**Beam and Warming Algorithm for Compressible Navier-Stokes Equations**

## I. Introduction

The Navier-Stokes equations in conservative form are given by

$$
\begin{aligned}
\partial_t \rho + \partial_j (\rho u_j) &= 0 , \\
\partial_t (\rho u_i) + \partial_j [\rho u_j u_i + p\delta_{ij} - \tau_{ij}] &= \rho F_i , \\
\partial_t(\rho E) + \partial_j [\rho u_j H - u_i\tau_{ij} - k\partial_j T] &= \rho u_i F_i + \dot{q} ,
\end{aligned} \tag{1}
$$

where

$$
\tau_{ij} = \mu [\partial_j u_i + \partial_i u_j] + \lambda \delta_{ij} \partial_k u_k , \tag{2}
$$

$\rho$: density

$u_i$: velocity component in $i$

$p$: pressure

$F_i$: external force in $i$

$E$: total internal energy $= e + \frac{u_i u_i}{2}$, where $e$ is the specific internal energy

$H$: total enthalpy $= h + \frac{u_i u_i}{2}$, where $h$ is the specific enthalpy

$T$: temperature

$k$: thermal conductivity

$\mu$: absolute viscosity

$\lambda$: second coefficient of viscosity ($= -\frac{2}{3}\mu$ upon invoking Stokes assumption)

A perfect gas is assumed, in which case,

$$
e = E - \frac{u_i u_i}{2} = C_v T \Rightarrow T = \frac{1}{C_v}\left[E - \frac{u_i u_i}{2}\right] . \tag{3}
$$

Furthermore, since $p = \rho RT$, then

$$p = \rho RT = \rho \frac{R}{C_v}\left[E - \frac{u_i\,u_i}{2}\right] = (\gamma - 1)\rho\left[E - \frac{u_i\,u_i}{2}\right]\ . \tag{4}$$

To simplify the subsequent development of the Beam and Warming Algorithm, consider a two-dimensional flow in cartesian coordinates. Expanding the Navier-Stokes equations gives

$$\begin{aligned}
\rho_t\ + &\quad (\rho u)_x + (\rho v)_y = 0\ ,\\
(\rho u)_t\ + &\quad \left(\rho u^2\right)_x + (\rho uv)_y + p_x - \left[(2\mu + \lambda)u_x + \lambda v_y\right]_x - \left[\mu\,(u_y + v_x)\right]_y = \rho F_x\ ,\\
(\rho v)_t\ + &\quad (\rho uv)_x + \left(\rho v^2\right)_y + p_y - \left[\mu\,(u_y + v_x)\right]_x - \left[(2\mu + \lambda)v_y + \lambda u_x\right]_y = \rho F_y\ ,\\
(\rho E)_t\ + &\quad (\rho uH)_x + (\rho vH)_y - \left[2\mu u u_x + \mu v\,(u_y + v_x) + \lambda u\,(u_x + v_y) + kT_x\right]_x\\
- &\quad \left[2\mu v v_y + \mu u\,(u_y + v_x) + \lambda v\,(u_x + v_y) + kT_y\right]_y = \rho u F_x + \rho v F_y + \dot{q}\ .
\end{aligned}$$

Rearrange the above equations according to

$$\begin{aligned}
\rho_t + (\rho u)_x + (\rho v)_y \quad &= \quad 0\ ,\\
(\rho u)_t + \left(\rho u^2 + p\right)_x + (\rho uv)_y \quad &- \quad \left[(2\mu + \lambda)u_x\right]_x - \left[\lambda v_y\right]_x\\
&- \quad \left[\mu v_x\right]_y - \left[\mu u_y\right]_y = \rho F_x\ ,\\
(\rho v)_t + (\rho uv)_x + \left(\rho v^2 + p\right)_y \quad &- \quad \left[\mu v_x\right]_x - \left[\mu u_y\right]_x\\
&- \quad \left[\lambda u_x\right]_y - \left[(2\mu + \lambda)v_y\right]_y = \rho F_y\ ,\\
(\rho E)_t + (\rho uH)_x + (\rho vH)_y \quad &- \quad \left[2\mu u u_x + \mu v v_x + \lambda u u_x + kT_x\right]_x\\
&- \quad \left[\mu v u_y + \lambda u v_y\right]_x\\
&- \quad \left[\mu u v_x + \lambda v u_x\right]_y\\
&- \quad \left[2\mu v v_y + \mu u u_y + \lambda v v_y + kT_y\right]_y\\
&= \quad \rho u F_x + \rho v F_y + \dot{q}\ .
\end{aligned} \tag{5}$$

