points at distance r at which the spin variable is in the same state as the initial marked point. The correlator f_2 counts the number of spins in different states.

The total number of points at geodesic distance r (the quantity $n(r, N)$ introduced earlier) is then the sum f_1 plus f_2.

$$n(r, N) = f_1(r, N) + f_2(r, N)$$

To recover the usual (unnormalised) spin-spin correlator for a general q-states Potts spin we form the difference

$$g_{un} = \left(1 - \frac{1}{q}\right) f_1\left(r, N\right) - \frac{1}{q} f_2\left(r, N\right)$$

We have also measured the normalized spin-spin correlator

$$g_n\left(r, N\right) = \left\langle \frac{\sum_{ij} \sigma_i \sigma_j \delta\left(d_{ij} - r\right)}{\sum ij \delta\left(d_{ij} - r\right)} \right\rangle \tag{3}$$

The quantity d_{ij} is precisely the geodesic distance as measured on the graph between points i and j.

The scaling ansatz applied to $n\left(r, N\right)$ implies

$$n\left(r, N\right) = N^p \phi\left(r / N^q\right) \tag{4}$$

The combination $l_G = N^q$ constitutes a dynamical length scale which appears *non-perturbatively* in the theory. It can be used to define a Hausdorff or fractal dimension $d_H = 1/q$ characterizing the quantum geometry. Notice that in this case the exponent p is not free - it is constrained by the fact that the integral of $n\left(r, N\right)$ over all geodesic distances recovers the total number of points N. This yields $p = 1 - 1/d_H$.

Similarly, for the spin-spin correlator we expect that

$$g_{un}\left(r, N\right) = N^{\frac{\gamma}{\nu d_H} - r} \psi\left(r / N^r\right) \tag{5}$$

The overall power is again determined from the constraint that the integral of $g_{un}\left(r, N\right)$ is just the usual spin susceptibility which scales as $\chi \sim N^{\frac{\gamma}{\nu d_H}}$ at criticality. The exponent r determines another linear scale associated with the critical spin correlations. In flat space of course this would be identical with the geometrical scale (here l_G) and $r = 1/d_H$. It is not clear on a dynamical lattice that this is necessarily so; one could imagine a scenario in which the geometrical scale varies anomalously with the spin scale $l_G \sim l_S^{\omega}$. The quantity ω would then constitute a new exponent characterizing the coupled matter-gravity theory.

This spin correlation length scale can also be extracted from the normalized correlator $g_n\left(r, N\right)$ assuming a similar scaling behavior occurs there.

3 Numerical Simulations

To investigate the validity of the scaling hypothesis we have performed Monte Carlo simulations on three models; pure gravity (central charge $c = 0$), the Ising model ($c = 1/2$) and 3-state Potts model ($c = 4/5$) coupled to gravity. In the microcanonical ensemble the partition function of these models is given by

$$Z(\beta, N) = \sum_{T \in \mathcal{T}} Z_M(\beta, N) \tag{6}$$

where $Z_M(N,\beta)$ describes the matter sector (absent for pure gravity) and which for a q-state Potts model is

$$Z_M(\beta, N) = \sum_{\{\sigma_i\}} \exp\left(\beta \sum_{<i,j>} (\delta_{\sigma_i,\sigma_j} - 1)\right). \tag{7}$$

$\sigma_i \in \{1, \cdots, q\}$ are the Potts spins, i denotes a lattice site and $< i, j >$ indicates that the sum is over neighboring pairs on the lattice.

The integration over manifolds is implemented as a sum over an appropriate class of triangulations $\mathcal{T}$. Since it has been observed that finite size effects in numerical determinations of critical exponents are generally smaller if one includes degenerate triangulations in $\mathcal{T}$, i.e. triangulations allowing two vertices connected by more than one link and vertices connected to itself [10][1], we will work in that ensemble.

In the simulations a standard link-flip algorithm was used to explore the space of triangulations and a Swendsen-Wang cluster algorithm for spin updates. Lattice sizes ranging from 500 to 32000 triangles were studied and typically 10^6 to 4×10^6 Monte Carlo sweeps performed for each lattice size (a sweep consists in flipping about N links and one SW update of the spin configuration).

