The Three-Dimensional Compressible Navier-Stokes Equations

The Navier-Stokes equations in conservative form are given by

$$\begin{aligned} \partial_t^* \rho^* + \partial_j^* \left(\rho^* u_j^*\right) &= 0 \,, \\ \partial_t^* (\rho^* u_i^*) + \partial_j^* \left[\rho^* u_i^* u_j^* + p^* \delta_{ij} - \tau_{ij}^*\right] &= \rho^* F_i^* \,, \\ \partial_t^* (\rho^* E^*) + \partial_j^* \left[\rho^* u_j^* H^* - u_i^* \tau_{ij}^* - k^* \partial_j^* T^*\right] &= \rho^* u_i^* F_i^* + \dot{q}^* \,, \end{aligned} \qquad (1)$$

where the superscript '*' denotes dimensional quantities and

$$\tau_{ij}^* = \mu^* \left[\partial_j^* u_i^* + \partial_i^* u_j^*\right] + \lambda^* \delta_{ij} \, \partial_k^* u_k^* \,, \qquad (2)$$

ρ^*: density

u_i^*: i^{th} velocity component, $i = 1, 3$

p^*: pressure

F_i^*: i^{th} component of the external force

E^*: total internal energy $= e^* + \frac{u_i^* u_i^*}{2}$, where e^* is the specific internal energy

H^*: total enthalpy $= h^* + \frac{u_i^* u_i^*}{2}$, where h^* is the specific enthalpy

T^*: temperature

k^*: thermal conductivity

μ^*: absolute (or dynamic) viscosity

λ^*: second coefficient of viscosity ($= -\frac{2}{3}\mu^*$ upon invoking Stokes assumption)

$\dot{q}^*$: volumetric heat generation

The specific enthalpy is related to the specific internal energy by

$$h^* = e^* + \frac{p^*}{\rho^*} \Rightarrow H^* = E^* + \frac{p^*}{\rho^*} \,. \qquad (3)$$

A perfect gas is assumed, in which case,

$$e^* = E^* - \frac{u_i^* \, u_i^*}{2} = C_v^* T^* \Rightarrow T^* = \frac{1}{C_v^*}\left[E^* - \frac{u_i^* \, u_i^*}{2}\right] \; . \tag{4}$$

Furthermore, for a perfect gas, $p^* = \rho^* R^* T^*$. Hence,

$$p^* = \rho^* \frac{R^*}{C_v^*}\left[E^* - \frac{u_i^* \, u_i^*}{2}\right] = (\gamma - 1)\rho^* \left[E^* - \frac{u_i^* \, u_i^*}{2}\right] \; , \tag{5}$$

where

C_v^*: specific heat at constant volume

R^*: gas constant

γ: ratio of specific heats $= \frac{C_p^*}{C_v^*}$

C_p^*: specific heat at constant pressure

Non-dimensional Form of the Equations

We non-dimensionalize the equations by defining the following reference values:

U_∞^*: reference velocity

L_∞^*: reference length scale

ρ_∞^*: reference density

μ_∞^*: reference absolute viscosity

k_∞^*: reference thermal conductivity

$C_{v\infty}^*$: reference specific heat

All non-dimensional quantities are denoted without the superscript '*'. Hence,

$$\begin{aligned} u_i &= \frac{u_i^*}{u_\infty^*} \; ; \quad x_i = \frac{x_i^*}{L_\infty^*} \; ; \quad \rho = \frac{\rho^*}{\rho_\infty^*} =; \\ \mu &= \frac{\mu^*}{\mu_\infty^*} \; ; \quad \lambda = \frac{\lambda^*}{\mu_\infty^*} \; ; \quad k = \frac{k^*}{k_\infty^*} \; ; \quad C_v = \frac{C_v^*}{C_{v\infty}^*} \; . \end{aligned} \tag{6}$$