Next, define the following vector functions:

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix}; \qquad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u H \end{bmatrix}; \qquad \mathbf{G} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v H \end{bmatrix};$$

$$\mathbf{U}_I = \begin{bmatrix} 0 \\ (2\mu + \lambda) u_x \\ \mu v_x \\ 2\mu u u_x + \mu v v_x + \lambda u u_x + k T_x \end{bmatrix}; \qquad \mathbf{U}_E = \begin{bmatrix} 0 \\ \lambda v_y \\ \mu u_y \\ \mu v u_y + \lambda u v_y \end{bmatrix};$$

$$\mathbf{V}_I = \begin{bmatrix} 0 \\ \mu u_y \\ (2\mu + \lambda) v_y \\ 2\mu v v_y + \mu u u_y + \lambda v v_y + k T_y \end{bmatrix}; \qquad \mathbf{V}_E = \begin{bmatrix} 0 \\ \mu v_x \\ \lambda u_x \\ \mu u v_x + \lambda v u_x \end{bmatrix};$$

and, finally,

$$\mathbf{H} = \begin{bmatrix} 0 \\ \rho F_x \\ \rho F_y \\ \rho u F_x + \rho v F_y + \dot{q} \end{bmatrix}. \tag{6}$$

With the above definitions, the Navier-Stokes equations may be written as

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \frac{\partial \mathbf{U}_I}{\partial x} + \frac{\partial \mathbf{U}_E}{\partial x} + \frac{\partial \mathbf{V}_I}{\partial y} + \frac{\partial \mathbf{V}_E}{\partial y} + \mathbf{H} \tag{7}$$

**Note:** $\frac{\partial \mathbf{U}_I}{\partial x}, \frac{\partial \mathbf{V}_I}{\partial y}$ do not contain cross-derivatives, and $\frac{\partial \mathbf{U}_E}{\partial x}, \frac{\partial \mathbf{V}_E}{\partial y}$ do contain cross-derivatives.

From equation (6), it follows that

$$\begin{aligned} \mathbf{F} &= \mathbf{F}(\mathbf{q})\,, & \mathbf{G} &= \mathbf{G}(\mathbf{q})\,, \\ \mathbf{U}_I &= \mathbf{U}_I(\mathbf{q}, \mathbf{q}_x)\,, & \mathbf{U}_E &= \mathbf{U}_E(\mathbf{q}, \mathbf{q}_y)\,, \\ \mathbf{V}_I &= \mathbf{V}_I(\mathbf{q}, \mathbf{q}_y)\,, & \mathbf{V}_E &= \mathbf{V}_E(\mathbf{q}, \mathbf{q}_x)\,. \end{aligned}$$

## II. Second-order Temporal Discretization

Let the equations be discretized at $t = (n+1)\Delta t$. A Taylor series expansion about $t + \Delta t$ (or $(n+1)\Delta t$) yields

$$\mathbf{q}^n = \mathbf{q}^{n+1} - \Delta t \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n+1} + \frac{\Delta t^2}{2} \left.\frac{\partial^2 \mathbf{q}}{\partial t^2}\right|_{n+1} + O\left(\Delta t^3\right) , \tag{8a}$$

$$\mathbf{q}^{n-1} = \mathbf{q}^{n+1} - 2\Delta t \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n+1} + 2\Delta t^2 \left.\frac{\partial^2 \mathbf{q}}{\partial t^2}\right|_{n+1} + O(\Delta t)^3 . \tag{8b}$$