3.1 Pure Gravity

We start with the results for pure gravity. Here we measured the point-point distributions $n(r,N)$ both on the direct and the dual lattice. On the dual lattice geodesic distances are measured as shortest paths going from one triangle to another. Having measured these distributions for different lattice sizes there are several ways we can use the scaling assumption (4) to extract d_H. We use two methods.

First we fitted a distribution (for a given lattice size) to an appropriately chosen function from which we located the maximum of the distribution r_0 and its maximal value $n(r_0)$. Then the scaling assumption implies that $r_0 \sim N^{1/d_H}$ and $n(r_0) \sim N^{1-1/d_H}$. As a function to fit to we chose the following

$$P_l(r) \exp(-ar^b) \tag{8}$$

The exponential is included in order to capture the long-distance behavior of the distribution and P_l is an l-order polynomial. The order of the polynomial is chosen in such way that we get a reasonably good fit; a $4th$ order polynomial turned out to be sufficient. We checked that the values of r_0 and $n(r_0)$ did not change appreciable if we increased the order of $P_l(r)$. The values of r_0 and $n(r_0)$ obtained in this way are plotted in Figs. 1a and 1b on log-log plots. As expected both quantities scale well with N (significantly better for the direct lattice), the Hausdorff dimensions extracted from the slopes are listed in Table 1.

Another way to extract d_H is to use the scaling relations directly to collapse distributions for different lattices sizes on the same curve using only a *single* scaling parameter.

[1]This corresponds to allowing tadpoles and self-energy diagrams in the dual lattice formulation.

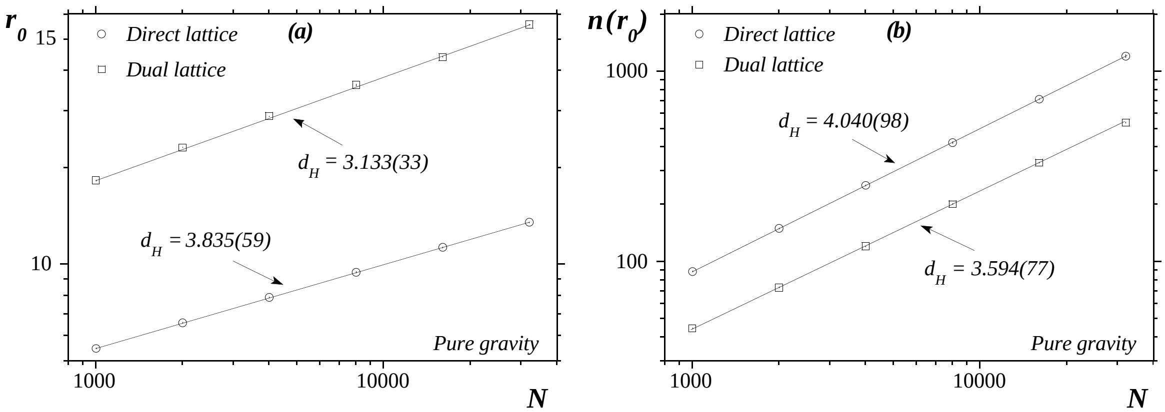

Figure 1: Volume scaling of (a) the location of the peak r_0 in the distributions $n(r, N)$ and (b) their maximal value $n(r_0)$ in the case of pure gravity. Data is shown both for the direct and dual lattices and the extracted values of d_H are included.

		Direct lattice		Dual lattice	
		d_H	$\tilde{\chi}^2$	d_H	$\tilde{\chi}^2$
(a)	126 – 250	3.640(60)	44.6	2.497(37)	49.2
	250 – 500	3.707(45)	13.0	2.715(40)	29.1
	500 – 1000	3.727(42)	8.0	2.871(38)	20.5
	1000 – 2000	3.770(38)	4.2	2.996(26)	22.6
	2000 – 4000	3.800(54)	2.3	3.111(39)	12.5
	4000 – 8000	3.804(55)	1.5	3.217(47)	9.7
	8000 – 16000	3.810(55)	0.97	3.264(34)	6.9
	16000 – 32000	3.830(50)	1.4	3.411(89)	4.8
(b)	1000 – 32000	3.790(30)	13.0	3.150(31)	85
(c)	position	3.835(59)	0.03	3.133(43)	10.45
	height	4.040(98)	0.09	3.594(77)	0.37

