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A Critical Examination of $\eta\pi^-$ Partial-Wave Analysis

—Version II—

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Abstract

A critical analysis is given of the assumptions necessary for a partial-wave analysis of the $\eta\pi^-$ system in the reaction $\pi^- p \to \eta\pi^- p$.

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1 Introduction

This note concerns the assumptions necessary for a partial-wave analysis of the $\eta\pi^-$ system in the reaction

$$\pi^- p \to \pi^- \eta\; p \tag{1}$$

As pointed out in a previous note,[1] one needs to make two assumptions—to carry out an amplitude analysis of the $\eta\pi^-$ system. The first assumption is that a state $|\ell\, m\rangle$ for which $|m| \geq 2$ is absent among the partial waves to be fitted. This is true, of course, in the limit of $-t = 0$, as the nucleon helicities give rise to the states with $m = 0$ or $m = \pm 1$ only. But this assumption can be dealt with—experimentally—since the moments $H(LM)$ with $M = 3$ or $M = 4$ could be checked, to see how important the states $|\ell\, m\rangle$ are in the data with $|m| \geq 2$.[b] The second assumption needed for a partial-wave analysis is that the density matrix has rank 1, i.e. the spin amplitudes do not depend on the nucleon helicities. Our justification, so far, has been that the fitted partial waves are very reasonable, that it can be fitted with a very simple mass-dependent formula, that Pomeron-exchange amplitudes are in genral independent of nucleon helicities, and so on...

The purpose of this note is to point out that, under a simple model for mass dependence of the partial waves, it is possible to *prove* that the spin density matrix has rank 1. Suppose that one has found a satisfactory fit under a rank-1 assumption. One can then show that, even if the problem involves both spin-nonflip and spin-flip at the nucleon vertex—i.e. it appears to be a rank-2 problem—the spin density matrix in reality has a rank of 1. Although this note is based on the results of our $\eta\pi^-$ analysis, the derivation does not depend on the decay channels; the conclusions apply equally well to any decay channel, e.g. the 1^{-+} state at 1.6 GeV coupling to $\pi\rho$ (a further discussion on this point is given at the end of Section 3).

This note relies on some technicalities generally well known, and so they have been presented without attribution. The reader may wish to consult a number of preprints and/or papers, which deal with them in some detail[2, 3, 4, 5, 6].

[b] A. Ostrovidov checked this out in our E852 data; the moments $H(33)$, $H(43)$ and $H(44)$ are all small in the $a_2(1320)$ region, and a fit including $|22\rangle$ shows a very small amount of this wave and is very broad. In particular, it does not affect our P_+ wave!

2 Partial Waves Produced via Natural-parity Exchange

Consider our $\eta\pi^-$ system produced via natural-parity exchange. It consists of just two waves D_+ and P_+ in the $a_2(1320)$ region. Without loss of generality, the decay amplitudes[1] can be considered real, i.e.

$$\begin{aligned} A_D(\Omega) &= \sqrt{\frac{5}{4\pi}}\,\sqrt{2}\,d^2_{10}(\theta)\,\sin\phi = -\sqrt{\frac{5}{4\pi}}\,\sqrt{3}\,\sin\theta\,\cos\theta\sin\phi \\ A_P(\Omega) &= \sqrt{\frac{3}{4\pi}}\,\sqrt{2}\,d^1_{10}(\theta)\,\sin\phi = -\sqrt{\frac{3}{4\pi}}\,\sin\theta\,\sin\phi \end{aligned} \tag{2}$$

Since one deals with the partial waves natural-parity exchange only, one can drop the subscript '+' from the waves, and the angular distribution resulting from the natural-parity exchange only is simply given by

$$\begin{aligned} I(\Omega) &\propto |D\,A_D(\Omega) + P\,A_P(\Omega)|^2 \\ &\propto \left(\frac{3}{4\pi}\right)\left|\sqrt{5}D\cos\theta + P\right|^2\,\sin^2\theta\,\sin^2\phi \\ &\propto \left(\frac{3}{4\pi}\right)\left[\,5|D|^2\,\cos^2\theta + 2\sqrt{5}\,\Re\{D^*P\}\,\cos\theta + |P|^2\,\right]\,\sin^2\theta\,\sin^2\phi \end{aligned} \tag{3}$$

The integration over the angles can be carried out easily, to obtain

$$\int I(\Omega)\,\mathrm{d}\Omega \propto |D|^2 + |P|^2 \tag{4}$$

as expected.

