Theory and Design of Multidimensional QMF Sub-Band Filters From 1-D Filters Using Transforms

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Abstract

The paper presents the general theory of designing multidimensional Quadrature Mirror Filters (QMF), for use in sub-band coding (SBC) systems, using the McClellan transform [1]. It was recently shown that McClellan transform could be used to generate 2-D diamond shape QMF filters [2]. In this paper we will formalize the proofs of the diamond shape case, and generalize it to other shapes, sampling rasters and dimensions. Examples are given of two dimensional diamond shape filters and three dimensional tetrad filters designed using the technique.

1 Introduction

Multidimensional sub-band coding systems are becoming popular for image representation and coding. This makes designing multidimensional QMF filters important. Such designs have been widely done for 1-D filters, but are more difficult for higher dimensions. The McClellan transform has been a popular method to generate multidimensional filters from a simple multidimensional kernel, and a 1-D filter. In [2] the McClellan transform was used to design a 2-D diamond shape QMF filter, with good magnitude reconstruction properties by transforming a 1-D QMF filter, using a particular form of the 2-D kernel. In this paper, we state and prove the theorems giving the necessary conditions on the 1-D filter and the 2-D kernel to get a 2-D diamond shape QMF filter. Moreover, we generalize the scheme to yield multidimensional 2 band QMF filters with different shapes and sampling rasters. This method reduces the design of multidimensional QMF filters of different shapes to the design of a 1-D QMF filter.

Section 2 presents a formulation of the procedure for the 2-D QMF filter bank, along with the theorems and proofs. Generalizations to other shapes and dimensions will be discussed in Section 3. Design examples for the 2-D diamond shape filters, and the 3-D tetrad shaped filters will be given in Section 4.

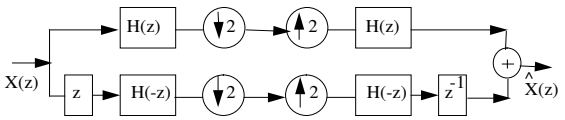

Figure 1: 1-D QMF filter bank

2 Generating 2-D QMF filter banks for a quincux sampling raster

In this section we show how a 1-D 2-band QMF filter bank can be transformed into a 2-D QMF filter bank suitable for a quincux sampling raster.

We first give a short description of the McClellan transformation in the z-domain. Given is a zero phase filter $H(z)$ and a 2-D kernel filter $K(\vec{z})$ where $\vec{z} = (z_1, z_2)$. The zero phase condition on $H(z)$ implies that there exists a filter $H_S(x)$ such that

$$H(z) = H_S(\frac{1}{2}(z + z^{-1})). \tag{1}$$

The McClellan transform $H^K(\vec{z})$ of $H(z)$ by $K(\vec{z})$ is now defined as

$$H^K(\vec{z}) = H_S(K(\vec{z})), \tag{2}$$

i.e. the 1-D *kernel* $S(z) = \frac{1}{2}(z + z^{-1})$ is replaced by the 2-D kernel $K(\vec{z})$. In order to control the shape of the resulting filter it is required that the values of $K(\vec{z})$ on the unit bi-circle $B = \{(z_1, z_2) : |z_1| = 1, |z_2| = 1\}$ form a subset of the range of $S(z)$ where z ranges over the unit circle. One easily checks that $K(\vec{z})$ has to be zero phase (in order to be real valued) and on B to be bounded between -1 and +1.

For the rest of this paper we assume that $H(z)$ is the lowpass filter $F_1(z)$ of a QMF analysis bank (see Figure 1). The highpass analysis filter $F_2(z)$, the lowpass synthesis filter $G_1(z)$, and the highpass synthesis filter $G_2(z)$ of this filter bank are then given by

$$\begin{aligned} F_1(z) = G_1(z) &\stackrel{\text{def}}{=} H(z), \\ z^{-1}F_2(z) = zG_2(z) = M(z) &\stackrel{\text{def}}{=} H(-z). \end{aligned} \tag{3}$$

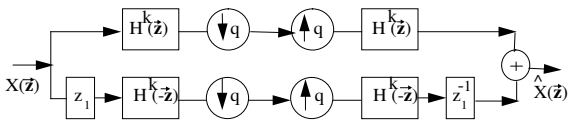

Figure 2: 2-D QMF filter bank

With this choice of filters the alias term cancels and the overall transfer function $T(z)$ is given by

$$T(z) = \frac{1}{2}(H^2(z) + H^2(-z)). \tag{4}$$

As $H(z)$ is a zero phase filter the resulting transfer function $T(z)$ is also zero phase. For perfect reconstruction purposes we assume of course that when z ranges over the unit circle $T(z) \approx 1$.