In addition, we non-dimensionalize the other quantities in equations (1)–(4) as follows:

$$
\begin{aligned}
t &= \frac{t^* U^*_\infty}{L^*} \ ; \quad p = \frac{p^*}{\rho^*_\infty\, U^{*2}_\infty} \ ; \quad F_i = \frac{F^*_i}{U^{*2}_\infty / L^*_\infty} \ ; \quad E = \frac{E^*}{U^{*2}_\infty} \\
H &= \frac{H^*}{U^{*2}_\infty} \ ; \quad \dot{q} = \frac{\dot{q}^*}{\rho^*_\infty\, U^{*3}_\infty \,/\, L^*_\infty} \ ; \quad \tau_{ij} = \frac{\tau^*_{ij}}{(U^*_\infty\, \mu^*_\infty)\,/\,L^*_\infty} \ .
\end{aligned} \tag{7}
$$

Upon substituting equations (6) and (7) into equations (1)–(2) and using equation (4), it may be easily shown that

$$
\begin{aligned}
\partial_t \rho + \partial_j \left(\rho u_j\right) &= 0 \ , \\
\partial_t \left(\rho u_i\right) + \partial_j \left[\rho u_i u_j + p\delta_{ij} - \frac{1}{\mathrm{Re}}\,\tau_{ij}\right] &= \rho F_i \ , \\
\partial_t(\rho E) + \partial_j \Bigg[\rho u_j H - \frac{1}{\mathrm{Re}}\, u_i \tau_{ij} &- \\
\frac{\gamma}{\mathrm{Re\,Pr}}\, k \partial_j \left\{ \frac{1}{C_v} \left(E - \frac{u_i u_i}{2} \right) \right\} \Bigg] &= \rho u_i F_i + \dot{q} \ ,
\end{aligned} \tag{8}
$$

where

$$
\tau_{ij} = \mu \left(\partial_j u_i + \partial_i u_j\right) - \frac{2}{3} \mu \delta_{ij} \partial_k u_k \ , \tag{9}
$$

and

$$
\begin{aligned}
\mathrm{Re} &= \frac{\rho^*_\infty u^*_\infty L^*_\infty}{\mu^*_\infty} \ , \\
\mathrm{Pr} &= \frac{\mu^*_\infty C^*_{p\infty}}{k^*_\infty} = \frac{\mu^*_\infty C^*_{v\infty} \gamma}{k^*_\infty} \ .
\end{aligned}
$$

Here, Re and Pr denote the Reynolds and Prandtl numbers, respectively. Finally, the non-dimensional form of equations (3) and (4) are given by

$$
H = E + \frac{p}{\rho} \ , \tag{10}
$$

$$
p = (\gamma - 1)\rho \left[E - \frac{u_i u_i}{2} \right] \ . \tag{11}
$$

Note: The Stokes' assumption has been invoked in equation (9).

Three-Dimensional Compressible Navier-Stokes Equations and the NAS Benchmark

Let the system of equations (8) be written as

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x_1} + \frac{\partial \mathbf{F}}{\partial x_2} + \frac{\partial \mathbf{G}}{\partial x_3} = \frac{\partial \mathbf{U}}{\partial x_1} + \frac{\partial \mathbf{V}}{\partial x_2} + \frac{\partial \mathbf{W}}{\partial x_3} + \mathbf{H} \,, \tag{12}$$

where

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho E \end{bmatrix};$$

$$\mathbf{E} = \begin{bmatrix} \rho u_1 \\ \rho u_1^2 + p \\ \rho u_1 u_2 \\ \rho u_1 u_3 \\ u_1(\rho E + p) \end{bmatrix}; \qquad \mathbf{F} = \begin{bmatrix} \rho u_2 \\ \rho u_1 u_2 \\ \rho u_2^2 + p \\ \rho u_2 u_3 \\ u_2(\rho E + p) \end{bmatrix}; \qquad \mathbf{G} = \begin{bmatrix} \rho u_3 \\ \rho u_1 u_3 \\ \rho u_2 u_3 \\ \rho u_3^2 + p \\ u_3(\rho E + p) \end{bmatrix};$$