The spin density matrix is given by

$$I(\Omega) \propto |D\,A_D(\Omega) + P\,A_P(\Omega)|^2 = \sum_{k,k'} \rho_{k,k'}\,A_k\,A^*_{k'} \tag{5}$$

where $\{k, k'\} = \{1, 2\}$ and '1' ('2') corresponds to D (P). From this definition, one sees that

$$\rho = \begin{pmatrix} |D|^2 & D\,P^* \\ D^*\,P & |P|^2 \end{pmatrix} \tag{6}$$

One can work out the eigenvalues of this 2×2 matrix:

$$\lambda = \{|D|^2 + |P|^2,\ 0\} \tag{7}$$

One of the two allowed eigenvalues is zero, i.e. the rank of this matrix is 1. This is the 'rank-1' assumption one makes to carry out the partial-wave analysis and is valid for a given mass bin.

Suppose now that the rank is 2, i.e.

$$I(\Omega) \propto |D_1\, A_{_D}(\Omega) + P_1\, A_{_P}(\Omega)|^2 + |D_2\, A_{_D}(\Omega) + P_2\, A_{_P}(\Omega)|^2 \tag{8}$$

where subscripts 1 and 2 stand for spin-nonflip and spin-flip amplitudes at the nucleon vertex for reaction (1). Comparing (3) and (8), one finds immediately

$$\begin{aligned} |D|^2 &= |D_1|^2 + |D_2|^2 \\ |P|^2 &= |P_1|^2 + |P_2|^2 \\ \Re\{P^*\, D\} &= \Re\{P_1^*\, D_1\} + \Re\{P_2^*\, D_2\} \end{aligned} \tag{9}$$

Let w be the effective mass of the $\eta\pi^-$ system. If the mass dependence is included explicitly in the formula, one should write, in the case of rank 1,

$$\frac{\mathrm{d}\sigma(w,\Omega)}{\mathrm{d}w\,\mathrm{d}\Omega} \propto |D(w)\, A_{_D}(\Omega) + P(w)\, A_{_P}(\Omega)|^2\, pq \tag{10}$$

where p is the breakup momentum of the $\eta\pi^-$ system in the overall CM system and q is the breakup momentum of η in the $\eta\pi^-$ rest frame. Note that both p and q depend on w. Note also that the w dependence of the partial waves D and P are given in the formula. Obviously, a similar expression could be written down for the case of rank 2.

One is now ready to make the one crucial assumption for a mass-dependent analysis of the D and P waves: one assumes that two resonances—in D and P waves, respectively—are produced in *both* spin-nonflip and spin-flip amplitudes. One may then write, for the rank-1 case,

$$\begin{aligned} D(w) &= a\, \mathrm{e}^{i\,\alpha}\, \mathrm{e}^{i\,\delta_a}\, \sin\delta_a \\ P(w) &= b\, \mathrm{e}^{i\,\delta_b}\, \sin\delta_b \end{aligned} \tag{11}$$

where a, b and the production phase α are all real and *independent* of the $\eta\pi^-$ mass. In addition, one can set $a \geq 0$ and $b \geq 0$ without loss of generality. δ_a and δ_b are the phase-shifts corresponding to the resonances and highly mass dependent. In its generic form, the Breit-Wigner formula is given the usual expression

$$\cot\delta = \frac{w_0^2 - w^2}{w_0\,\Gamma_0} \tag{12}$$

where w_0 and Γ_0 are the standard resonance parameters. In this note, the width is considered independent of w. Likewise, the barrier factor dependence for D and P is ignored.[c]

The formulas (11) are generalized to the case of rank 2, as follows:

$$\begin{aligned} D_1(w) &= a_1\, \mathrm{e}^{i\,\alpha_1}\, \mathrm{e}^{i\,\delta_a}\, \sin\delta_a \\ P_1(w) &= b_1\, \mathrm{e}^{i\,\delta_b}\, \sin\delta_b \\ D_2(w) &= a_2\, \mathrm{e}^{i\,\alpha_2}\, \mathrm{e}^{i\,\delta_a}\, \sin\delta_a \\ P_2(w) &= b_2\, \mathrm{e}^{i\,\delta_b}\, \sin\delta_b \end{aligned} \tag{13}$$