Table 1: Extracted values of d_H from $n(r, N)$ in the case of pure gravity. The values in *(a)* are obtained by collapsing data for two consecutive lattices sizes on a single curve using *one* scaling parameter. *(b)* is the same except data from all lattice sizes between 1000 and 32000 triangles are used. In *(c)* the values are obtained from the volume scaling of r_0 and $n(r_0)$ separately. The quality of the fit is indicated by $\tilde{\chi}^2$ and the errors (in *(a)* and *(b)*) are obtained from where $\tilde{\chi}^2$ changes by unit of one.

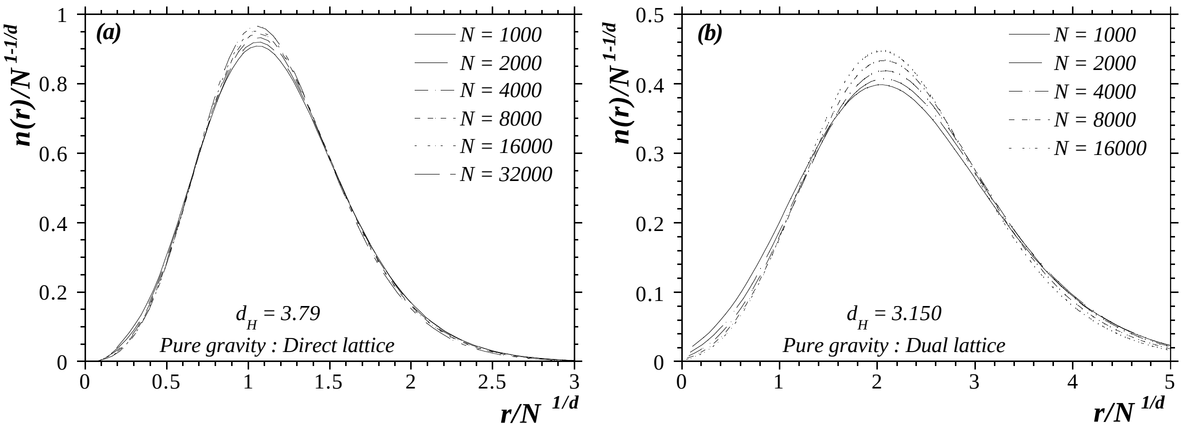

Figure 2: Scaling plots for the point-point distributions $n(r, N)$ in the case of pure gravity; *(a)* the direct and *(b)* dual lattice. Shown are the curves fitted to distributions after a scaling with a single parameter d_H had been applied. The value of d_H is chosen so as it minimized the total chi-square of the fits.

This we have done including all the data (for $N \geq 1000$) and also, to explore the finite size corrections, only using pairs of datasets (N and $2N$). The same functional form Eq. 8 was used in the fits. The results are shown in Table 1, together with the quality of the fits ($\tilde{\chi}^2 = \chi^2/dof$) The errors quoted indicate where $\tilde{\chi}^2$ changes by one unit. In Figs. $2a$ and $2b$ we show an overall scaling plots for $n(r, N)$, both for the direct and dual lattices.

Looking at the data there are a few things we would like to point out. We start with the direct lattice. Fig. $2a$ shows that the scaling hypothesis is indeed well satisfied for the distribution $n(r, N)$, this is also evident from the low values of $\tilde{\chi}^2$ for the fits (Table 1). The values of d_H obtained from the scaling of r_0 and $n(r_0)$ and collapsing the data are close to the expected value of $d_H = 4$. Noting that these results are obtained on moderately small lattices shows how superior this way of extracting d_H is to earlier numerical attempts.

But we also notice that there is a systematic increase in the value of d_H with lattice size. Even though this effect is too small compared to the uncertainty in the measured values to allow reliable extrapolation to infinite volume d_H, it indicates that the difference between measured and expected values of d_H is due to finite-size effects. The improvement of the $\tilde{\chi}^2$ values of the fits with increasing lattice size also implies diminishing deviations from scaling.