$$\mathbf{U} = \mathrm{Re}^{-1} \begin{bmatrix} 0 \\ \tau_{11} \\ \tau_{21} \\ \tau_{31} \\ \alpha_1 \end{bmatrix}; \qquad \mathbf{V} = \mathrm{Re}^{-1} \begin{bmatrix} 0 \\ \tau_{12} \\ \tau_{22} \\ \tau_{32} \\ \alpha_2 \end{bmatrix}; \qquad \mathbf{W} = \mathrm{Re}^{-1} \begin{bmatrix} 0 \\ \tau_{13} \\ \tau_{23} \\ \tau_{33} \\ \alpha_3 \end{bmatrix};$$

and

$$\mathbf{H} = \begin{bmatrix} 0 \\ \rho F_1 \\ \rho F_2 \\ \rho F_3 \\ \rho u_k F_k + \dot{q} \end{bmatrix}.$$

The quantities α_i, $i = 1, 3$ are given by

$$\alpha_1 = u_k \tau_{k1} + \frac{\gamma k}{\mathrm{Pr}} \partial_1 \left\{ \frac{1}{C_v} \left(E - \frac{u_i u_i}{2} \right) \right\} ,$$

$$\alpha_2 = u_k \tau_{k2} + \frac{\gamma k}{\mathrm{Pr}} \partial_2 \left\{ \frac{1}{C_v} \left(E - \frac{u_i u_i}{2} \right) \right\} ,$$

$$\alpha_3 = u_k \tau_{k3} + \frac{\gamma k}{\mathrm{Pr}} \partial_3 \left\{ \frac{1}{C_v} \left(E - \frac{u_i u_i}{2} \right) \right\} .$$

Let

$$\mathbf{U} = \mathbf{U}_I + \mathbf{U}_E ,$$

$$\mathbf{V} = \mathbf{V}_I + \mathbf{V}_E ,$$

$$\mathbf{W} = \mathbf{W}_I + \mathbf{W}_E ,$$

so that $\frac{\partial \mathbf{U}_I}{\partial x_1}$, $\frac{\partial \mathbf{V}_I}{\partial x_2}$, and $\frac{\partial \mathbf{W}_I}{\partial x_3}$ involve no cross derivatives, and $\frac{\partial \mathbf{U}_E}{\partial x_1}$, $\frac{\partial \mathbf{V}_E}{\partial x_2}$, $\frac{\partial \mathbf{W}_E}{\partial x_3}$ do involve cross derivatives.

As demonstrated in the development of the Beam and Warming algorithm, only those terms which do not involve cross-derivatives are handled implicitly. Cross-derivative terms are handled explicitly. The focus of the NAS benchmark is on the implicit solution algorithm, and not on the evaluation of the right-hand side. Therefore, in effect, a solution of the following equation is sought:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x_1} + \frac{\partial \mathbf{F}}{\partial x_2} + \frac{\partial \mathbf{G}}{\partial x_3} - \frac{\partial \mathbf{U}_I}{\partial x_1} - \frac{\partial \mathbf{V}_I}{\partial x_2} - \frac{\partial \mathbf{W}_I}{\partial x_3} = \mathbf{H}^* , \qquad (13)$$

where

$$\mathbf{H}^* = \mathbf{H} + \frac{\partial \mathbf{U}_E}{\partial x_1} + \frac{\partial \mathbf{V}_E}{\partial x_2} + \frac{\partial \mathbf{W}_E}{\partial x_3} . \qquad (14)$$

In a real CFD application, equation (4) would be evaluated at each time step (for reference, see equation (22) of BW). However, in the NAS benchmark, the individual contributions to $\mathbf{H}^*$ are ignored, but in order to guarantee the existence of a steady-state solution, the function $\mathbf{H}^*$ is specified in the following manner. Let $\mathbf{q}^*$ denote the steady-state solution (which, for the NAS benchmark, is given). Then at steady state,

$$\mathbf{H}^* = \frac{\partial \mathbf{E}^*}{\partial x_1} + \frac{\partial \mathbf{F}^*}{\partial x_2} + \frac{\partial \mathbf{G}^*}{\partial x_3} - \frac{\partial \mathbf{U}_I^*}{\partial x_1} - \frac{\partial \mathbf{V}_I^*}{\partial x_2} - \frac{\partial \mathbf{W}_I^*}{\partial x_3} . \qquad (15)$$

If $\mathbf{H}^*$ is specified according to the above identity, a steady-state solution of equation (13) exists. Thus, for the NAS benchmark, we need to consider only those terms that do not involve cross-derivatives.