Once again, a_i, b_i and α_i are real, $a_i \geq 0$ and $b_i \geq 0$, and *independent* of w. One finds, using (9),

$$\begin{aligned} a^2 &= a_1^2 + a_2^2 \\ b^2 &= b_1^2 + b_2^2 \\ ab\cos(\alpha+\delta_a-\delta_b) &= a_1 b_1 \cos(\alpha_1+\delta_a-\delta_b) + a_2 b_2 \cos(\alpha_2+\delta_a-\delta_b) \end{aligned} \tag{14}$$

A plot of $\cos(\alpha+\delta_a-\delta_b)$ as a function of w is shown in Fig. 1 for three values of α, i.e. $0°$, $45°$ and $90°$. The resonance parameters for a and b (see Table) have been taken from our E852 paper[7],[d] and the normalized absolute squares of the Breit-Wigner forms are given in Fig. 2, as well as the 'normalized' interference term. The same quantities, as they appear in our paper, are shown in Fig. 3. This figure shows how important the interference term is compared to the P-wave term. Note also how rapidly the interference term varies as a function of w in the $a_2(1320)$ region. This term, of course, is intimately related to the asymmetry in the Jackson angle and vanishes when integrated over the angle, i.e. it does not contribute to the mass spectrum [see (3) and (4)]. Fig. 4 shows the contour plot of the intensity distribution in w vs. $\cos\theta$; note variation of the asymmetry as a function w.

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[c] Although simplified formulas are used in this note, the results given in this note do *not* change even when correct formulas are used. Note that, to go over to a correct formulation for each wave, one needs to substitute the absolute value of the Breit-Wigner formula as follows:

$$\sin\delta(w) \to B(q)\,\left[\frac{\Gamma_0}{\Gamma(w)}\right]\,\sin\delta(w)$$

where $B(q)$ is the barrier factor and $\Gamma(w)$ is the mass-dependent width. It should be noted that the correction factors are all real, by definition.

[d] The value of α as given in this paper is $37.46°$; for the purpose of illustration, one may consider $\alpha = 45°$ close enough.

Table 1. Parameters* taken from our E852 paper[7]:

Partial Wave	Mass (GeV)	Width (GeV)	Production Phase	Magnitude
D	1.317	0.127	$\alpha = 37.46°$	$a = 1.0$
P	1.370	0.385	$\alpha = 0°$	$b = 0.151$

* Refer to (11) for the notations.
The waves are normalized such that $D(w) = 1$ at $w = 1.317$ GeV.

For the last equation in (14) to be true for any mass, the coefficient of $\cos(\delta_a - \delta_b)$ or $\sin(\delta_a - \delta_b)$ on the left-hand side must be equal to that on the right-hand side, so that

$$\begin{aligned} ab\cos\alpha &= a_1 b_1 \cos\alpha_1 + a_2 b_2 \cos\alpha_2 \\ ab\sin\alpha &= a_1 b_1 \sin\alpha_1 + a_2 b_2 \sin\alpha_2 \end{aligned} \tag{15}$$

Take a sum of the squares of the two formulas above and introduce the first two equations of (14):

$$\begin{aligned} & 2a_1 b_1 a_2 b_2 \ \cos\alpha_1 \cos\alpha_2 + 2a_1 b_1 a_2 b_2 \ \sin\alpha_1 \sin\alpha_2 \\ =& a_1^2 b_2^2 + a_2^2 b_1^2 \\ =& a_1^2 b_2^2 (\cos^2\alpha_1 + \sin^2\alpha_1) + a_2^2 b_1^2 (\cos^2\alpha_2 + \sin^2\alpha_2) \end{aligned} \tag{16}$$

which is recast into

$$0 = (a_1 b_2 \ \cos\alpha_1 - a_2 b_1 \ \cos\alpha_2)^2 + (a_1 b_2 \ \sin\alpha_1 - a_2 b_1 \ \sin\alpha_2)^2 \tag{17}$$

It is clear that each term must be set to zero, so that

$$\begin{aligned} \left(\frac{a_1}{b_1}\right) \cos\alpha_1 &= \left(\frac{a_2}{b_2}\right) \cos\alpha_2 \\ \left(\frac{a_1}{b_1}\right) \sin\alpha_1 &= \left(\frac{a_2}{b_2}\right) \sin\alpha_2 \end{aligned} \tag{18}$$

