A Description of Cook's Code for Finding Solutions to the Initial Value Equations in General Relativity That Correspond to Two Black Holes in Stable, Quasi-circular Orbit

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1 Introduction

One problem that faces the Grand Challenge is the general solution for the initial data that corresponds to a two black hole system. While the general problem remains unsolved, a class of solutions can be found that correspond to two black holes with arbitrary radius, each with arbitrary linear momentum and arbitrary spin angular momentum. The result is a 14 parameter space of solutions (1 for radius of black hole 1, 1 for radius of black hole 2, 3 for linear momentum of black hole 1, 3 for linear momentum of black hole 2, 3 for spin angular momentum of black hole 1, 3 for spin angular momentum of black hole 2 = 14 total parameters). This paper is a description (for the non-specialist) of the code written by Cook to solve the problem in a coordinate system that is naturally adapted to a two-black hole spacetime, called Cadez coordinates. First, the initial value constraint equations are presented. Simplifications are introduced, thus simplifying the form of the initial value constraints. These simplified equations are then solved. Three of the four equations (the three momentum constraints) can be solved analytically as an infinite power series. The fourth equation (the Hamiltonian constraint), a quasi-linear elliptic equation, is then solved using a FAS (Full Approximation Storage) multigrid scheme, which is described. The final step for obtaining initial data that corresponds to two black holes in stable, quasi-circular orbit is employing the effective potential method.

This paper is a compilation of previous work. All the information contained in this paper can be found in the articles listed in the bibliography.

2 Initial Value Equations: Simplifications

The initial value constraints are differential equations involving the twelve functions q_{ij} and K_{ij} of the 3 spatial coordinates, $x^i = (x^1, x^2, x^3)$. (These coordinates could be Cartesian , spherical, cylindrical, etc.) Remember that both q_{ij} and K_{ij} can be thought of as 3 X 3 symmetric matrices of functions of the coordinates, x^i (the indices i and j will always range from 1 to 3). The momentum and Hamiltonian constraints (in covariant notation) are respectively

$$q^{jk}\mathcal{D}_j K_{ki} - \mathcal{D}_i K = 0 \tag{1}$$

and

$$R + K^2 - q^{ik}q^{jl}K_{ij}K_{kl} = 0 \tag{2}$$

where $\mathcal{D}$ is the covariant derivative operator, K is the trace of the extrinsic curvature, and R is the scalar curvature. This covariant language has great advantages when doing analytical work, but it hides the numerical complexity. Written out as explicit derivatives of the dependent variables, q_{ij} and K_{ij}, the momentum and Hamiltonian constrains are respectively

$$q^{jk}(\frac{\partial K_{ik}}{\partial x^j} - \frac{\partial K_{jk}}{\partial x^i} + K_{jm}\Gamma^m_{ik} - K_{im}\Gamma^m_{jk}) = 0$$

and

$$q^{ij}(\frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{jk}}{\partial x^i} + \Gamma^m_{ij}\Gamma^k_{mk} - \Gamma^m_{jk}\Gamma^k_{im}) + (K_{ij}q^{ij})^2 - K_{ik}K_{jm}q^{ij}q^{km} = 0$$

where q^{ij} (note the superscripted indices) is the inverse of the 3–metric, q_{ij}. That is,

$$\begin{bmatrix} q^{11} & q^{12} & q^{13} \\ q^{12} & q^{22} & q^{23} \\ q^{13} & q^{23} & q^{33} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix}^{-1}$$

The Christoffel symbols, Γ^k_{ij} are simply combinations of derivatives of the 3–metric:

$$\Gamma^k_{ij} = \frac{1}{2}q^{km}(\frac{\partial q_{im}}{\partial x^j} + \frac{\partial q_{jm}}{\partial x^i} - \frac{\partial q_{ij}}{\partial x^m})$$

Also recall that we are employing the Einstein summation convention: any repeated indices imply a summation over those indices.

