Block-Diagonal-Bordered Sparse Matrices

Identify parallelism when factoring BDB form sparse matrices

1. partition the matrix into:
   - a block-diagonal matrix
   - an upper border
   - a lower border
   - a last block
2. calculate the Shur complement with these partitions
3. prove the theorem — the LU factors of a BDB matrix are also in BDB form
4. prove the lemma — the LU factors of a BD matrix are also BD form
5. identify the parallelism

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Calculate the Shur complement

Partition the **A**, **L**, and **U** matrices

$$\mathbf{A} = \begin{pmatrix} \mathcal{A}_{1,1} & \mathcal{A}_{1,2} \\ \mathcal{A}_{2,1} & \mathcal{A}_{2,2} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{1,1} & \mathbf{0} \\ \mathcal{L}_{2,1} & \mathcal{L}_{2,2} \end{pmatrix} \begin{pmatrix} \mathcal{U}_{1,1} & \mathcal{U}_{1,2} \\ \mathbf{0} & \mathcal{U}_{2,2} \end{pmatrix} = \mathbf{LU}$$

$\mathcal{A}_{1,1}$, $\mathcal{L}_{1,1}$, and $\mathcal{U}_{1,1}$ are of size $n_1 \times n_1$
$\mathcal{A}_{1,2}$ and $\mathcal{U}_{1,2}$ are of size $n_1 \times n_2$
$\mathcal{A}_{2,1}$ and $\mathcal{L}_{2,1}$ are of size $n_2 \times n_1$
$\mathcal{A}_{2,2}$, $\mathcal{L}_{2,2}$, and $\mathcal{U}_{2,2}$ are of size $n_2 \times n_2$

Perform the matrix multiplication

$$\mathbf{A} = \begin{pmatrix} \mathcal{A}_{1,1} & \mathcal{A}_{1,2} \\ \mathcal{A}_{2,1} & \mathcal{A}_{2,2} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{1,1}\mathcal{U}_{1,1} & \mathcal{L}_{1,1}\mathcal{U}_{1,2} \\ \mathcal{L}_{2,1}\mathcal{U}_{1,1} & \mathcal{L}_{2,1}\mathcal{U}_{1,2} + \mathcal{L}_{2,2}\mathcal{U}_{2,2} \end{pmatrix}$$

Equate blocks and identify how to solve for the partitions in **LU**

$$\mathcal{A}_{1,1} = \mathcal{L}_{1,1}\mathcal{U}_{1,1} \;\Rightarrow\; \mathcal{L}_{1,1}\mathcal{U}_{1,1} = \mathcal{A}_{1,1}$$

$$\mathcal{A}_{1,2} = \mathcal{L}_{1,1}\mathcal{U}_{1,2} \;\Rightarrow\; \mathcal{U}_{1,2} = \mathcal{L}_{1,1}^{-1}\mathcal{A}_{1,2}$$

$$\mathcal{A}_{2,1} = \mathcal{L}_{2,1}\mathcal{U}_{1,1} \;\Rightarrow\; \mathcal{L}_{2,1} = \mathcal{A}_{2,1}\mathcal{U}_{1,1}^{-1}$$

$$\mathcal{A}_{2,2} = \mathcal{L}_{2,1}\mathcal{U}_{1,2} + \mathcal{L}_{2,2}\mathcal{U}_{2,2} \;\Rightarrow\; \mathcal{L}_{2,2}\mathcal{U}_{2,2} = \mathcal{A}_{2,2} - \mathcal{L}_{2,1}\mathcal{U}_{1,2}$$

Block-Diagonal-Bordered Sparse Matrices

Theorem — the LU factors of a BDB matrix are also BDB form

Define:

$$\mathbf{A}_{BDB} = \begin{pmatrix} \mathcal{A}_{1,1} & \mathcal{A}_{1,2} \\ \mathcal{A}_{2,1} & \mathcal{A}_{2,2} \end{pmatrix} = \begin{pmatrix} A_{1,1} & & \mathbf{0} & & A_{1,m} \\ & A_{2,2} & & & A_{2,m} \\ \mathbf{0} & & \ddots & & \vdots \\ & & & A_{m-1,m-1} & A_{m-1,m} \\ A_{m,1} & A_{m,2} & \cdots & A_{m,m-1} & A_{m,m} \end{pmatrix}$$

$$\mathcal{A}_{1,1} = \mathbf{A}_{BD} = \begin{pmatrix} A_{1,1} & & \mathbf{0} & \\ & A_{2,2} & & \\ \mathbf{0} & & \ddots & \\ & & & A_{m-1,m-1} \end{pmatrix}$$

$$\mathcal{A}_{1,2} = \begin{pmatrix} A_{1,m} \\ A_{2,m} \\ \vdots \\ A_{m-1,m} \end{pmatrix}$$

$$\mathcal{A}_{2,1} = \begin{pmatrix} A_{m,1} & A_{m,2} & \cdots & A_{m,m-1} \end{pmatrix}$$

$$\mathcal{A}_{2,2} = A_{m,m}$$

Block-Diagonal-Bordered Sparse Matrices

Theorem — the LU factors of a BDB matrix are also BDB form (continued)