Next, plow these back into (15), to deduce that

$$\begin{aligned} \left(\frac{a}{b}\right) \cos\alpha &= \left(\frac{a_1}{b_1}\right) \cos\alpha_1 = \left(\frac{a_2}{b_2}\right) \cos\alpha_2 \\ \left(\frac{a}{b}\right) \sin\alpha &= \left(\frac{a_1}{b_1}\right) \sin\alpha_1 = \left(\frac{a_2}{b_2}\right) \sin\alpha_2 \end{aligned} \tag{19}$$

One takes—alternately—a sum of the squares of the two formulas above, or a division of the second over the first, and obtains, remembering that a's and b's are non-negative real quantites,

$$\begin{aligned} \frac{a}{b} &= \frac{a_1}{b_1} = \frac{a_2}{b_2} \\ \tan\alpha &= \tan\alpha_1 = \tan\alpha_2 \end{aligned} \tag{20}$$

The last equation above demands that α_1 and α_2 are determined up to $\pm\pi$, but they have to satisfy (19). It is clear that one must set $\alpha = \alpha_1 = \alpha_2$. Next, one introduces two new real variables $x \geq 0$ and $y \geq 0$, given by

$$\begin{aligned} x &= \frac{a_1}{a} = \frac{b_1}{b} \\ y &= \frac{a_2}{a} = \frac{b_2}{b} \end{aligned} \tag{21}$$

with the constraint $x^2 + y^2 = 1$.

Now one can prove that the case of rank 2 is reduced to that of rank 1. Indeed, one sees immediately that

$$\begin{pmatrix} D_1 \\ P_1 \end{pmatrix} = x \begin{pmatrix} D \\ P \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} D_2 \\ P_2 \end{pmatrix} = y \begin{pmatrix} D \\ P \end{pmatrix} \tag{22}$$

and (8) becomes identical to (3).

3 Discussions

It is shown in this note that the problem of two resonances in D_+ and P_+ in the $\eta\pi^-$ system in (1) is—effectively—a rank-1 problem. For this to be true, the following conditions have to be met:

(a) There exist two distinct resonances with different masses and/or widths. Note that the crucial step, from (14) to (15), depends on that fact that $\delta_a - \delta_b$ is non-zero and is mass dependent.

(b) There exists a satisfactory rank-1 fit with two resonances in a given mass region, in which each amplitude for D_+ or P_+ has the following general form

$$\mathcal{M}_k(w,\Omega) = r_k\, e^{i\,\alpha_k}\, e^{i\,\delta_k(w)}\, f_k(w)\, A_k(\Omega) \tag{23}$$

where $k = \{1, 2\}$ and '1' ('2') corresponds to D_+ (P_+). $\delta_k(w)$ is the Breit-Wigner phase and highly mass dependent, while r_k and α_k are *mass independent* in the fit. Of course, one of the two α_k's can be set to zero without loss of generality, so that there are three independent parameters, e.g. r_1, r_2 and α_1 (these were denoted a, b and α, respectively, in the previous section). $f_k(w)$ contains the absolute value of the Breit-Wigner form, plus any other mass-dependent factors introduced in the model. $A_k(\Omega)$ carries the information on the rotational property of a partial wave k.

(c) Same two resonances in D_+ and P_+ are produced in both spin-nonflip and spin-flip amplitudes, with the same general form as given above—but with arbitrary r_k's and α_k's for each spin-nonflip and spin-flip amplitudes. In this note, it is shown that only one set of r_k's and α_k's, i.e. r_1, r_2 and α_1, is required for both spin-nonflip and spin-flip amplitudes. (This is indeed a remarkable result; the rank-2 problem entails a set of six parameters, but it has been shown that the set is reduced to that consisting of just three.) Therefore, the distribution function in both w and Ω is given by

$$\frac{\mathrm{d}\sigma(w, \Omega)}{\mathrm{d}w\, \mathrm{d}\Omega} \propto |\sum_k \mathcal{M}_k(w, \Omega)|^2\, pq \tag{24}$$

independent of the nucleon helicities.

In another words, the spin density matrix has rank 1. The key ingredients for this remarkable result are that both spin-nonflip and spin-flip amplitudes harbor two resonances in D_+ and P_+ and that the production phase is mass-independent. It should be emphasized that the derivation given in this note does *not* depend on the existence of a good mass fit; it merely states that any fit with a mass-independent production phase is necessarily a rank-1 fit. Of course, the point is moot, if there exists no satisfactory fit in this model.