The assumptions for simplifying the equations are as follows:

1. Set the trace of the extrinsic curvature, K = $q^{ij}K_{ij}$, equal to 0.

2. Assume the 3–metric, q_{ij} is conformally flat, i.e. can be written as $q_{ij} = \psi^4 f_{ij}$, where ψ is called the conformal factor (it must be strictly positive), and f_{ij} is a flat 3–metric.

The momentum and Hamiltonian constraints now take the form

$$f^{jk}\partial_j A_{ik} = 0 \tag{3}$$

and

$$\nabla^2\psi = -\frac{1}{8}\psi^{-7} f^{ik} f^{jl} A_{ij} A_{kl} \tag{4}$$

where ∇^2 is the ordinary, flat-space Laplacian, ψ is the conformal factor, and A_{ij} is called the trace-free conformal background extrinsic curvature. A_{ij} can be thought of as five independent functions, arranged as a symmetric 3 X 3 matrix that is trace free, i.e., $f^{ij}A_{ij} = 0$.

Once these equations are solved for A_{ij} and ψ, the original variables, which are

$$K_{ij} = \psi^{-2} A_{ij} \tag{5}$$

and

$$q_{ij} = \psi^4 f_{ij} \tag{6}$$

are guarantied to be solutions to the initial value constraints, eqs. (1) and (2).

3 Solving the Momentum Constraints

The solution of the momentum constraint (3) (under certain isometry conditions) has been found [1], and an accurate and efficient numerical scheme for evaluating the formal infinite series solution has been found [2]. In the construction of a solution to the momentum constraints, values of the coordinate radius of each hole, $a_{1,2}$, and coordinate center of each hole, $C_{1,2}$, are fixed, along with the linear momentum, $P_{1,2}$ and spin $S_{1,2}$ of each hole.

4 Solving the Hamiltonian Constraint

Given a solution of the momentum constraint, (3), the Hamiltonian constraint (4) can be solved as a quasi-linear elliptic boundary-value problem for the conformal factor ψ, given appropriate boundary conditions. The inner boundary conditions at the throat of each hole, whose location is

$B_\alpha = \{\vec{x} : |\vec{x} - \vec{C_\alpha}| = a_\alpha\}$ is given by the isometry condition, and is written as

$$n^i_\alpha \partial_i \psi|_{B_\alpha} = -\frac{psi}{2r_\alpha}|_{B_\alpha}$$

where $r_\alpha = |\vec{x} - \vec{C_\alpha}|$ and $n_\alpha = (\vec{x} - \vec{C_\alpha})/r_\alpha$.

The outer boundary condition follows directly from asymptotic and conformal flatness:

$$\psi \to 1 \quad \texttt{as} \quad r \to \infty$$

Using a truncated multipole expansion for ψ is used to approximate the above outer boundary condition:

$$\frac{\partial \psi}{\partial r} = \frac{1 - \psi}{r}$$

4.1 Numerical Solution of the Hamiltonian Constraint

The coordinates used in the numerical solution of the Hamiltonian Constraint (4) are Cadez coordinates. Because this coordinate system has constant coordinate surfaces which are boundary conforming, the set of finite-difference equations which we obtain from the constraint and its boundary conditions are particularly simple and well behaved. The primary concern with any finite-difference scheme based on Cadez coordinates is that it must deal effectively with coordinate singularities. The Cadez coordinate system is a numerically generated coordinate system based on a complex-plane transformation from cylindrical coordinates (see [2]). The three Cadez coordinates (η, ξ, ϕ) are based in cylindrical coordinates (ρ, ϕ, z). $\eta(\rho, z)$ is a radial-like coordinate and $\xi(\rho, z)$ is an angular coordinate. The azimuthal coordinate ϕ is unchanged from cylindrical coordinates. The coordinate system is logically composed of three regions (see. Figure):

$$\texttt{region 1} \quad \equiv \begin{cases} \eta^+ \leq \eta < \eta_s, \\ 0 \leq \xi \leq \xi_s, \\ 0 \leq \phi < 2\pi \end{cases}$$

$$\texttt{region 2} \quad \equiv \begin{cases} \eta^- \leq \eta < \eta_s, \\ \xi_s \leq \xi \leq \pi, \\ 0 \leq \phi < 2\pi \end{cases}$$

$$\texttt{region 3} \quad \equiv \begin{cases} \eta_s \leq \eta < \infty, \\ 0 \leq \xi \leq \pi, \\ 0 \leq \phi < 2\pi \end{cases}$$

This three-region nature is the consequence of a coordinate singularity at (η_s, ξ_s) which itself is a consequence of demanding that constant-η surfaces be coincident with each of the two throats while at the same time demanding that constant-η surfaces surround both holes at large radius.

