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# NUMERICAL RELATIVITY: EVOLVING SPACETIME

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**Abstract**

The construction of numerical solutions of Einstein's General Relativity equations is formulated as an initial-value problem. The space-plus-time (3+1) decomposition of the spacetime metric tensor is used to discuss the structure of the field equations. The resulting evolution system is shown to depend in a crucial way of the coordinate gauge. The mandatory use of singularity avoiding coordinate conditions (like maximal slicing or similar gauges) is explained. A brief historical review of Numerical Relativity is included, showing the enormous effort in constructing codes based in these gauges, which lead to non-hyperbolic evolution systems, using "ad hoc" numerical techniques. A new family of first order hyperbolic evolution systems for the vacuum Einstein field equations in the harmonic slicing gauge is presented. This family depends on a symmetric $3 \times 3$ array of parameters which can be used to scale the dynamical variables in future numerical applications.

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## 1 Introduction

In this section, we give a review of basic theoretical framework; we introduce the concept of Numerical Relativity, together with the standard $3+1$ formalism, and a short discussion of coordinate conditions (following Ref. 1). A brief historical review of the field can be found in Section 2. Our work on the harmonic slicing conditions and hyperbolic balance laws systems will be discussed in Section 3.

### 1.1 What is Numerical Relativity?

As good relativists, we must start this review section by writing the beautiful Einstein Equations[a] (for a full exposition, see any standard Relativity text[2]):

$$G^{\mu\nu} = 8\pi T^{\mu\nu} \tag{1}$$

As we know, General Relativity can be interpreted as a theory of the geometry of the spacetime and the Einstein Equations are the mathematical expression of the fundamental notion that "matter tells spacetime how to curve and the curvature of the spacetime tells matter how to move". We represent the spacetime by a four-dimensional differentiable manifold, $\mathcal{M}$, endowed with a metric represented by a symmetric tensor $g_{\mu\nu}$. At each spacetime "point", that is, at each event of the spacetime, the metric tells us the way to determine distances to nearby events:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

The information about the curvature of the spacetime is given by the Einstein tensor $G^{\mu\nu}$, which is a function of the metric and its first and second derivatives. The stress-energy tensor $T^{\mu\nu}$ describes the total non-gravitational energy content of the spacetime and contains additional fields representing the various sources of energy and matter.

Okay, all this is well-known. What happens when we try to solve the Einstein Equations? The fact is that no general analytical techniques are available for obtaining strong-field, highly dynamical solutions in the

[a] In the following, unless otherwise stated, we use units such that $c = G = 1$. Greek indices run from 0 to 3 and latin ones from 1 to 3

absence of spatial symmetries. In such situations, the only recourse is the numerical solution of the Einstein Equations. In fact, the equations are completely intractable for any but the simplest physical scenarios. Numerical methods provide the only available means to derive detailed information about the nature of the gravitational interaction for many problems of theoretical and astrophysical interest. Then, Numerical Relativity is simply the field which is devoted to study the numerical solution of the Einstein Equations with the computer.

Many of the most interesting problems in General Relativity involve the generation and propagation of gravitational radiation from dynamic gravitational configurations. Violent astrophysical events,[3,4] such as binary coalescence, supernovae explosions, stellar collapse or the interaction of black holes are the ideal candidates for numerical study. Vigorous experimental and theoretical investigation of methods for detecting gravitational waves is under way, coming from the early sixties. The latest generation of detectors, known as Laser Interferometer Gravity wave Observatories (LIGOs) should be sensitive enough[5] to have a high probability of detecting gravitational waves. The day that this detection will happen, maybe before the end of the century,[6] Numerical Relativity will be the only tool to compare the observed waveforms with the predictions of the General Relativistic theory, opening a new window to the universe.

General Relativity has been slow to incorporate modern computational methods developed in the last decade. The lack of a preferred spacetime coordinate system makes Numerical Relativity more complicated and more difficult to interpret than similar research in Hydrodynamics or Electromagnetism, where fields propagate on a fixed background spacetime. Einstein Equations produce a larger number of dynamical variables than do other partial differential equations used in the simulation of physical systems. This places enormous demands of computer time and memory on even the largest and fastest supercomputers.