In the Introduction, it was pointed out that the results presented here can be applied equally well to the 1^{-+} state at 1.6 GeV coupling to $(\pi\rho)^-$. One recalls that, because of the Bose symmetrization resulting from two identical particles (π^-'s) in the isobar model, the imaginary part of the density matrix can be determined as well as the real part, i.e.

$$\Im\{P^*\, D\} = \Im\{P_1^*\, D_1\} + \Im\{P_2^*\, D_2\} \tag{25}$$

using the notation of Section 2. This implies that

$$ab \sin(\alpha + \delta_a - \delta_b) = a_1 b_1 \sin(\alpha_1 + \delta_a - \delta_b) + a_2 b_2 \sin(\alpha_2 + \delta_a - \delta_b) \tag{26}$$

Once again, one must realize that the formula above is true for any mass w, and therefore there must exist relationships in the coefficients of $\cos(\delta_a - \delta_b)$ and of $\sin(\delta_a - \delta_b)$. It is remarkable indeed that the resulting formulas are identical to (15), and one comes to the conclusion that the rank condition derived for $\eta\pi^-$ applies to $(\pi\rho)^-$ as well.

References

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FIGURES

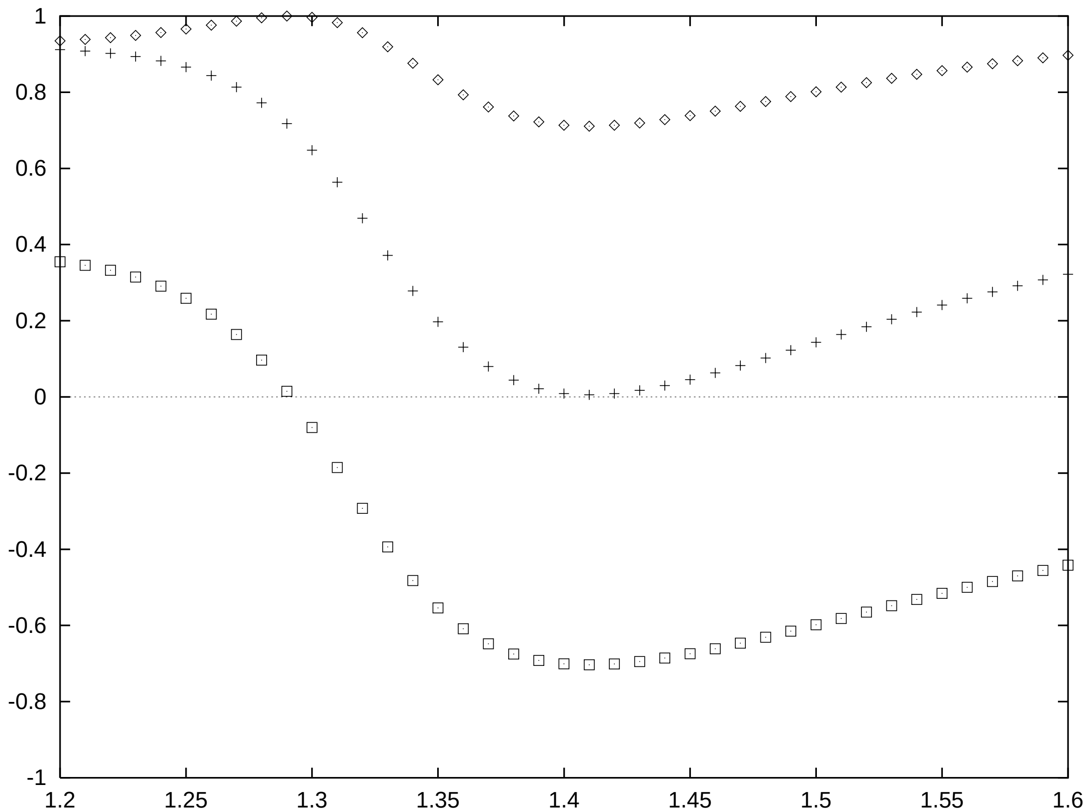

Figure 1: $\cos(\alpha + \delta_a - \delta_b)$ as a function of w from 1.2 to 1.6 GeV for $\alpha = 0°(\diamond)$, $\alpha = 45°(+)$ and $\alpha = 90°(\square)$.