4.2 FAS: The Full Approximation Storage Algorithm

The general spirit of the FAS method is to obtain a solution to a set of equations on a fine grid by solving the equations on a coarser grid, where it is possible to solve the equations more efficiently. Let the differential equations which we are trying to solve be denoted by

$$\mathcal{L}u = f$$

where $\mathcal{L}$ is some differential operator on functions u, with source terms f. Consider a two level system consisting of a fine grid G^k and a coarser grid G^{k-1}. Each grid has its respective finite difference approximation to $\mathcal{L}$, denoted as $\mathcal{L}^k$ and $\mathcal{L}^{k-1}$. Let us take two operators which smoothly take values from one grid to the other. The prolongation operator p^k_{k-1} takes values from the coarse grid to the finer grid and the restriction operator r^{k-1}_k takes values from the fine grid to the coarser grid.

Assume we have an estimate for a solution on the fine grid k : $\bar{u}^k$. The FAS algorithm tells us how to update the estimate:

$$\bar{u}^k = \bar{u}^k + p^k_{k-1}(\tilde{u}^{k-1} - r^{k-1}_k \bar{u}^k)$$

where $\tilde{u}^{k-1}$ is obtained by solving the following equation on the coarser grid G^{k-1}:

$$\mathcal{L}^{k-1}\tilde{u}^{k-1} = f^{k-1} + \mathcal{L}^{k-1}(r^{k-1}_k \bar{u}^k) - r^{k-1}_k \mathcal{L}^k \bar{u}^k$$

At this point, the general philosophy can be seen. The fine grid solution is found by taking an estimate of the solution on the fine grid and using this to generate a correction to the finite difference equations on the next coarser grid. The solution on the coarser grid is then used to correct the solution on the fine grid. This requires the solution of the equations on the coarser grid. If there are still too many equations on the coarser grid to solve them efficiently, then a step can be made to an even coarser grid. This promotion of the problem to coarser and coarser grids is continued until the problem is on a grid with few enough points to be solved efficiently.

4.3 Finite Difference Equations

For the actual finite difference equations, see section III A in [3].

5 The Effective Potential Method

In the previous sections, the method of finding solutions to the initial value equations has been sketched out. These solutions depend on 14 parameters. Because of the circularizing effects of gravitational radiation damping, it is expected that the orbits of most tight binary systems to have small eccentricities. Therefore, a method is needed which can discern which data sets, within the very large parameter space of binary black hole initial data sets, correspond to binary black holes in quasi-circular orbit. To accomplish this, the effective binding energy E_b is defined as

$$E_b \equiv E - M_1 - M_2$$

where E is the ADM energy

$$E = -\frac{1}{2\pi} \oint_\infty \partial^i \psi \; d^2 S_i$$

and $M_{1,2}$ are the masses of the two black hole,

$$M^2 = {M_{\mathtt{ir}}}^2 + \frac{S^2}{4{M_{\mathtt{ir}}}^2}$$

The irreducible mass is approximated by $M_{\mathtt{ir}} \approx \sqrt{A/(16\pi)}$ (A is the area of the apparent horizon of the hole). S is the magnitude of the spin of the black hole.