### 1.2 $3+1$ Decomposition

For the purpose of obtaining numerical solutions to the Einstein Equations, is is convenient to break the lack of distinction between time and space. Time is distinguished by the fact that causality is defined (locally) with respect to only one of the four coordinates. We will recast the system (1) in terms of a space-plus-time decomposition of the manifold. In this way, the dynamic nature of the gravitational field equations becomes manifest, and the task of solving the Equations is reformulated as the construction of spacetime from the evolution of initial data. That is, formulated as an initial-value, or Cauchy, problem. This approach, known as the $3+1$ formalism, forms the basis of most of the work in Numerical Relativity and was made popular by Arnowitt, Deser and Misner.[7] They reinterpreted within a Hamiltonian formalism the previous work from Lichnerowicz[8] and Choquet-Bruhat.[9] Other approaches are based in a $2+2$ decomposition[10,11] and also in the $(2+1)+1$ formalism.[12]

In the $3+1$ approach, the manifold $\mathcal{M}$ is to be pictured as an infinite foliation of 3-dimensional space-like hypersurfaces $\{\Sigma\}$. Each hypersurface $\Sigma$ is itself a differentiable manifold with an intrinsic geometric structure induced by the embedding spacetime. Then, the view of the 4-geometry as the time development of the 3-geometry naturally arises.

Introducing coordinates $x^\mu \equiv \{x^0, x^i\} = \{t, x^i\}$ on the manifold, such that $t$ is constant on any hypersurface, the spacetime displacement between events $\xi$ and $\xi'$ having coordinates $x^\mu$ and $x^\mu + dx^\mu$ respectively, may be written as:

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \end{aligned} \tag{3}$$

where we have introduced four *kinematical* quantities grouped as a scalar function, $\alpha$, and a 3-vector, $\beta^i$.

If we are observers moving in a direction normal to the hypersurface (Eulerian observers), then we will measure an elapsed proper time:

$$d\tau = \alpha dt \tag{4}$$

so $\alpha$ is called the *lapse* function. If we were generic, non-Eulerian, observers we would find that, when reaching the nearby hypersurface, we must shift our position by $\beta^i dt$ to arrive at an event which has the same spatial coordinates as the event from which we departed. Thus, $\beta^i$ is called the *shift* vector. Given an slice with some coordinate system, a prescription for specifying the lapse function and the shift vector on that and all the other slices will determine the coordinate labelling of the spacetime. The arbitrariness in the choice of the lapse and the shift, as well as the choice of the initial slice and its internal coordinates, is the $3+1$ expression of the coordinate freedom in General Relativity.

### 1.3 Structure of the Einstein Equations

A closer look to the Einstein field equations (1) shows that the higher order time derivatives occurring in the Einstein tensor $G^{\mu\nu}$ are the second order time derivatives of the space metric components $g_{ij}$ and the first order time derivatives of the remaining components $g_{0\mu}$. These higher order terms appear linearly in the six space components of the Einstein tensor, but do not occur in the four remaining components $G^{0\mu}$. The corresponding equations

$$G^{0\mu} = 8\pi T^{0\mu} \tag{5}$$

are known as the *energy and momentum constraints* (for $\mu = 0,\ i$, respectively).

Let us note, however, that the stress-energy tensor conservation:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{6}$$

ensures that the constraints (5) are actually first integrals of the remaining Einstein Equations. Any spacetime obtained from an initial data set which satisfies the constraint equations will automatically satisfy the constraint equations forever.

The problem is that when we actually implement the evolution equations numerically, we are interested in knowing if they are satisfied to the same numerical order of approximation as the one of the numerical scheme used in the discretization of the evolution equations, given that the constraints are PDEs themselves.