Prove: $\mathbf{A}_{BDB} = \mathbf{L}_{BDB}\mathbf{U}_{BDB}$

where:

$$\mathbf{L}_{BDB} = \begin{pmatrix} L_{1,1} & & & & \\ & L_{2,2} & & & \mathbf{0} \\ \mathbf{0} & & \ddots & & \\ & & & L_{m-1,m-1} & \\ L_{m,1} & L_{m,2} & \cdots & L_{m,m-1} & L_{m,m} \end{pmatrix}$$

and

$$\mathbf{U}_{BDB} = \begin{pmatrix} U_{1,1} & & & \mathbf{0} & U_{1,m} \\ & U_{2,2} & & & U_{2,m} \\ & & \ddots & & \vdots \\ \mathbf{0} & & & U_{m-1,m-1} & U_{m-1,m} \\ & & & & U_{m,m} \end{pmatrix}$$

The matrix partitions $\mathcal{A}_{1,2}$ and $\mathcal{A}_{2,1}$ have simply been further partitioned, and matrix partition $\mathcal{A}_{2,2}$ has been left unchanged. For $\mathbf{A}_{BDB} = \mathbf{L}_{BDB}\mathbf{U}_{BDB}$, then the LU factors of the block diagonal matrix $\mathcal{A}_{1,1}$ must also be block diagonal. This is proven in an accompanying lemma.

Block-Diagonal-Bordered Sparse Matrices

Lemma — the LU factors of a BD matrix are also BD form

Let:

$$\mathbf{A}_{BD} = \begin{pmatrix} \mathcal{A}_{1,1} & \mathbf{0} \\ \mathbf{0} & \mathcal{A}_{2,2} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{1,1} & \mathbf{0} \\ \mathcal{L}_{2,1} & \mathcal{L}_{2,2} \end{pmatrix} \begin{pmatrix} \mathcal{U}_{1,1} & \mathcal{U}_{1,2} \\ \mathbf{0} & \mathcal{U}_{2,2} \end{pmatrix} = \mathbf{L}_{BD}\mathbf{U}_{BD}$$

Prove $\mathbf{L}_{BD}$ and $\mathbf{U}_{BD}$ are block diagonal, ie., $\mathcal{L}_{2,1}$ and $\mathcal{U}_{1,2} = \mathbf{0}$

By applying the Shur Complement:

$$\mathcal{A}_{1,2} = \mathcal{L}_{1,1}\mathcal{U}_{1,2} = \mathbf{0} \Rightarrow \mathcal{U}_{1,2} = \mathcal{L}_{1,1}^{-1}\mathbf{0} = \mathbf{0}$$
$$\mathcal{A}_{2,1} = \mathcal{L}_{2,1}\mathcal{U}_{1,1} = \mathbf{0} \Rightarrow \mathcal{L}_{2,1} = \mathbf{0}\mathcal{U}_{1,1}^{-1} = \mathbf{0}$$

If $\mathbf{A}_{BD}$ is non-singular and has a numerical factor, then $\mathcal{L}_{1,1}^{-1}$ and $\mathcal{U}_{1,1}^{-1}$ must exist and be non-zero: thus

$$\mathbf{A}_{BD} = \begin{pmatrix} \mathcal{A}_{1,1} & \mathcal{A}_{1,2} \\ \mathcal{A}_{2,1} & \mathcal{A}_{2,2} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{1,1} & \mathbf{0} \\ \mathbf{0} & \mathcal{L}_{2,2} \end{pmatrix} \begin{pmatrix} \mathcal{U}_{1,1} & \mathbf{0} \\ \mathbf{0} & \mathcal{U}_{2,2} \end{pmatrix} = \mathbf{L}_{BD}\mathbf{U}_{BD}$$

This lemma can be applied recursively to a block-diagonal matrix with any number of diagonal blocks to prove that the LU factorization of a block-diagonal matrix preserves the block structure.

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Available Parallelism in a BDB Form Matrix

As a result of the theorem $\mathbf{A}_{BDB} = \mathbf{L}_{BDB}\mathbf{U}_{BDB}$, it is simple to identify available parallelism by simply performing the matrix multiplication in a manner similar to the Shur complement.

Diagonal Blocks: $\mathcal{A}_{1,1} = \mathcal{L}_{1,1}\mathcal{U}_{1,1} \Rightarrow \begin{cases} A_{1,1} = L_{1,1}U_{1,1} \\ A_{2,2} = L_{2,2}U_{2,2} \\ \vdots \end{cases}$

Lower Border: $\mathcal{A}_{2,1} = \mathcal{L}_{2,1}\mathcal{U}_{1,1} \Rightarrow \begin{cases} A_{m,1} = L_{m,1}U_{1,1} \\ A_{m,2} = L_{m,2}U_{2,2} \\ \vdots \end{cases}$

Upper Border: $\mathcal{A}_{1,2} = \mathcal{L}_{1,1}\mathcal{U}_{1,2} \Rightarrow \begin{cases} A_{1,m} = L_{1,1}U_{1,m} \\ A_{2,m} = L_{2,2}U_{2,m} \\ \vdots \end{cases}$

Last Block:

$$\mathcal{A}_{2,2} - \mathcal{L}_{2,1}\mathcal{U}_{1,2} = \mathcal{L}_{2,2}\mathcal{U}_{2,2} \Rightarrow \left\{ A_{m,m} - \sum_{i=1}^{(m-1)} L_{m,i}U_{i,m} = L_{m,m}U_{m,m} \right.$$

Assign $A_{i,i}$, $A_{m,i}$, and $A_{i,m}$ $(1 \leq i \leq (m-1))$ to the same processor and there are *no* communications until the last block is factored. At that time, only sums of sparse matrix $\times$ sparse matrix products are sent to the processors that hold the appropriate data in the last block.