Subtracting equation 8(b) from four times equation 8(a) yields

$$4\mathbf{q}^n - \mathbf{q}^{n-1} = 3\mathbf{q}^{n+1} - 2\Delta t \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n+1} + O\left(\Delta t^3\right) ,$$

or, upon rearranging the above,

$$\mathbf{q}^{n+1} - \mathbf{q}^n = \frac{2}{3}\Delta t \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n+1} + \frac{1}{3}\left[\mathbf{q}^n - \mathbf{q}^{n-1}\right] + O\left(\Delta t^3\right) . \tag{9}$$

Denote

$$\begin{aligned} \Delta \mathbf{q}^n &= \mathbf{q}^{n+1} - \mathbf{q}^n , \\ \frac{\partial}{\partial t}\left(\Delta \mathbf{q}^n\right) &= \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n+1} - \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n} . \end{aligned}$$

Upon substituting the above into equation (9),

$$\Delta \mathbf{q}^n = \frac{2}{3}\Delta t \frac{\partial}{\partial t}\Delta \mathbf{q}^n + \frac{2}{3}\Delta t \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n} + \frac{1}{3}\Delta \mathbf{q}^{n-1} + O\left(\Delta t^3\right) . \tag{10}$$

**Note:** (1) Equation (9) indicates that $\left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n+1}$ is second order accurate in $t$. (2) To achieve the above second-order accuracy, the time steps $\Delta t$ must be uniform.

A general form of the temporal discretization which encompasses equation (10) is given by

$$\begin{aligned} \Delta \mathbf{q}^n &= \frac{\theta \Delta t}{1+\alpha} \frac{\partial}{\partial t}\left(\Delta \mathbf{q}^n\right) + \frac{\Delta t}{1+\alpha} \left.\frac{\partial \mathbf{q}}{\partial t}\right|_{n} + \frac{\alpha}{1+\alpha}\Delta \mathbf{q}^{n-1} \\ &+ O\left(\left(\theta - \frac{1}{2} - \alpha\right)\Delta t^2, \Delta t^3\right) . \end{aligned} \tag{11}$$

The choice $\theta = 1$, $\alpha = 1/2$ recovers the second-order accurate equation (10).

We now work with the general form equation (11). Upon substitution of equation (7) into equation (11) yields

$$\begin{aligned}
\Delta \mathbf{q}^n &= \frac{\theta \Delta t}{1+\alpha} \left[ \frac{\partial}{\partial x} \left\{ -\Delta \mathbf{F}^n + \Delta \mathbf{U}_I^n + \Delta \mathbf{U}_E^n \right\} \right. \\
&\qquad + \frac{\partial}{\partial y} \left\{ -\Delta \mathbf{G}^n + \Delta \mathbf{V}_I^n + \Delta \mathbf{V}_E^n \right\} + \Delta \mathbf{H}^n \left. \vphantom{\frac{\partial}{\partial x}} \right] \\
&+ \frac{\Delta t}{1+\alpha} \left[ \frac{\partial}{\partial x} \left\{ -\mathbf{F}^n + \mathbf{U}_I^n + \mathbf{U}_E^n \right\} + \frac{\partial}{\partial y} \left\{ -\mathbf{G}^n + \mathbf{V}_I^n + \mathbf{V}_E^n \right\} + \mathbf{H}^n \right] \\
&+ \frac{\alpha}{1+\alpha} \Delta \mathbf{q}^{n-1} + O\left( \left( \theta - \frac{1}{2} - \alpha \right) \Delta t^2, \, \Delta t^3 \right), \qquad (12)
\end{aligned}$$

where the following notation is adopted:

$$\begin{aligned}
\mathbf{F}^n &= \mathbf{F}(\mathbf{q}^n) \ , \\
\Delta \mathbf{F}^n &= \mathbf{F}\left(\mathbf{q}^{n+1}\right) - \mathbf{F}(\mathbf{q}^n) \ . \qquad (13)
\end{aligned}$$

Similarly, the same notation is used for the other vector functions.