To decrease the parameters that define a solution to the initial value equations, we impose the following conditions:

$$\vec{P_1} + \vec{P_2} = 0$$

$$\vec{P_{1,2}} \cdot (\vec{C_2} - \vec{C_1}) = 0$$

The first condition demands that the configuration be in a center of momentum frame. The second condition should be satisfied for configurations representing a quasi-circular orbit. These conditions decrease the number of parameters to 9. These 9 dimensionless parameters are α, β, P/a_1, $\vec{S_1}/a_1^2$, and $\vec{S_2}/a_1^2$, where

$$\alpha \equiv \frac{a_1}{a_2}$$

$$\beta \equiv \frac{|\vec{C}_2 - \vec{C}_1|}{a_1}$$

However, the effective potential depends on the 9 dimensionless *physical* parameters $X \equiv M_1/M_2$, l/m, $J/\mu m$, $\vec{S}_1/M_1^2$, and $\vec{S}_2/M_2^2$ where l is the shortest proper separation between the apparent horizons of the two black holes, $m \equiv M_1 + M_2$ is the total mass, $\mu \equiv M_1 M_2/m$ is the reduced mass, and $J = |\vec{J}|$ is the magnitude of the orbital angular momentum of the system which is defined as

$$\vec{J} \equiv \left(\vec{C}_1 - \frac{\vec{d}}{E}\right) \times \vec{P}_1 + \left(\vec{C}_2 - \frac{\vec{d}}{E}\right) \times \vec{P}_2$$

In the above definition of orbital angular momentum, $\vec{d}$ is the dipole moment of the energy distribution, defined as

$$d_i = -\frac{1}{2\pi} \oint_\infty \left[x_i \partial^j \psi + (1-\psi)\delta_i^j\right] d^2 S_j$$

Finally, the dimensionless effective potential is given by the binding energy as E_b/μ.

Finding quasi-circular orbits is now a conceptually easy task. We compute E_b/μ as a function of l/m while holding the remaining eight *physical* parameters constant. A minimum in E_b/μ then corresponds to a “stable” quasi-circular orbit. Though conceptually straightforward, the computational task of locating quasi-circular orbits is difficult. The main difficulty arises from the fact that the physical parameters that must be held fixed are not independently correlated with their respective initial-data parameters. That is, holding eight of the nine initial-data parameters fixed while varying β will not result in an effective potential curve. We see then that, in general, the problem of determining *one* quasi-circular orbit is quite involved. It requires finding the roots of eight functions, each of which depends on eight parameters, at each value of the separation at which the effective potential is evaluated. It turns out that only the magnitude of the two initial-data spin-vector parameters needs to be varied to hold the physical parameters fixed. As a result, for a fixed value of β the following four equations must be satisfied:

$$X_{(\alpha, P/a_1, S_1/a_1^2, S_2/a_1^2; \beta)} = X_0 \tag{7}$$

$$\left[\frac{J}{\mu m}\right]_{(\alpha, P/a_1, S_1/a_1^2, S_2/a_1^2; \beta)} = \left[\frac{J}{\mu m}\right]_0 \tag{8}$$

$$\left[\frac{S_1}{M_1^2}\right]_{(\alpha,P/a_1,S_1/a_1^2,S_2/a_1^2;\beta)} = \left[\frac{S_1}{M_1^2}\right]_0 \tag{9}$$

$$\left[\frac{S_2}{M_2^2}\right]_{(\alpha,P/a_1,S_1/a_1^2,S_2/a_1^2;\beta)} = \left[\frac{S_2}{M_2^2}\right]_0 \tag{10}$$

When these four equations are satisfied, we have a value of the effective potential E_b/μ at some value of the physical separation l/m. Changing the value of β and resolving Eqns. (7) - (10) results in another value of the effective potential at a different separation.

6 Skeleton of the Code

Cook's code can be broken down into several modules.

1. This module will solve the momentum constraint (3) for the A_{ij} using the 14 input data α, β, $\vec{P_{1,2}}$, and $\vec{S_{1,2}}$. The method of solution is evaluating the analytic solution which exists in the form of an infinite power series.

2. This module will solve the Hamiltonian constraint (4) for the conformal factor ψ using the FAS algorithm outlined in section (4).

3. This module will solve equations (7) -(10) for specific values of the physical parameters.

4. The main module will find the minimum of the effective potential as a function of l/m, keeping all other physical parameters fixed. The initial data that corresponds to this minimum will be initial data for two black holes in stable, quasi-circular orbit.

References

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