It is also intriguing that the scaling of the peak heights seems to give better values of d_H (the exact one for the direct lattice). It is plausible that the heights of the peaks are less sensitive to the discretization as they take continuous values, as opposed to the geodesic distances which are discrete in this approach.

On the dual lattice we observe much larger finite size deviations. This is evident both from Fig. $2b$ and the values of $\tilde{\chi}^2$ in Table 1. This is not hard to understand. The short distance behavior of $n(r, N)$ is dominated by a power growth $r^{d_H - 1}$. But as the order of vertices on the dual lattice are fixed to be three, the growth of $n(r, N)$ is bounded by the function $3 \times 2^{r-1}$. If $d_H = 4$ this means that up to $r = 9$ the distribution $n(r, N)$ cannot

	Ising model		3-state Potts model	
Exponent	Measured	Exact	Measured	Exact
$\beta/\nu d_H$	0.167(3)	1/6	0.199(4)	1/5
$\gamma/\nu d_H$	0.653(8)	2/3	0.608(6)	3/5
$1/\nu d_H$	0.318(12)	1/3	0.382(30)	2/5

Table 2: Comparing critical exponents, obtained using finite sizes scaling in β_c, to exact values, for the Ising and 3-state Potts models coupled to gravity

grow fast enough to display the correct fractal structure. Only when the lattices are big enough so that the first 9 steps are negligible can the dual lattice be used to extract d_H. This constraint on the growth is not present on the direct lattice, which is why that is much better suited for extracting d_H.

3.2 Coupling to matter

To see how the point-point distributions (and d_H) change as we include coupling to matter we look at both the Ising and 3-state Potts models coupled to gravity. These models are chosen because in both cases the exact solution of the models is known[2]; knowing the exact critical coupling makes the simulations much easier.

As shown in the case of pure gravity it is preferable to measure on the direct lattice and so we have placed the spins on the vertices. In that case the critical couplings are (as we include degenerate triangulations):

$$\beta_c = \frac{1}{2}\log\left[\frac{13+\sqrt{7}}{14-\sqrt{7}}\right] \ \ \text{(Ising)} \qquad \text{and} \qquad \beta_c = \frac{1}{2}\log\left[\frac{41+\sqrt{47}}{47-2\sqrt{47}}\right] \ \ (3-\text{statePotts}). \quad (9)$$

To verify that these are indeed the correct couplings we have performed a standard finite size scaling analysis of some observables related to the spin models; the average magnetization $\mathcal{M} \sim N^{-\beta/\nu d_H}$, the magnetic susceptibility $\chi \sim N^{\gamma/\nu d_H}$, and the derivative of Binders cumulant $\partial BC/\partial\beta \sim N^{1/\nu d_H}$ The measured critical exponents are shown in Table 2, together with the exact values with which they agree very well. The main reason is, of course, that we know β_c, but also including degenerate triangulations and placing the spins on vertices reduces finite-size effects dramatically.

Now to the distribution functions. As mentioned in section 2, having the spins on vertices allows us to measure several combinations of distributions; $f_1(r,N)$, $f_2(r,N)$, $n(r,N)$ and $g_{un}(r,N)$. We have analyzed these distributions in the same way as for pure

[2] The 3-state Potts model coupled to gravity has just recently been solved using matrix model techniques [11]. The numerical simulations we do here verify that the solution is correct. To obtain the critical coupling from [11] one has to do some reformulation. This leads to $\beta_c^* = 1/2\log[(45-\sqrt(45))/(\sqrt(47)-2)]$. This is for the spins placed on triangles. To get the coupling for spins on vertices we use the duality transformation for the q-state Potts model $\left(e^{2\beta_c}-1\right)\left(e^{2\beta_c^*}-1\right) = q$ [17].