Now assume that $K(\vec{z})$ is a diamond shaped lowpass kernel. As an example one can assume $K(\vec{z})$ to be given by

$$K(\vec{z}) = 1/4(z_1 + z_1^{-1} + z_2 + z_2^{-1}), \tag{5}$$

or in more conventional notation by

$$(K_{mn})_{|m|,|n|\leq 1} = \begin{pmatrix} 0 & 1/4 & 0 \\ 1/4 & 0 & 1/4 \\ 0 & 1/4 & 0 \end{pmatrix}. \tag{6}$$

Consider the 2-D filters $\tilde{F}_1(\vec{z}) = H^K(\vec{z})$, $\tilde{F}_2(\vec{z}) = z_1 M^K(\vec{z})$, $\tilde{G}_1(\vec{z}) = H^K(\vec{z})$ and $\tilde{G}_2(\vec{z}) = z_1^{-1} M^K(\vec{z})$. It is easy to see that the first and third filter are diamond shaped lowpass filters, and that the second and fourth are diamond shaped highpass filters. The diamond shape is determined by the kernel $K(\vec{z})$, the quality of the filters is determined by the filter $H(\vec{z})$. In the context of a quincux sampling raster these four filters build in a canonical way a 2-D analysis/synthesis system (see Figure 2). We are interested in the conditions on $K(\vec{z})$ such that the 2-D analysis/synthesis system given above achieves alias cancellation and perfect reconstruction. Moreover, the question of what restrictions are necessary to ensure that $\tilde{F}_1(\vec{z})$ is zero phase is of some importance to image processing. The following two theorems give an answer to these questions.

Theorem 1 *Zero phase of $K(\vec{z})$ implies zero phase of $\tilde{F}_1(\vec{z})$.*

Remark 1 As any useful application of the McClellan transform requires K to be zero phase (see above) $\tilde{F}_1(\vec{z})$ is ensured to be zero phase automatically.

Proof: With notations as before we have

$$\tilde{F}_1(\vec{z}) = H_S(K(\vec{z})). \tag{7}$$

The zero phase condition on $\tilde{F}_1(\vec{z})$ can be stated as invariance under the substitution $(z_1, z_2) \leftarrow (z_1^{-1}, z_2^{-1})$. With this formulation it is immediately clear that $K(\vec{z})$ being zero phase implies $\tilde{F}_1(\vec{z})$ being zero phase. □

Theorem 2 *If $K(\vec{z})$ satisfies*

$$K(-\vec{z}) = -K(\vec{z}) \tag{8}$$

the four transform filters form an alias free analysis/synthesis bank for a quincux sampling lattice. Moreover, the transfer function $\tilde{T}(\vec{z})$ is given by the McClellan transform $T^K(\vec{z})$ of the 1-D transfer function $T(z)$. In particular if $T(z)$ is (almost) perfect reconstructing, $T^K(\vec{z})$ is (almost) perfect reconstructing.

Proof: The output $\hat{X}(\vec{z})$ of the analysis/synthesis bank when supplied with an input signal $X(\vec{z})$ is given by

$$\begin{aligned} 2\hat{X}(\vec{z}) &= \{(H^K(\vec{z}))^2 + (M^K(\vec{z}))^2\}X(\vec{z}) + \\ &\quad \{H^K(-\vec{z})H^K(\vec{z}) - M^K(-\vec{z})M^K(\vec{z})\}X(-\vec{z}) \\ &= 2\tilde{T}(\vec{z})X(\vec{z}) + 2\tilde{A}(\vec{z})X(-\vec{z}). \end{aligned} \tag{9}$$

The alias term is given by the second term of the last expression above. In order to prove alias cancellation we have to show $\tilde{A}(\vec{z})$ equals 0. We first examine the transform M^K. Considering the equalities

$$\begin{aligned} M(z) &= H(-z) \\ &= H_S(S(-z)) \\ &= H_S(-S(z)) \end{aligned} \tag{10}$$

it follows that

$$M_S(x) = H_S(-x). \tag{11}$$

Rewriting the alias factor $\tilde{A}(\vec{z})$ using equation (8) and (11) we find

$$\begin{aligned} 2\tilde{A}(\vec{z}) &= H^K(-\vec{z})H^K(\vec{z}) - M^K(-\vec{z})M^K(\vec{z}) \\ &= H_S(K(-\vec{z}))H_S(K(\vec{z})) - \\ &\quad H_S(-K(-\vec{z}))H_S(-K(\vec{z})) \\ &= H_S(-K(\vec{z}))H_S(K(\vec{z})) - \\ &\quad H_S(K(\vec{z}))H_S(-K(\vec{z})) \\ &= 0. \end{aligned} \tag{12}$$