First consider the solution $\mathbf{q}$ and the inviscid fluxes (for reference, see page 46 of NAS benchmark).

Present Notation **NAS Benchmark Notation**

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho E \end{bmatrix} \qquad \equiv \qquad \mathbf{U} = \begin{bmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \\ u^{(4)} \\ u^{(5)} \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} \rho u_1 \\ \rho u_1^2 + p \\ \rho u_1 u_2 \\ \rho u_1 u_3 \\ u_1(\rho E + p) \end{bmatrix} \qquad \equiv \qquad -\mathbf{E} = \begin{bmatrix} u^{(2)} \\ \left[u^{(2)}\right]^2 \,/\, \left[u^{(1)}\right] + \phi \\ u^{(2)} u^{(3)} \,/\, u^{(1)} \\ u^{(2)} u^{(4)} \,/\, u^{(1)} \\ u^{(2)} \,/\, u^{(1)} \left[u^{(5)} + \phi\right] \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \rho u_2 \\ \rho u_1 u_2 \\ \rho u_2^2 + p \\ \rho u_2 u_3 \\ u_2(\rho E + p) \end{bmatrix} \qquad \equiv \qquad -\mathbf{F} = \begin{bmatrix} u^{(3)} \\ u^{(2)} u^{(3)} \,/\, u^{(1)} \\ \left[u^{(3)}\right]^2 \,/\, \left[u^{(1)}\right] + \phi \\ u^{(3)} u^{(4)} \,/\, u^{(1)} \\ u^{(3)} \,/\, u^{(1)} \left[u^{(5)} + \phi\right] \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \rho u_3 \\ \rho u_1 u_3 \\ \rho u_2 u_3 \\ \rho u_3^2 + p \\ u_3(\rho E + p) \end{bmatrix} \qquad \equiv \qquad -\mathbf{G} = \begin{bmatrix} u^{(4)} \\ u^{(2)} u^{(4)} \,/\, u^{(1)} \\ u^{(3)} u^{(4)} \,/\, u^{(1)} \\ \left[u^{(4)}\right]^2 \,/\, \left[u^{(1)}\right] + \phi \\ u^{(4)} \,/\, u^{(1)} \left[u^{(5)} + \phi\right] \end{bmatrix}$$

$$p = (\gamma - 1)\left[\rho E - \frac{\rho u_i u_i}{2}\right] \equiv$$

$$\phi = k_2 \left[u^{(5)} - \frac{1}{2} \left\{ \frac{\left[u^{(2)}\right]^2 + \left[u^{(3)}\right]^2 + \left[u^{(4)}\right]^2}{u^{(1)}} \right\} \right] ,$$

$$\Rightarrow k_2 = \gamma - 1\ . \tag{16}$$

Next, consider the momentum equations. It follows from equation (9) that

$$\partial_j\,\tau_{ij} = \begin{bmatrix} \partial_1\left(\frac{4}{3}\mu\partial_1 u_1\right) + \partial_2\left(\mu\partial_2 u_1\right) & + \partial_3\left(\mu\partial_3 u_1\right) & + \text{c.d.} \\ \partial_1\left(\mu\partial_1 u_2\right) \quad + \partial_2\left(\frac{4}{3}\mu\partial_2 u_2\right) & + \partial_3\left(\mu\partial_3 u_2\right) & + \text{c.d.} \\ \partial_1\left(\mu\partial_1 u_3\right) \quad + \partial_2\left(\mu\partial_2 u_3\right) & + \partial_3\left(\frac{4}{3}\mu\partial_3 u_3\right) & + \text{c.d.} \end{bmatrix}\ , \tag{17}$$

where 'c.d.' denotes cross-derivative terms. Next, consider the energy equation. Upon substituting for τ_{ij} from equation (9), it may be shown that