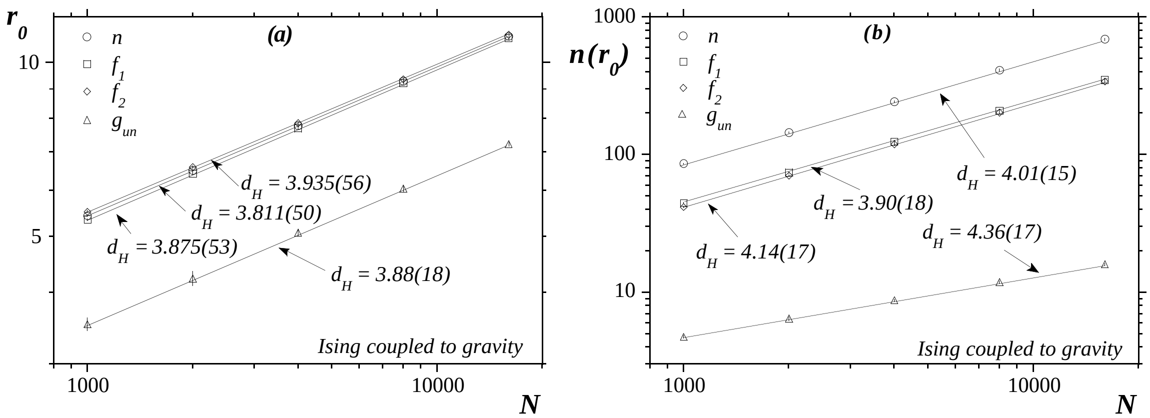

Figure 3: Volume scaling for r_0 and $n(r_0)$ for the distributions we measured for Ising model coupled to gravity. The same scaling behavior is used to extract d_H from the slope as in the case of pure gravity, except for $g_{un}(r_0)$. There we used $n(r_0) \sim \gamma/\nu d_H - 1/d_H$, substituting the exact values for $\gamma/\nu d_H$.

gravity. In Figs. 3a and b we show the scaling with volume of r_0 and $n(r_0)$, obtained from fitting the distributions to the functional form Eq. (8). These plots are for the Ising model but plots for the 3-state Potts model are very similar. The extracted Hausdorff dimensions, for $n(r,N)$ and $g_{un}(r,N)$, are shown in Table 3. As for pure gravity we also scaled all the data (for $N \geq 1000$), and for pairs of distributions, on a single curve. Resulting optimal values of d_H are listed in Table 3. The quality of the scaling is shown in Figs. 4a and b where we show scaling plots for $n(r,N)$ and $g_{un}(r,N)$ (for the Ising model). Again the value of d_H that minimizes $\tilde{\chi}^2$ is used to scale the data.

In the case of the spin models we also measured the normalized spin-spin correlation function $g_n(r,N)$. At the critical point $g_n(r,N)$ is expected to have the following behavior

$$g_n(r,N) \sim \frac{e^{-m(N)\, r}}{r^{\eta}} \ , \tag{10}$$

were the mass gap $m(N)$ vanishes in the infinite volume limit. Surprisingly we have only been able to see the exponential behavior of the spin-spin correlator, not the power underneath it (on a log plot we have a straight line for some range of r). This we have used to extract the mass gap for different lattice sizes. As the mass gap is related to the correlation length $m = 1/\xi$, and ξ is the only length scale in the system, it is reasonable to expect that the way in which $m(N)$ scales to zero with lattice size is related to the Hausdorff dimension, i.e. $1/m \sim N^{1/d_H}$. This gives another method of extracting the Hausdorff dimension.

Looking at the data it is clear that the scaling hypothesis is just as well satisfied as in the case of pure gravity. What is surprising is that the extracted values of d_H, with two exceptions, are almost the same as for pure gravity. The exceptions, for both models, are the scaling of the peak height of $g_{un}(r,N)$ and d_H obtained from the mass gap, both indicating larger values of d_H. Why is it that we do not seem see any effects of the back