Free evolution schemes (see, for instance, Refs. 13–15) impose the constraints only on the initial data and then forget about them or check their "drifting" as the evolution proceeds. Other algorithms impose the constraints at regular intervals or abandon an evolution equation to derive one of the variables in terms of the others through the constraints (see, for instance, Refs. 16–19)

Some authors[20,21] suggest that free evolution schemes suffer from an inconsistency with respect to the constraint equations and these are satisfied to a lower order than the numerical scheme. Recently, Choptuik[22] has put forward an argument that would imply that there should be no inconsistencies for convergent difference schemes. Nevertheless, empirically, schemes that impose the constraints tend to be less prone to numerical instabilities than free evolution ones. So, this remains an open question.

Now, we can summarize the two basic stages for the construction of the spacetime as an initial value problem:

1. Construct Initial data on a given hypersurface.
2. Then, evolve these data.

In the following subsections we will review both stages.

### 1.4 The Initial Data Problem

The first stage, the *initial data problem*, involves deciding what topology the initial surface is to have, the introduction of a spatial coordinate system and a specification of an initial data set which satisfies Eqs.(5). Because constraints must be solved this part of the procedure is, in general, far from trivial.

An elegant and general treatment of this problem, largely due to O'Murchadha and York[23] and York,[24,25] has provided considerable insight into the theoretical nature of the initial data problem, as well as the basis for most of the recent work involving the construction of initial data sets (for a review, see Cook[26]). We will not discuss this procedure here. Let us simply say that it involves the use of a background metric $\hat{g}$ conformally related by a factor $\psi^4$ to the 3-dimensional metric:

$$g_{ij} = \psi^4 \hat{g}_{ij} \tag{7}$$

and this is worked explicitly in terms of the transverse traceless part of the conformally related extrinsic curvature. Then, careful scaling of the conformal factor makes possible to determine the general solution for many matter distributions and, specially, for any configuration of black holes.[27,26]

### 1.5 Evolving spacetime

We will briefly discuss the process of constructing numerical space-time models. In order to avoid complications related with the behaviour of matter fields, we shall limit ourselves in this work to vacuum space-times, where the stress-energy tensor $T^{\mu\nu}$ vanishes.

We shall start from a very convenient form of writing down the equations (1), due to Fock[28]:

$$(\Box g^{\mu\nu} + \partial^\mu \Gamma^\nu + \partial^\nu \Gamma^\mu) - \Gamma^{\mu\rho\sigma}\Gamma^\nu_{\rho\sigma} = 0 \ , \tag{8}$$

where the symbol $\Box$ stands for the d'Alembert differential operator acting on functions, namely

$$\Box f \equiv g^{\mu\nu}\partial^2_{\mu\nu} f - \Gamma^\mu \partial_\mu f \ , \tag{9}$$

and we have noted

$$\Gamma^\mu \equiv g^{\rho\sigma}\Gamma^\mu_{\rho\sigma} \ , \tag{10}$$

where $\Gamma^\mu_{\rho\sigma}$ are the connection coefficients associated to the spacetime metric $g_{\mu\nu}$.

One must now remember that there are four constraint equations contained in the system (8). This system must be then completed with four evolution equations for the metric coefficients $g^{0\mu}$ (which amount to the lapse $\alpha$ and shift $\beta^i$). These four supplementary degrees of freedom allow the arbitrary specification of a coordinate system in the spacetime so that their evolution equations are known as *coordinate conditions.*

An interesting choice is provided by the harmonic coordinates condition

$$\Box x^\mu = -\Gamma^\mu = 0, \tag{11}$$

which ensures that the evolution system one obtains by dropping the derivatives of $\Gamma^\mu$ in (8) is hyperbolic.

Hyperbolic systems are very appealing to the numerical relativist because there is a huge arsenal of ready-to-run numerical methods which have been developed and fully tested in Computational Fluid Dynamics. In spite of that, nobody has yet developed a Numerical Relativity code based in the harmonic coordinates condition (11). We think that the main reason for this is that, until very recently, numerical codes where devised to model only problems with one or more degrees of symmetry. In these cases, the spacetime metric takes a simpler form (with a reduced number of independent components) when written in adapted coordinate systems in which one or more coordinates can not be harmonic.