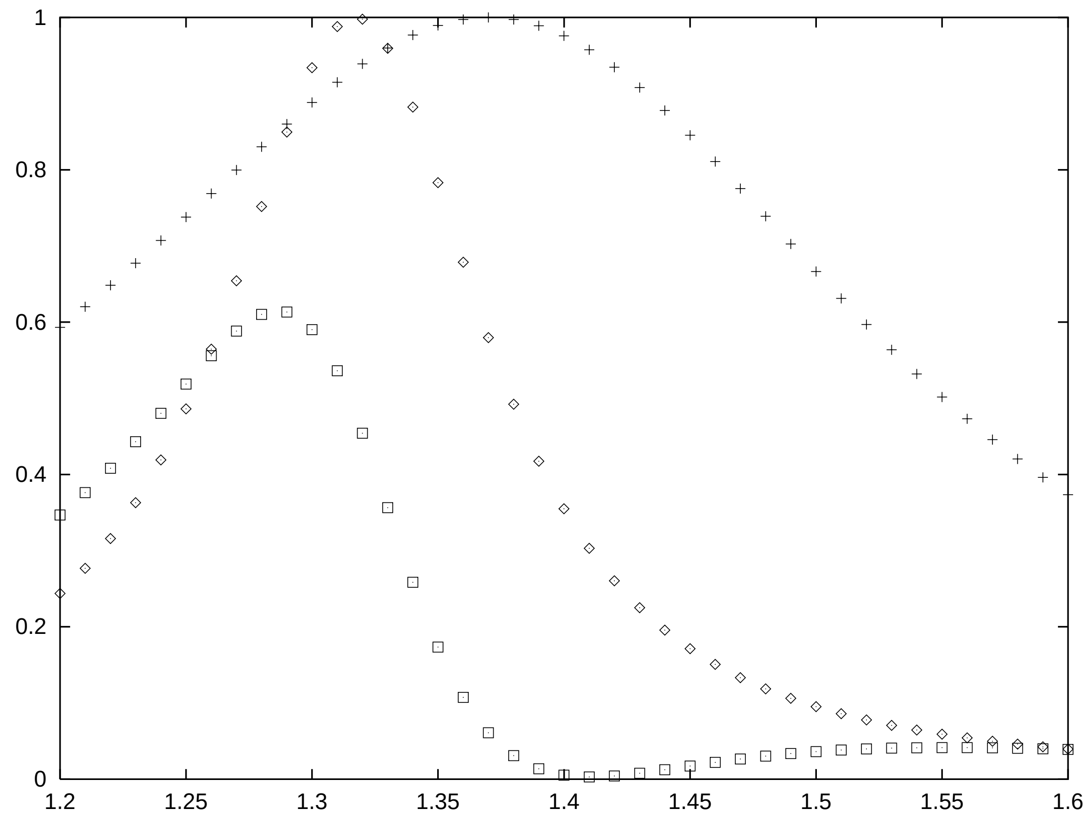

Figure 2: $\sin^2\delta_a$ ($\diamond$), $\sin^2\delta_a$ ($+$) and $\sin\delta_a\,\sin\delta_b\,\cos(\alpha+\delta_a-\delta_b)$ ($\square$) as a function of w from 1.2 to 1.6 GeV, using $\alpha = 45°$.