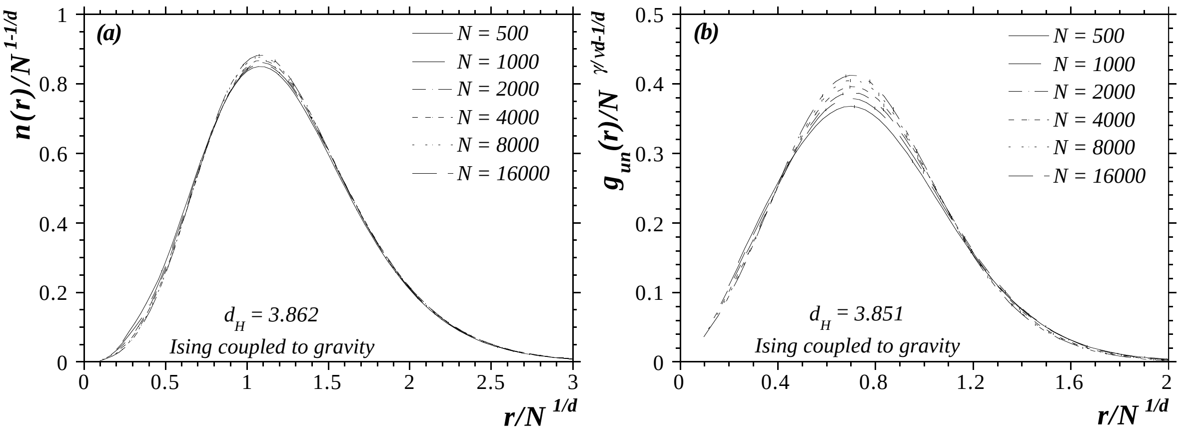

Figure 4: Collapsing the data for $n(r, N)$ and $g_{un}(r, N)$ on a single curve using *one* scaling parameter in the case of an Ising model coupled to gravity.

reaction of matter on the fractal dimension?

A possible explanation would be that the critical region is slightly shifted away from the infinite volume critical coupling at the finite volumew we simulate. This is, for example, observed in measurements of the string susceptibility [12], where measured values of γ_s peak away from β_c. To check this we have measured d_H for the Ising model over an interval of β. Within errors the extracted value of d_H did not change over this interval.

This leaves us with the impression that the extracted values of d_H are contradictory to some extent. In the case of pure gravity we saw that the scaling of the peak height gave better results. If we believe this we get different values for d_H depending on which point-point correlator we examine. Looking at $n(r, N)$ we get $d_H \approx 3.9$ for both models, and observe no back reaction from the matter. $g_{un}(r, N)$ on the other hand indicates $d_H > 4$, and indeed gives results that might be consistent with the values predicted in [7]. This is supported by the scaling of the mass gap of the spin-spin correlator. We will return to this in the discussion section.

4 Hausdorff Dimension - Analytic results

In this section we briefly review the continuum and matrix model derivations of the intrinsic Hausdorff dimension (d_H) of the surfaces generated by the coupling of $2d$ gravity to matter [7, 14, 15, 6]. There are several potentially inequivalent ways to define an appropriate measure of the fractal dimensionality of random surfaces. In the original paper of [14] two methods were proposed. In the first method one determines a power-like relation between two gauge-invariant observables with dimensions of volume (V) and length (L) respectively, with d_H determined by $V \propto L^{d_H}$. The volume is measured by the cosmological term and the length by the anomalous dimension of a test fermion which couples to

	Ising model				3-state Potts model			
	$n(r,N)$		$g_{un}(r,N)$		$n(r,N)$		$g_{un}(r,N)$	
	d_H	$\tilde{\chi}^2$	d_H	$\tilde{\chi}^2$	d_H	$\tilde{\chi}^2$	d_H	$\tilde{\chi}^2$
(a)								
500-1000	3.758(53)	2.6	3.76(12)	0.93	3.752(63)	0.68	4.01(26)	2.5
1000-2000	3.802(55)	0.77	3.75(15)	1.0	3.787(65)	0.29	4.11(18)	1.0
2000-4000	3.833(56)	1.0	3.73(12)	2.5	3.864(63)	1.0	4.04(22)	3.2
4000-8000	3.893(61)	0.88	3.69(09)	3.9	3.870(73)	0.15	4.11(19)	0.41
8000-16000	3.870(87)	0.35	3.80(10)	0.99	3.820(97)	0.58	4.14(15)	0.56
(b)								
1000-16000	3.862(74)	1.4	3.851(53)	4.5	3.831(32)	2.4	3.966(64)	12.5
(c)								
position	3.875(53)		3.88(19)		3.879(29)		4.141(58)	
height	4.01(15)		4.36(18)		3.900(41)		4.424(35))	
mass gap	4.51(20)				4.56(43)			

Table 3: Extracted values of d_H for the Ising and 3-state Potts models coupled to gravity. The values are obtained in the same way as for pure gravity (Table 1).