Even in the generic case (no symmetries), we can see that hyperbolicity should not be the only criterion. A given coordinate choice may cause our code to crash because it hits a singularity (due to the formation of a black hole, for instance). The spacetime slicing must stay non-singular throughout the calculation, even though a portion of a slice may be inside a black hole and another part is in nearly flat spacetime. As we are going to see, we need a "clever" coordinate choice in the numerical construction of the spacetime.

### 1.6 Coordinate conditions

The choice of the lapse, the *time slicing* is the fundamental kinematical choice because it determines which sequence of data represents spacetime. So, to simplify, we simply choose some set of world lines orthogonal to a family of spacelike hypersurfaces to define our Eulerian observers. That is, we impose 3 of the 4 degrees of freedom by choosing a zero shift vector:

$$\beta^i = 0 \tag{12}$$

to ensure that the congruence of time coordinate lines is as regular as the time slicing itself.

The simplest way of completing our coordinate choice would be the *geodesic slicing*, that is, $\alpha = 1$ everywhere. This would imply that the Eulerian observers are freely falling and that the slices would be geodesically parallel. It is now well known that this condition fails to give a family of slices that adequately cover the spacetime region of interest.[23] The attractive nature of the gravitational interaction makes these observers to have the tendency to focus to a caustic.

Lichnerowicz[8] realized the unsuitable features of the geodesic slicing and suggested a possible remedy: the *maximal slicing*. He pointed out that $\alpha$ can be chosen in such a way that the volume element of the Eulerian observers is time independent and their world lines are like those of an incompressible and irrotational "test fluid". This leads to an elliptic equation which must be solved at every hypersurface which, in the vacuum, reduces to:

$$\Delta\alpha = k\alpha, \tag{13}$$

where $k$ is a positive definite quantity. The resulting lapse function goes to zero rapidly ("collapse of the lapse") in the strong field region while it keeps nearly constant in the nearly Minkowskian outer zones. This behaviour has been termed *singularity avoidance.*

Since its introduction, the maximal slicing has been extensively used in Numerical Relativity and there are many codes that make use of it, ranging from problems involving colliding black holes, collapsing stars, imploding gravity waves, etc. See, for instance, Refs. 3,17,18,29–39

It has been generalized by York[40] to deal with cosmological models in which the volume element of the Eulerian observers is allowed to depend on time (*constant mean curvature slicing*). See, for instance, Refs. 13,41–46. For a review of other time slicings and a full discussion of spatial gauges (polar slicing, minimal shear, diagonal gauge, radial gauge, isothermal gauge, etc.) see Piran.[20]

It is worth to note at this point that all these singularity avoiding gauges lead to a complete evolution system which is not hyperbolic. In the third section, we will take just the opposite way to study the singularity avoidance properties of both the harmonic coordinates condition and a variant of it, which we term *harmonic slicing*, due to Choquet-Bruhat and Ruggeri[47] which also leads to a hyperbolic evolution system.

## 2 A brief history of Numerical Relativity

So far, we have reviewed the $3+1$ formalism and we have taken a quick look to the problem of coordinate conditions. We have given some references concerning the use of some slicings but we have not talked about the historical background. Here, we will give a brief history of Numerical Relativity.

### 2.1 First birth: the sixties

Numerical Relativity is still in an early stage of development compared to fields such as Computational Fluid Dynamics, which have more than forty years of experience. We can place its birth in the mid sixties and seventies. In fact, Numerical Relativity has born twice.

In 1964, Hahn and Lindquist[48] tried to calculate a black hole collision. Lack of computer resources and a poor theoretical background (the notion of black hole and horizon had not been defined yet), combined with their use of geodesic slicing, made this pioneering work only partially successful. In contrast, in 1966 May and White[49] succeeded in the spherical (i.e., 1-D) collapse problem, treated in a Lagrangian approximation with fairly simple techniques. Nevertheless, their slicing could not avoid the central singularity that formed during the collapse to a black hole and, therefore, their code was limited to the hydrodynamics involved in the neutron star formation.

Since then, Numerical Relativity has been split in two different branches. The success of May and White is the responsible for the astrophysical branch of Numerical Relativity; what is called *Numerical Relativistic Hydrodynamics*. May and White work,[49,50] together with the modelling of spherical supernova explosions performed by Wilson,[51] the detailed studies of relativistic spherical collapse problems by Matsuda and Sato,[52] and the Eulerian code of Wilson in 1972, form the basis of all the later work.