$$\begin{aligned} & \partial_j\left[u_i\tau_{ij} + \frac{\gamma k}{\Pr}\partial_j\left\{\frac{1}{C_v}\left(E - \frac{u_i u_i}{2}\right)\right\}\right] \\ = \ & \partial_1\left[\frac{\mu}{2}\left(1 - \frac{\gamma k}{\Pr\mu C_v}\right)\partial_1\left(u_1^2 + u_2^2 + u_3^2\right) + \frac{\mu}{6}\partial_1\left(u_1^2\right) + \frac{\gamma k}{\Pr C_v}\partial_1 E\right] \\ + \ & \partial_2\left[\frac{\mu}{2}\left(1 - \frac{\gamma k}{\Pr\mu C_v}\right)\partial_2\left(u_1^2 + u_2^2 + u_3^2\right) + \frac{\mu}{6}\partial_2\left(u_2^2\right) + \frac{\gamma k}{\Pr C_v}\partial_2 E\right] \\ + \ & \partial_3\left[\frac{\mu}{2}\left(1 - \frac{\gamma k}{\Pr\mu C_v}\right)\partial_3\left(u_1^2 + u_2^2 + u_3^2\right) + \frac{\mu}{6}\partial_3\left(u_3^2\right) + \frac{\gamma k}{\Pr C_v}\partial_3 E\right] + \text{c.d.}\ , \\ & \text{where } C_v \text{ is assumed const.} \end{aligned} \tag{18}$$

Now we can construct the functions $\mathbf{U}_I$, $\mathbf{V}_I$, and $\mathbf{W}_I$ using equations (17) and (18).

Present Notation		**NAS Benchmark Notation**

$$\mathbf{U}_I = \mathrm{Re}^{-1} = \begin{bmatrix} 0 \\ \frac{4}{3}\mu\partial_1 u_1 \\ \mu\partial_1 u_2 \\ \mu\partial_1 u_3 \\ u_I^5 \end{bmatrix} \quad \equiv \quad T = \begin{bmatrix} 0 \\ \frac{4}{3}k_3 k_4\frac{\partial}{\partial\xi}\left(\frac{u^{(2)}}{u^{(1)}}\right) \\ k_3 k_4\frac{\partial}{\partial\xi}\left(\frac{u^{(3)}}{u^{(1)}}\right) \\ k_3 k_4\frac{\partial}{\partial\xi}\left(\frac{u^{(4)}}{u^{(1)}}\right) \\ t^{(5)} \end{bmatrix}$$

where

$$u_I^5 = \frac{\mu}{2}\left(1 - \frac{\gamma k}{\mathrm{Pr}\mu C_v}\right)\partial_1\left(u_1^2 + u_2^2 + u_3^2\right) + \frac{\mu}{6}\partial_1\left(u_1^2\right) + \frac{\gamma k}{\mathrm{Pr} C_v}\partial_1 E \ ,$$

and

$$\begin{aligned} t^{(5)} &= \frac{1}{2}(1 - k_1k_5)\frac{\partial}{\partial \xi}\left[\frac{\left[u^{(2)}\right]^2 + \left[u^{(3)}\right]^2 + \left[u^{(4)}\right]^2}{\left[u^{(1)}\right]^2}\right] \\ &+ \frac{1}{6}\frac{\partial}{\partial \xi}\left[\frac{\left[u^{(2)}\right]^2}{\left[u^{(1)}\right]^2}\right] + k_1k_5\frac{\partial}{\partial \xi}\left[\frac{u^{(5)}}{u^{(1)}}\right] \ . \end{aligned}$$

Similarly, the correspondence between $\mathbf{V}_I$ and $\mathbf{V}$, and, $\mathbf{W}_I$ and $\mathbf{W}$, may be constructed.