**Note:** Equation (12) is a nonlinear equation in $\Delta \mathbf{q}^n$. Once an appropriate spatial differencing scheme is employed, the equations could be solved by using a Newton-Raphson method; however, this is an expensive iterative procedure.

Three simplifications to equation (12) are now made to allow for an efficient solution procedure.

**Step 1:**
From equation (12), it follows that $\Delta \mathbf{q}^n \sim O(\Delta t)$. Also,

$$[\Delta \mathbf{q}^n]_x = \left[ \mathbf{q}^{n+1} - \mathbf{q}^n \right]_x = \mathbf{q}_x^{n+1} - \mathbf{q}_x^n = [\Delta \mathbf{q}_x]^n \sim O(\Delta t) \ . \qquad (14)$$

Similarly, for the $y$-derivative. Let

$$\begin{aligned}
\Delta \mathbf{q}_x^n &= [\Delta \mathbf{q}^n]_x = [\Delta \mathbf{q}_x]^n \ , \\
\Delta \mathbf{q}_y^n &= [\Delta \mathbf{q}^n]_y = [\Delta \mathbf{q}_y]^n \ ,
\end{aligned}$$

and expand the vector functions $\mathbf{F}$ and $\mathbf{G}$ about $\mathbf{q}^n$. Thus,

$$\mathbf{F}^{n+1} = \mathbf{F}^n + \Delta\mathbf{q}^n \left.\frac{\partial\mathbf{F}}{\partial\mathbf{q}}\right|_n + O\left(\Delta\mathbf{q}^n\right)^2 \ ,$$

$$\mathbf{G}^{n+1} = \mathbf{G}^n + \Delta\mathbf{q}^n \left.\frac{\partial\mathbf{G}}{\partial\mathbf{q}}\right|_n + O\left(\Delta\mathbf{q}^n\right)^2 \ .$$

Using the notation in equation (13) and $\Delta\mathbf{q}^n \sim O(\Delta t)$,

$$\begin{aligned} \Delta\mathbf{F}^n &= \mathbf{A}^n\Delta\mathbf{q}^n + O\left(\Delta t^2\right) \ , && \text{(15a)} \\ \Delta\mathbf{G}^n &= \mathbf{B}^n\Delta\mathbf{q}^n + O\left(\Delta t^2\right) \ , && \text{(15b)} \end{aligned}$$

where

$$\mathbf{A}^n = \left.\frac{\partial\mathbf{F}}{\partial\mathbf{q}}\right|_n \ ; \quad \mathbf{B}^n = \left.\frac{\partial\mathbf{G}}{\partial\mathbf{q}}\right|_n \ . \tag{16}$$

Similarly, the functions $\mathbf{U}_I$ and $\mathbf{V}_I$ are expanded according to

$$\begin{aligned} \mathbf{U}_I^{n+1} &= \mathbf{U}_I^n + \Delta\mathbf{q}^n \left.\frac{\partial\mathbf{U}_I}{\partial\mathbf{q}}\right|_n + \Delta\mathbf{q}_x^n \left.\frac{\partial\mathbf{U}_I}{\partial\mathbf{q}_x}\right|_n + O\left(\Delta t^2\right) \ , \\ \mathbf{V}_I^{n+1} &= \mathbf{V}_I^n + \Delta\mathbf{q}^n \left.\frac{\partial\mathbf{V}_I}{\partial\mathbf{q}}\right|_n + \Delta\mathbf{q}_y^n \left.\frac{\partial\mathbf{V}_I}{\partial\mathbf{q}_y}\right|_n + O\left(\Delta t^2\right) \ , \end{aligned} \tag{17}$$