the gravitational field but generates no back reaction. This yields

$$d_H = 2\frac{\sqrt{25-c}+\sqrt{13-c}}{\sqrt{25-c}+\sqrt{1-c}}\ . \tag{11}$$

In the second method one considers the diffusion of a test fermion field and determines d_H by the short-time come-back probability $p(\tau) \propto \tau^{-d_H/2}$. The authors were able to determine the Hausdorff dimension in a double power series expansion in $\epsilon = D-2$ and $\frac{-1}{c}$, where D is the classical dimensionality of the surface and c is the central charge of the matter coupled to gravity. In [7] this second method was applied instead to a scalar field – one considers the diffusion equation for a random walk on the ensemble of $2d$ manifolds determined by the Liouville action. This yields

$$d_H = -2\frac{\alpha_1}{\alpha_{-1}} = 2\frac{\sqrt{25-c}+\sqrt{49-c}}{\sqrt{25-c}+\sqrt{1-c}}\ , \tag{12}$$

where $e^{\alpha_1\phi}$ corresponds to the cosmological constant operator, which has dimension one, and $e^{\alpha_{-1}\phi}$ corresponds to the Liouville dressing of the Laplacian, which requires it to be of conformal dimension -1.

In the matrix-model/dynamical triangulation approach the transfer matrix formulation can be used to obtain an expression for the Hausdorff dimension in the case of pure gravity [15, 6]. One finds $d_H = 4$ in agreement with Eq. (12) for $c = 0$.

For the case of pure gravity this result can be compared with the matrix model result obtained in [15]. Using matrix model results it is possible to derive the expression

$$\rho(L;D)=\frac{3}{7\sqrt{\pi}}\frac{1}{D^2}\left(x^{-5/2}+\frac{1}{2}x^{-3/2}+\frac{14}{3}x^{-1/2}\right)e^{-x}, \tag{13}$$

where $\rho(L;D)$ is the number of boundaries separated by geodesic distance D from a loop of length L with one marked point, and the scaling variable $x=\frac{L}{D^2}$. Now one can consider the quantities $<L^n>=\int_a^\infty dLL^n\rho(L;D)$ where a is the lattice constant. From Eq. (13) it can be shown that:

$$\begin{aligned} <L^0> &\simeq const\times D^3a^{-3/2}+const\,Da^{-1/2}+const\,D^0 &(14)\\ <L^1> &\simeq const\times D^3a^{-1/2}+const\,D^2 &(15)\\ <L^n> &\simeq const\times D^{2n}\quad (n\geq 2). &(16)\end{aligned}$$

Then, using the definition $<L^0>\propto r^{d_H-1}$, one can read off the Hausdorff dimension $d_H=4$, which agrees with the continuum result and our numerical results based on scaling arguments. This result is not universal because of the explicit lattice dependence in $<L^0>$. One obtains the same result, however, from the second and higher moments provided one assumes that $<L^2>$ scales like the area A. The result thus appears to be universal.

The general situation is, however, far from clear. One case where there is an obvious discrepancy seems to be the $(2,2k-1)$ series of minimal models coupled to gravity. It is possible to extend the continuum Liouville theory analysis to these models after taking into account the fact that these non-unitary models possess operators in the matter sector with negative conformal dimensions. It is also possible to use the results obtained in [16] to calculate the Hausdorff dimensions for models (with 'k' even). We find that the results thus obtained do not agree with each other except for the cases $k=1,2$.

The expression for the distribution of loops at a geodesic distance 'D' for the $(2,2k-1)$ models coupled to gravity (for even 'k') was computed in [16]. They find that

$$\rho(L;D)\simeq\frac{1}{D^{\frac{1}{\sigma}}}\left[\frac{\gamma_1}{\gamma_2\Gamma(\sigma)}x^{-\sigma-2}(2\sigma+1+x)+\frac{x^\sigma}{\Gamma(\sigma+1)}\right]e^{-x}, \tag{17}$$

where $\sigma=k-3/2$ and γ_1,γ_2 are 'k' dependent constants. Using the same arguments as in the case of pure gravity we can compute $d_H=(2k-1)/(2k-3)+1$.