This proves alias cancellation. For the transfer function we find

$$\begin{aligned} 2\tilde{T}(\vec{z}) &= (H^K(\vec{z}))^2 + (M^K(\vec{z}))^2 \\ &= H_S^2(K(\vec{z})) + M_S^2(K(\vec{z})) \\ &= H_S^2(K(\vec{z})) + H_S^2(-K(\vec{z})). \end{aligned} \tag{13}$$

In a similar manner as for equation (11) we can prove that

$$T_S(z) = \frac{H_S^2(z) + H_S^2(-z)}{2}. \tag{14}$$

Rewriting with equation (13) and (14) we find

$$\tilde{T}(\vec{z}) = T^K(\vec{z}). \tag{15}$$

If $T(z)$ is (almost) perfect reconstructing the function $T_S(x)$ will be (approximately) 1 for real values between -1 and +1. As $K(\vec{z})$ has values between -1 and +1 the function $\tilde{T}(\vec{z})$ will also be (approximately) 1 on its domain. This proves the theorem. □

Remark 2 The condition of equation (8) can be worded more informally by saying that $K(\vec{z})$ lives on the non-zero coset of the quincux sampling lattice, i.e. all its non-zero coefficients occur at points which are not member of the quincux sampling lattice.

Remark 3 The proofs of the two theorems above are derived from simple algebraic manipulations (compare [2]). One can check that the essential ingredient of the proof is the equality $S(-z) = -S(z)$. This equality guarantees that the polyphase decomposition of the 1-D filter bank is preserved when extracting $S(z)$.

Remark 4 The phase shift z_1 in the highpass branch of the filterbank can be replaced by any odd degree monomial in the two variables z_1 and z_2.

3 Generalizations

Due to the simple formulation in the z-domain the results of the previous section can easily be generalized from the quincux sampling raster to other sampling rasters. An obvious application in 2-D is for subsampling by 2 only in the horizontal or vertical direction. Also the extension to M dimensional sampling rasters is straightforward. One example is 3-D quincux sampling with tetrad shaped filters. All this is summarized in the following theorem.

Theorem 3 *Let Λ an M-dimensional sampling raster of index 2 (i.e. a sampling raster with only one non-zero coset). Let $K(\vec{z})$ be a symmetric kernel, having values between -1 and 1 (and therefore zero phase) and living on the non-zero coset of Λ. Let $H(z)$ be as in the previous section and let $m(\vec{z})$ be any monomial living on the non-zero coset of Λ. Applying the McClellan transform as in the previous section*

$$\begin{aligned} \tilde{F}_1(\vec{z}) &= \tilde{G}_1(\vec{z}) \stackrel{\text{def}}{=} H^K(\vec{z}), \\ m(\vec{z}^{-1})\tilde{F}_2(\vec{z}) &= m(\vec{z})\tilde{G}_2(\vec{z}) \stackrel{\text{def}}{=} M^K(\vec{z}) \end{aligned} \tag{16}$$

yields filters such that the resulting filter bank is free of aliasing. Moreover the transfer function $\tilde{T}(\vec{z})$ is equal to $T^K(\vec{z})$ and consequently the (almost) perfect reconstructing property is preserved. Finally, as $K(\vec{z})$ is zero phase the filter $\tilde{F}_1(\vec{z})$ is also zero phase.

The proof of this theorem is a straightforward generalization of the proofs of the previous section. In the next section the theorems are applied to some examples.

4 Design Example

Starting with an odd length symmetric 1-D QMF filter, and the 2-D kernel of equation (5), we designed a 2-D diamond shape filter. Figure 3 shows the magnitude response of the 2-D lowpass filter, Figure 4 that of the mirror highpass filter and Figure 5 shows the magnitude response of the resulting analysis/synthesis system, all on the same scale. The non-zero coefficients of one quadrant of the symmetric lowpass filter are given in Table 1. The highpass filter is mirror of the lowpass filter.

h(m,n)	m=0	1	2
n=0	0.621729	0.161889	-0.0126949
1	0.161889	-0.0353769	-0.0162751
2	-0.0126949	-0.0162751	0.00749029
3	-0.00542504	0.00499353	
4	.00124838		
h(m,n)	m= 3	4	
n=0	-0.00542504	.00124838	
1	0.00499353		