or,

$$\begin{aligned} \Delta\mathbf{U}_I^n &= \mathbf{P}\Delta\mathbf{q}^n + \mathbf{R}\Delta\mathbf{q}_x^n + O\left(\Delta t^2\right) \ , \\ \Delta\mathbf{V}_I^n &= \mathbf{Q}\Delta\mathbf{q}^n + \mathbf{S}\Delta\mathbf{q}_y^n + O\left(\Delta t^2\right) \ , \end{aligned} \tag{18}$$

where

$$\begin{aligned} \mathbf{P} &= \left.\frac{\partial\mathbf{U}_I}{\partial\mathbf{q}}\right|_n \ ; \quad \mathbf{Q} = \left.\frac{\partial\mathbf{V}_I}{\partial\mathbf{q}}\right|_n \ , \\ \mathbf{R} &= \left.\frac{\partial\mathbf{U}_I}{\partial\mathbf{q}_x}\right|_n \ ; \quad \mathbf{S} = \left.\frac{\partial\mathbf{V}_I}{\partial\mathbf{q}_y}\right|_n . \end{aligned} \tag{19}$$

Substituting equations (15)–(19) into equation (12) gives

$$
\begin{aligned}
\Delta \mathbf{q}^n &= \frac{\theta\Delta t}{1+\alpha}\left[\frac{\partial}{\partial x}\left\{-\mathbf{A}^n\Delta\mathbf{q}^n + \mathbf{P}\Delta\mathbf{q}^n + \mathbf{R}\Delta\mathbf{q}_x^n + \Delta\mathbf{U}_E^n\right\}\right.\\
&+ \left.\frac{\partial}{\partial y}\left\{-\mathbf{B}^n\Delta\mathbf{q}^n + \mathbf{Q}\Delta\mathbf{q}^n + \mathbf{S}\Delta\mathbf{q}_y^n + \Delta\mathbf{V}_E^n\right\} + \Delta\mathbf{H}^n\right]\\
&+ \frac{\Delta t}{1+\alpha}\left[\frac{\partial}{\partial x}\left\{-\mathbf{F}^n + \mathbf{U}_I^n + \mathbf{U}_E^n\right\} + \frac{\partial}{\partial y}\left\{-\mathbf{G}^n + \mathbf{V}_I^n + \mathbf{V}_E^n\right\} + \mathbf{H}^n\right]\\
&+ \frac{\alpha}{1+\alpha}\Delta\mathbf{q}^{n-1} + 0\left(\left(\theta - \frac{1}{2} - \alpha\right)\Delta t^2\,,\,\Delta t^3\right)\ . \qquad (20)
\end{aligned}
$$

**Note:** $\Delta\mathbf{U}_E^n$ and $\Delta\mathbf{V}_E^n$ are left unchanged and are not linearized like the other functions for the following reason:

Suppose indeed that $\Delta\mathbf{U}_E^n$ and $\Delta\mathbf{V}_E^n$ were linearlized in the same fashion as in equation (18). Then, upon transferring the first term, i.e., $\frac{\theta\Delta t}{1+\alpha}[\quad]$, from the right-hand side to the left-hand side, an implicit system of equations for the unknown $\Delta\mathbf{q}^n$ would be set up. Furthermore, suppose a simple second-order central difference scheme is employed to discretize the system of equations. Then, for all terms other than $\frac{\partial}{\partial x}[\Delta\mathbf{U}_E]^n$ and $\frac{\partial}{\partial y}[\Delta\mathbf{V}_E]^n$, a five-point stencil would result; however, since the two terms above involve cross-derivatives, a nine-point stencil is created. (In three dimensions, it would be a nineteen-point stencil). The solution procedure would then be quite expensive. In order to circumvent this difficulty, the terms $\Delta\mathbf{U}_E^n$ and $\Delta\mathbf{V}_E^n$ are treated in the following manner:

**Step 2:**

A Taylor series expansion about $t$ gives

$$
\mathbf{U}_E^{n+1} = \mathbf{U}_E^n + \Delta t\,\frac{\partial \mathbf{U}_E}{\partial t}\bigg|_n + O\left(\Delta t^2\right)\ ,
$$

$$
\mathbf{U}_E^{n-1} = \mathbf{U}_E^n - \Delta t\,\frac{\partial \mathbf{U}_E}{\partial t}\bigg|_n + O\left(\Delta t^2\right)\ .
$$

Hence, for a uniform time step,

$$
\Delta\mathbf{U}_E^n = \Delta\mathbf{U}_E^{n-1} + O\left(\Delta t^2\right)\ ,
$$

and, similarly,

$$
\Delta\mathbf{V}_E^n = \Delta\mathbf{V}_E^{n-1} + O\left(\Delta t^2\right)\ . \qquad (21)
$$

In effect, the cross-derivative terms are handled "explicitly." With the simplification in equation (21), equation (20) now takes the following form:

$$\begin{aligned}
\Delta \mathbf{q}^n \; &- \frac{\theta \Delta t}{1+\alpha}\left[\frac{\partial}{\partial x}\left\{-\mathbf{A}^n \Delta \mathbf{q}^n + \mathbf{P}\Delta \mathbf{q}^n + \mathbf{R}\Delta \mathbf{q}^n_x\right\}\right. \\
&\left. + \frac{\partial}{\partial y}\left\{-\mathbf{B}^n \Delta \mathbf{q}^n + \mathbf{Q}\Delta \mathbf{q}^n + \mathbf{S}\Delta \mathbf{q}^n_y\right\}\right] \\
&= \mathbf{H}^{*n} + \; O\left(\left(\theta - \frac{1}{2} - \alpha\right)\Delta t^2 \, , \, \Delta t^3\right) \; , \qquad (22a)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{H}^{*n} &= \frac{\theta \Delta t}{1+\alpha}\left[\frac{\partial}{\partial x}\left(\Delta \mathbf{U}^{n-1}_E\right) + \frac{\partial}{\partial y}\left(\Delta \mathbf{V}^{n-1}_E\right) + \Delta \mathbf{H}^n\right] \\
&+ \frac{\Delta t}{1+\alpha}\mathbf{H}^n + \frac{\alpha}{1+\alpha}\Delta \mathbf{q}^{n-1} \; . \qquad (22b)
\end{aligned}$$

**Note:** The forcing function $\mathbf{H}^{*n}$ of the temporally discretized equations is not the same as the physical forcing function $\mathbf{H}^n$.

An expansion of equation (22a) gives

$$\begin{aligned}
\Delta \mathbf{q}^n \quad - \quad & \frac{\theta \Delta t}{1+\alpha}\Big[ - \mathbf{A}^n_x \Delta \mathbf{q}^n - \mathbf{A}^n \Delta \mathbf{q}_x + \mathbf{P}^n_x \Delta \mathbf{q}^n + \mathbf{P}^n \Delta \mathbf{q}_x \\
+ \quad & \mathbf{R}^n_x \Delta \mathbf{q}^n_x + \mathbf{R}^n \Delta \mathbf{q}^n_{xx} - \mathbf{B}^n_y \Delta \mathbf{q}^n - \mathbf{B}^n \Delta \mathbf{q}^n_y \\
+ \quad & \mathbf{Q}^n_y \Delta \mathbf{q}^n + \mathbf{Q}^n \Delta \mathbf{q}_y + \mathbf{S}^n_y \Delta \mathbf{q}^n_y + \mathbf{S}^n \Delta \mathbf{q}^n_{yy}\Big] = \mathbf{H}^{*n} \; .
\end{aligned}$$