### 2.2 Second birth: the seventies

The second branch, *Numerical Relativistic Geometrodynamics*, that is, what we usually mean simply by *Numerical Relativity*, is devoted to the left hand side of the Einstein Equations. Progress in this branch stagnated until the mid seventies, when DeWitt's group attacked the two black hole problem. Following the work of Cadez,[53] in 1975 Smarr[29] and Eppley[54,30] succeeded in evolving with the maximal slicing condition the head-on collision of two Misner black holes of equal mass and their coalescence, with an estimate of the gravitational waveform emitted. They found that the remnant is a single Schwarzschild black hole and the emitted energy in form of gravitational radiation was a small fraction ($10^{-3}$) of the total mass of the formed black hole.

Furthermore, this ambitious work was responsible for the firm stablishment of the theoretical framework, leading to the first numerical tests with the maximal slicing condition,[55] the introduction of the initial value problem formulation,[23,24,56] the theoretical formalism for the collision of two black holes,[53,29,57] and the precise statement of kinematical conditions in the construction of spacetime.[1] Finally, in 1979 a rigorous definition of Numerical Relativity was attempted by Eardley and Smarr,[31] who pointed out that Numerical Relativity can in principle construct any globally hyperbolic solution to Einstein Equations.

In some sense, the work of Smarr and Eppley was a bit ahead of its time. They used computer resources – a CDC-6600 with a peak speed of 7 MFLOPS and only 40 KWords of main memory and a CDC-7600 with 30 MFLOPs and 65KW – that nowadays seem completely insufficient for the involved 2-D computations. Their computations took more than a week of computer time to complete (for a review, see Hobill and Smarr[58]) and ran on crudely zoned grids. The same lack of computer power characterizes the first 2-D axisymmetric code which solved both the metric and the hydrodynamic equations, developed in 1978 by Wilson and Smarr,[32] applied to study the head-on collision of polytropic neutron stars. At the same time, codes using cylindrical coordinates[41] and planar symmetry[13] were developed.

### 2.3 Infancy: 1980–1986

This situation was highly improved with the increase in computer resources caused by the release of the first CRAY supercomputer in 1978. This marked the explosion in applications of Numerical Relativistic 2-D codes during the early eighties. The extension to 2-D was crucial for the study of gravitational radiation, because it is well known that spherically symmetric systems do not emit gravitational radiation. Until then, most of the work in gravitational radiation emission had been based in newtonian gravity and the quadrupole formula.

In 1981, the first general relativistic rotating stellar collapse was calculated by Nakamura and Sato,[59] who found that black hole formation only occurred below a certain amount of total angular momentum. Unfortunately, their code was not suitable for evaluating emission of gravitational radiation. In 1983, Bardeen and Piran[35,60] overcame several theoretical difficulties and introduced the use of the polar slicing. They succeeded in computing the emitted waveform and their work was extended in 1985 by Piran and Stark,[61] who developed a general code for rotating gravitational collapse which made use of a radial gauge and a mixture of polar and maximal slicing. In this code, the treatment of the hydrodynamic terms followed Wilson[32] and they used a staggered (in space and time) grid. They found that the waveform has a characteristic signature corresponding to the excitation of the lowest normal mode of the formed black hole. Besides, the energy transmitted in gravitational radiation is only $7 \cdot 10^{-4}$ of the mass of the star.

Meanwhile, following the work of Wilson[32] and Dykema,[33] Evans[34] developed an axisymmetric code specialized for vacuum which was extended to include matter with some modern treatment of the transport terms.[62] The metric equations were treated with a staggered mesh and second-order upwind differencing, and special treatment of the coordinate singularities by means of "numerical regularization". In 1986, the code was applied[17] to the study of non-rotating gravitational collapse. It showed the total breakdown in the usefulness of Newtonian gravity and the quadrupole formula for gravitational radiation estimation. It also stressed the importance of detailed microphysics and better parametrizations of the nuclear equation of state. In this computation, a CRAY-XMP was used, resulting in a code which ran at around 50 MFLOPS and took 60–120 $\mu$s per zone-cycle.