Discrepancies

(A) Physical Properties

- Comparison of elements 2–4 (i.e., the momentum equations) indicates that
$$k_3k_4 = \mathrm{Re}^{-1}\mu \ . \tag{19}$$

- Comparison of the energy equation indicates inconsistencies. The two forms may be reconciled if $t^{(5)}$ is correctly rewritten as
$$\begin{aligned} t^{(5)} &= k_3k_4\Bigg\{\frac{1}{2}(1 - k_1k_5)\frac{\partial}{\partial \xi}\left[\frac{\left[u^{(2)}\right]^2 + \left[u^{(3)}\right]^2 + \left[u^{(4)}\right]^2}{\left[u^{(1)}\right]^2}\right] \\ &+ \frac{1}{6}\frac{\partial}{\partial \xi}\left[\frac{\left[u^{(2)}\right]^2}{\left[u^{(1)}\right]^2}\right] + k_1k_5\frac{\partial}{\partial \xi}\left[\frac{u^{(5)}}{u^{(1)}}\right]\Bigg\} \ , \end{aligned}$$

  where
$$k_1k_5 = \frac{\gamma k}{\mathrm{Pr}\mu C_v} \ . \tag{20}$$

Note: Although the above factor of k_3k_4 is missing in the definitions of $t^{(5)}$, $v^{(5)}$, and $w^{(5)}$ (c.f. see pages 47–48 NAS-BM), it is correctly taken into account subsequently in the analytical evaluation of the various Jacobian matrices (c.f., see $n_{51}\dots n_{55}$; $q_{51}\dots q_{55}$; $s_{51}\dots s_{55}$ on pages 55–56 NAS-BM).

Discussion of the values of k_i, $i = 1, 5$

Although the values of k_i are immaterial insofar as the parallel implementation of the computational algorithm is concerned, we discuss here how well these values correspond to those encountered in a "real" CFD application code.

The suggested values in the NAS benchmark are as follows:

$$k_1 = 1.4,\;\; k_2 = 0.4,\;\; k_3 = 0.1,\;\; k_4 = 1.0,\;\; k_5 = 1.4.$$

However, the above appear in only three groups, viz., $k_2, (k_1k_5)$ and (k_3k_4). The first constant k_2 was found to be $k_2 = \gamma - 1$ (c.f. equation (16)). Since the ratio of specific heats, γ, for a diatomic gas is 1.4 (such as air which is composed mainly of $N_2 + O_2$), the above value for k_2 is appropriate.

Next, consider k_1k_5 which is given by equation (20). Recall that $\mathrm{Pr} = \frac{\mu^*_\infty C^*_{p\infty}}{k^*_\infty}$, and, hence,

$$k_1k_5 = \frac{\gamma}{\mathrm{Pr}_{\mathrm{local}}}\;,$$

where

$$\mathrm{Pr}_{\mathrm{local}} = \frac{\mu^* C^*_p}{k^*}\;.$$

For a very wide range of temperatures, $\mathrm{Pr} \simeq 0.72$ (for air). Since $\mathrm{Pr}^{-1} \simeq 1.4$, then the values of k_1 and k_5 specified by the NAS benchmark are appropriate.

On the other hand, for most flows of interest, $\mathrm{Re} \sim \mathrm{O}\left(10^6\right)$ or higher. Hence, the relatively high values of k_3k_4 does not correspond well to the values in a typical "real" CFD code. However, since second-order central differences are used to discretize the governing equations, a low value of k_3k_4 would produce non-physical oscillatory solutions. Therefore, a relatively high value of k_3k_4 is chosen to circumvent such numerical difficulties in the solution of the synthetic PDE's.

(B) Additional terms in the NAS benchmark

All such terms are associated with the constants $d_\xi^{(m)}$, $d_\eta^{(m)}$ and $d_\zeta^{(m)}$, $m = 1, 5$. An exact correspondence with the original N-S equations would be possible if the above constants are all zero; however, they are O(1) in the NAS benchmark. Upon inspection of these terms, it is clear that they all represent an added second-order dissipation although this has not been explicitly stated in the text of the NAS-BM. In practice, a blend of second and fourth-order dissipation functions are usually incorporated into the discretized equations to suppress high-frequency, non-physical oscillations. However, such a second-order dissipation function with constants of O(1) would produce incorrect (excessively diffused) solutions in a real CFD application. Note, however, that the inclusion of these terms does not alter the computational algorithm to solve these equations nor does it alter the communication pattern.