Denote

$\delta_y$, $\delta_x$ : generic first-order derivative

$\delta_{yy}$, $\delta_{xx}$ : generic second-order derivative

With the above notation, the discretized equation may be written compactly according to

$$\begin{aligned}
\Big[\mathbf{I} \quad + \quad & \frac{\theta \Delta t}{1+\alpha}\Big[(\mathbf{A}^n_x - \mathbf{P}^n_x) + (\mathbf{A}^n - \mathbf{P}^n - \mathbf{R}^n_x)\,\delta_x - \mathbf{R}^n \delta_{xx} \\
+ \quad & \left(\mathbf{B}^n_y - \mathbf{Q}^n_y\right) + \left(\mathbf{B}^n - \mathbf{Q}^n - \mathbf{S}^n_y\right)\delta_y - \mathbf{S}^n \delta_{yy}\Big]\Big]\Delta \mathbf{q}^n = \mathbf{H}^{*n} \; . \; (23)
\end{aligned}$$

If three-point differences are used, the above discretization would lead to a block pentadiagonal system of equations.

**Step 3:** Approximate Factorization

Denote the operators $\alpha_x$ and $\alpha_y$ according to

$$\alpha_x = (\mathbf{A}_x^n - \mathbf{P}_x^n) + (\mathbf{A}^n - \mathbf{P}^n - \mathbf{R}_x^n)\,\delta_x - \mathbf{R}^n \delta_{xx}$$

$$\alpha_y = \left(\mathbf{B}_y^n - \mathbf{Q}_y^n\right) + \left(\mathbf{B}^n - \mathbf{Q}^n - \mathbf{S}_y^n\right)\delta_y - \mathbf{S}^n \delta_{yy}$$

Then, equation (23) may be written as

$$\left\{\mathbf{I} + \frac{\theta\Delta t}{1+\alpha}(\alpha_x + \alpha_y)\right\}\Delta\mathbf{q}^n = \mathbf{H}^{*n} + O\left(\left(\theta - \frac{1}{2} - \alpha\right)\Delta t^2\,,\,\Delta t^3\right)\ . \tag{24}$$

But,

$$\left\{\mathbf{I} + \frac{\theta\Delta t}{1+\alpha}\,\alpha_x\right\}\left\{I + \frac{\theta\Delta t}{1+\alpha}\,\alpha_y\right\} = \left\{\mathbf{I} + \frac{\theta\Delta t}{1+\alpha}(\alpha_x + \alpha_y)\right\} + O\left(\Delta t^2\right)\ .$$

Since $\Delta\mathbf{q}^n \sim O(\Delta t)$, equation (24) may be written as

$$\left\{\mathbf{I} + \frac{\theta\Delta t}{1+\alpha}\,\alpha_x\right\}\left\{I + \frac{\theta\Delta t}{1+\alpha}\,\alpha_y\right\}\Delta\mathbf{q}^n = \mathbf{H}^{*n} + O\left(\left(\theta - \frac{1}{2} - \alpha\right)\Delta t^2\,,\,\Delta t^3\right)\ . \tag{25}$$

Equations (24) and (25) have the same formal temporary accuracy.

The factorization in equation (25) permits an efficient solution procedure consisting of a set of two block-tridiagonal system which is solved very efficiently by the Block Thomas algorithm. Specifically, solve

$$\left\{\mathbf{I} + \frac{\theta\Delta t}{1+\alpha}\,\alpha_x\right\}\Delta\mathbf{q}^* = \mathbf{H}^{*n}\ ,$$

followed by

$$\left\{\mathbf{I} + \frac{\theta\Delta t}{1+\alpha}\,\alpha_y\right\}\Delta\mathbf{q}^n = \Delta\mathbf{q}^*\ .$$

**Comment:**

- Cross-derivative terms are expensive to handle implicitly.
- If possible, it is best to handle them "explicitly" by moving them over to the right-hand side as demonstrated by the Beam and Warming Algorithm.
- A linearized (frozen coefficient assumption) stability analysis has shown the above algorithm to be stable for $\alpha = 1/2$. However, stable calculations for nonlinear equations is not guaranteed. This is usually a problem when nonorthogonal grids are used where the cross-derivative terms may not be small. Furthermore, stability requirements are far more stringent in three dimensions.
- The calculations at the first time step cannot be $O\left(\Delta t^2\right)$.