These successes with axisymmetric codes, together with several detailed 1-D computations such as the planar cosmology code of Centrella and Wilson,[42] the analysis of gravitational radiation emitted by normal mode perturbations in Schwarzschild[63] and Kerr geometries,[64] and the first relativistic stellar dynamics computations of Shapiro and Teukolsky[18] in which the Einstein Equations are solved for the dynamic evolution of a collisionless gas of particles using a combination of Numerical Relativity techniques with N-body particle simulations, contributed to the firm stablishment of the field. In Shapiro and Teukolsky,[18] we can find the definition of the "holy grail" of Numerical Relativity: a code that (i) avoids singularities, (ii) handles black holes, (iii) maintains high accuracy, and (iv) is capable of running forever. We can advance the result that, up to this date, no code has met all these strong requirements.

This situation led to propose a series of test bed calculations[65] which were considered to be crucial to calibrate future progress in Numerical Relativity. It was a set of 19 independent tests involving: the Riemann shock tube, spherical accretion onto a black hole, stability and equilibrium of star clusters, Newtonian homogeneous test, homogeneous collapse of a 4/3 polytrope, pulsation sequences of fluid stars, Oppenheimer-Snyder collapse, nonspherical dust infall onto a Schwarzschild black hole, linearized quadrupole waves, gravitational radiation damping of neutron stars, spatially homogenous cosmologies, ...etc. All of them have analytical solutions. Finally, they also proposed a test with a "numerical" (computed with the available axisymmetric codes) solution: the 2-D head-on collision of two neutron stars. Furthermore, the importance of a synergism between numerical and analytic relativity was stressed by Schutz,[66] and the study of matched analytic-numerical solutions of wave equations performed in 1985 by Anderson and Hobill[67] opened a new approach to gravitational radiation extraction.

In relativistic hydrodynamics, the treatment of the hydrodynamic terms introduced by Wilson[32,33] marked the posterior work. An excellent review can be found in Martí,[68] which contains a detailed numerical comparison of the works of Wilson and Smarr,[32] the study of accretion in a fixed metric background due to Hawley, Smarr and Wilson,[69] the work of Hawley,[70] the advanced code of Norman and Winkler,[71] and the contributions of the Kyoto school.[59]

### 2.4 Adolescence: 1987–1992

In 1987, the work of Nakamura, Oohara and Kojima[72] marks the beginning of a a new phase in Numerical Relativity. They constructed the first fully threedimensional (3-D) code. The availability of advanced supercomputers such as a FACOM V400 in Kyoto made possible this demonstration code. We use the term "demonstration" because they used fairly simple numerical techniques and a geodesic slicing ($\alpha = 1$), so they could only apply it to a simple problem: the evolution of pure gravitational waves. The test involved a grid of $80^3$ points in cartesian coordinates, requiring 40MW (around 60 Words per zone), which were evolved 3000 time steps, taking 3 hours to complete.

In the same work, they also gave results of a high resolution 2-D axisymmetric code for the general relativistic collapse of rotating stars applied to study the collapse of supermassive stars, deformed stars, stars with magnetic fields and neutron stars. This involved a extension of the regularization techniques developed for 1-D codes[73] and a careful analysis of gravitational waves from perturbation of spherically symmetric space-times, following the Regge-Wheeler-Zerilli formalism.[74] Finally, in 1989 Nakamura and Ooraha[75] developed a full 3-D newtonian hydrodynamic code (which used artificial viscosity techniques) to study the coalescence of binary neutron stars.

It is interesting to point out that the different work lines of the Kyoto's Numerical Relativity group are very representative of the work in the rest of the world. First, the existence of the first temptative tests in 3-D, both for the metric and for the hydrodynamic terms. Second, the development of highly accurate 2-D codes. Finally, the use of 1-D codes to explore theoretical advances and new numerical methods.