The continuum result of Kawamoto can also be extended to this case, with the difference being that the cosmological constant is not the dressing of the identity operator but of the operator with the lowest conformal dimension. Similarly the dressing condition for the Laplacian is that the Liouville field has dimension $-1-\Delta_{min}$.

Then one obtains:

$$\begin{aligned} \alpha_+ &= \frac{-k}{\sqrt{2k-1}} &(18)\\ \alpha_{-1-\Delta_{\min}} &= \frac{-1}{\sqrt{2k-1}}\left(2k+1-\sqrt{32k-15}\right) &(19)\\ d_H &= \frac{4k}{-2k-1+\sqrt{32k-15}}. &(20)\end{aligned}$$

It is possible to replace the dressing of the Laplacian with the condition that the dressing of the Laplacian involves the identity operator and not the minimal dimension operator, in which case we obtain:

$$d_H = \frac{8}{-2k - 1 + \sqrt{(4k^2 + 20k - 7)}}. \tag{21}$$

Thus for this class of models we find an obvious discrepancy between the matrix model and the continuum formulations. These models are not, unfortunately, amenable to numerical simulations to resolve this disagreement.

5 Discussion

We have studied a class of correlation functions defined along geodesic paths in the dynamical triangulation formulation of two-dimensional gravity. The critical nature of this theory is revealed in the observation that these correlators satisfy a scaling property. The origin of this scaling behavior can be attributed to the existence of a dynamically generated length scale in two-dimensional gravity. Furthermore the power relation between this linear scale and the total volume allows us to extract a fractal dimension characterizing the typical quantum geometry. For pure gravity we estimate $d_H = 3.83(5)$ which is close to the analytic prediction $d_H = 4$. Our numerical method constitutes by far the most reliable method yet investigated for extracting this fractal (Hausdorff) dimension.

Encouraged by this result we have studied two simple spin models coupled to quantum gravity - the $q = 2$ and $q = 3$ Potts models. As we have indicated there is are no truly reliable analytic predictions concerning the nature of the fractal geometry for these values of the matter central charge. The inclusion of matter fields allows us to define two independent correlation functions which we have termed f_1 and f_2. The usual geometrical correlator counting the number of sites at geodesic distance r is just the sum $f_1 + f_2$ whilst the weighted difference $(q-1)f_1 - f_2$ yields the (unnormalised) spin correlator.

For both types of correlation function in either the Ising or 3-state Potts cases we see good evidence for scaling. From the geometrical correlators the Hausdorff dimension we extract is statistically consistent with its value for pure gravity. Taken at face value this would seem to indicate that the backreaction of the critical spin system on the geometry is insufficiently strong to alter the Hausdorff dimension for these values of the central charge. This is supported by our best overall scaling fits to the spin correlator which yield comparable values for d_H.

However, if we use just the scaling of the peak height to estimate a value for d_H the picture is somewhat different - now a shift in d_H is observed to values somewhat above four. Indeed these estimates for d_H are not inconsistent with the predictions of the formula derived in [7]. Since the peak scaling appears to suffer from smaller finite size effects in the case of pure gravity (using just this we extract $d_H = 4.040(98)$) than other quantities it is possible that it also a more reliable channel in which to look for signs of backreaction in the case of spin models. These estimates for d_H are also favored by examining the scaling of the spin correlation length extracted from the normalised correlation function. However without good theoretical reasons for believing in such a favored channel it is probably more sensible to ascribe the differences in our estimates for d_H to the presence of rather large scaling violations at these lattice sizes.

One alternative scenario might be that the observed effects are due to the presence of two linear scales; the geometrical scale and another characterizing the critical spin correlations. Thus two fractal dimensions might be possible; one the (true) Hausdorff dimension associated with the geometry, and another revealed only in the spin channel. If this were so then the numerical estimates of these exponents would favor a situation in which the spin correlation length diverged *more slowly* with volume than the gravitational (geometrical) scale. This might serve as a partial explanation of the observed exponential behavior of the (normalised) spin correlator at the critical point - unlike flat space critical models the correlation length in a dynamical lattice can never reach the typical linear size of the lattice.

In the absence of any explicit transfer matrix type solutions for these unitary minimal models it would seem that further high resolution numerical work will be needed to resolve these important issues.

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