Table 1: One quadrant coefficients of the diamond shape lowpass filter

It was shown in Section 3 that the alias cancellation and perfect reconstruction condition for the 3-D quincux sampling with tetrad shaped filters is formally equivalent to the condition in the 2-D case. Using the same design procedure as above, we generated a 3-D tetrad shaped filter. We started with the zero phase 3-D kernel:

$$K(\vec{z}) = 1/12 \sum_{\substack{\sum i_j = 1 (mod\ 2) \\ -1 \le i_j \le 1}} z_1^{i_1} z_2^{i_2} z_3^{i_3}. \tag{17}$$

It is easy to see that since $K(\vec{z})$ is zero phase, the resulting 3-D filters will also be zero phase (theorem 1,3). It is also clear that K lives on the non-zero coset hence (theorem 2, 3) the transfer function $\tilde{T}(\vec{z}) = T^K(\vec{z})$, will be perfect reconstructing.

The "2-D slices" of the 3-D reconstruction function $T^K(\vec{z})$, are very similar to figure 5.

5 Conclusion

We have presented the generalized theory of designing multidimensional 2-band QMF filters from 1-D QMF filters using McClellan transform. The method

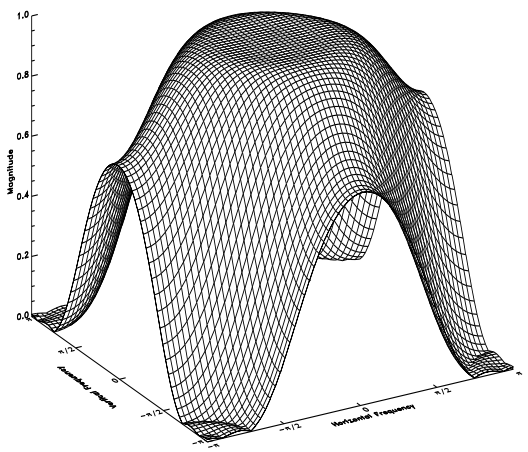

Figure 3: Frequency response of the lowpass filter

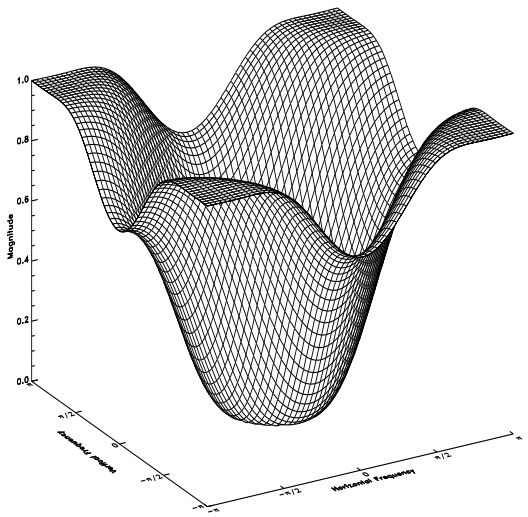

Figure 4: Frequency response of the highpass filter

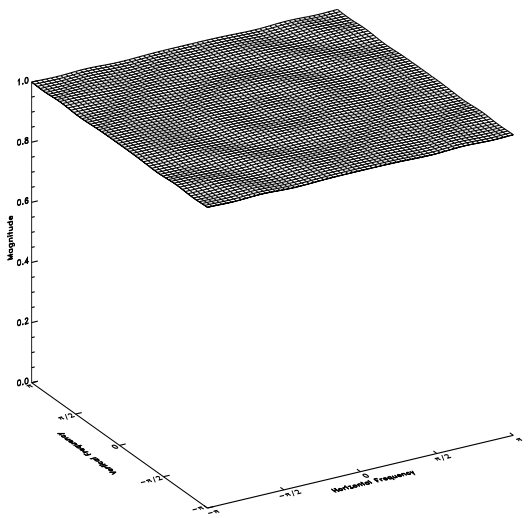

Figure 5: Frequency response of the total system

reduces the design of multidimensional QMF filters to the design on a 1-D QMF filter, and the choice of a transformation kernel. The various theorems imposing the conditions on the 1-D filters and the transformation kernel have been stated and proved. Generalizations were made to extend the theory to different subsampling matrices, and dimensions. For illustrative purposes, 2-D diamond shape filters were designed. Using similar procedure, 3-D tetrad shaped filters were also designed. Both filters are suitable for sub-band coding systems with quincux sub and upsampling matrices in 2 and 3 dimensions respectively.

Acknowledgements

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References

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