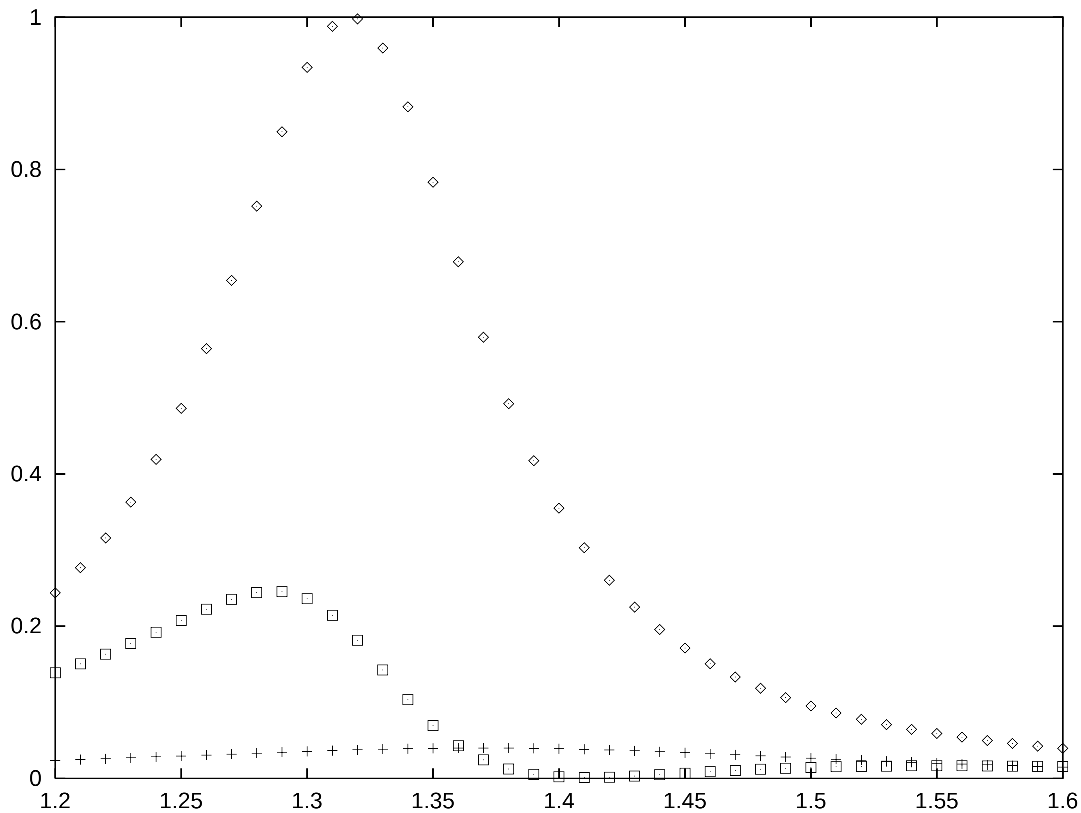

Figure 3: $a^2 \sin^2\delta_a$ ($\diamond$), $b^2 \sin^2\delta_a$ ($+$) and $2\,a\,b\sin\delta_a\,\sin\delta_b\,\cos(\alpha+\delta_a-\delta_b)$ ($\boxdot$) as a function of w from 1.2 to 1.6 GeV, where one has assumed that $a = 1.0$, $b = 0.20$ and $\alpha = 45°$.

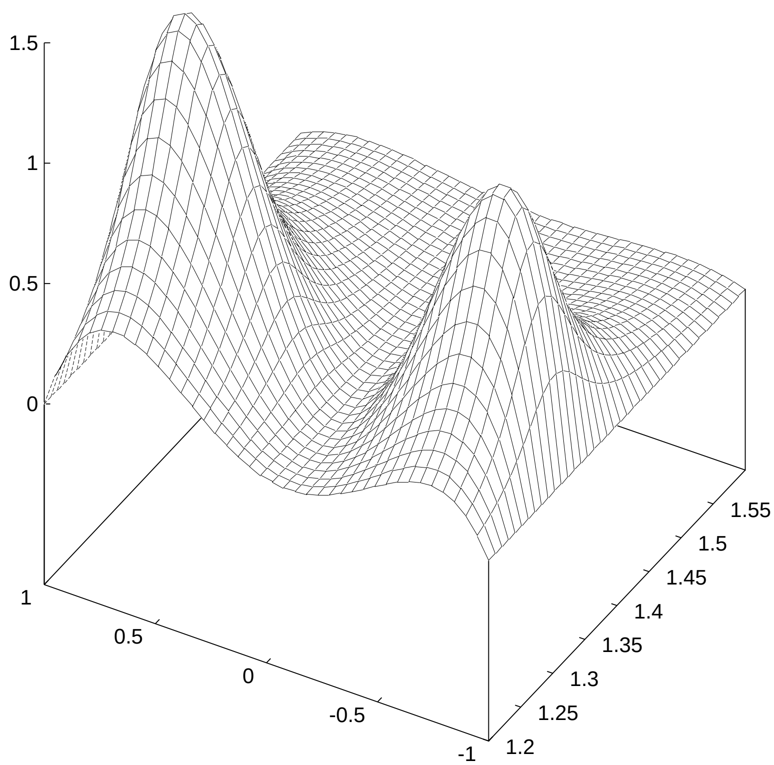

Figure 4: Angular distribution in $\cos\theta$ as a function of w from 1.2 to 1.6 GeV, where one has assumed that $a = 1.0$, $b = 0.151$ and $\alpha = 37.46°$.