#### *2.4.1 1-D codes*

Nowadays, practically every Numerical Relativity group has its own 1-D code. Many of them are applied to develop state-of-the-art numerical techniques, test new coordinate conditions, or perform preliminar tests for multidimensional extension.

There are many 1-D cosmology codes. Goldwirth, Ori and Piran[76] developed a code to investigate cosmic censorship and stressed the importance of 1-D codes as "laboratories" for new ideas. Holcomb[46] developed a spherically symmetric cosmology code which failed to be extended to several dimensions. Anninos, Centrella and Matzner[44,45] use a planar-symmetry code to explore basic nonlinear physics of vacuum. There are also new hydrodynamic 1-D codes and even May and White code is still vigent: Seidel[4] uses a modified lagrangian code to analyze the gravitational radiation from even-parity perturbations of stellar collapse. See Martí[68] for a review of the work performed by the Valencia group with highly accurate hydrodynamical codes and shock capturing techniques.

Choptuik[77] has developed a code which uses adaptative mesh refinement[78] and applies it to study scalar field collapse. Bernstein *et al.*[38] have performed advanced tests with black hole spacetimes using different numerical methods and slicing conditions. Gómez *et al.*[79] and Gómez and Winicour[11] explore the possibilities of numerical evolution on null cones using 1-D codes and developing new numerical techniques. Recently, Shibata and Nakamura[80] have proposed a new slicing (Conformal slicing) condition suitable for 3-D work based on studies with a 1-D code. Seidel and Suen[81] are working towards a singularity-proof scheme in Numerical Relativity with preliminar tests in 1-D, by choosing a horizon-locking coordinate using the shift vector and introducing a numerical scheme that respects the causal structure of the spacetime.

We can see that 1-D codes must not be underestimated. Insight may be obtained even with 0-D codes, such as the study of the chaotic behaviour in the mixmaster universe due to Hobill *et al.*,[82] in which the equations are simply ODEs.

### *2.4.2 2-D codes*

In 1989, Evans studies the enforcement of the momentum constraints in his 2-D code.[19] The same year, Abrahams[83] develops an advanced gauge invariant waveform extraction method. In 1990, Evans code is enhanced with the waveform extraction techniques of Abrahams and both of them apply it to study the oscillations of relativistic stars,[84] and the head-on collisions of compact stars.[85] Their code continues using maximal slicing, elliptic quasi-isotropic gauge conditions. Now, typical runs involve 180 radial zones and 32 angular, achieving a speed of 90 MFLOPS on a CRAY-YMP, and take 6 CPU hours to complete 80000 time steps. The code includes advanced gauge invariant waveform extraction. Recently,[86] they have applied it to study the black hole formation by an imploding gravitational wave (a "geon")

The group of the National Center for Supercomputing Applications (NCSA) has developed[39] a highly accurate 2-D vacuum code for the study of black hole interactions with gravitational waves. It allows easy implementation of different spatial gauges, time slicing conditions and numerical evolution schemes. It inherits the tests and experience of the 1-D code described in Bernstein *et al.*.[38] The initial data corresponds to a single black hole superimposed with time-symmetric gravitational waves (Brill waves). This code, as Evans and Abrahams code, also uses the waveform extraction gauge-invariant method of Abrahams.[83] They have performed extensive comparisons with black hole perturbation theory, finding excellent agreement. Typical runs involve 201 radial zones by 53 angular ones, requiring 7000 time steps to complete. This takes 4MW of memory and 8 hours of CPU time on a CRAY-2. Currently,[87] the code is being used to repeat, with higher accuracy, the pioneering computations of Smarr[29] and Eppley.[54]

The approach of Shapiro and Teukolsky to stellar dynamics has also been fruitful for 2-D studies. In 1987 they extended their 1-D mean-field particle simulation code to a Newtonian axisymmetric code, performing many applications.[37] Recently, they have developed a general relativistic version. One of the first applications has been the study of naked singularities,[88] in which they claim that naked singularities do form during the numerical evolution.

### *2.4.3 3-D codes*

Following the work of Matzner,[89,90] and the cosmology code of Kurki-Suonio, Matzner and Centrella,[91] the Texas relativity group has developed the first 3-D cosmology code.[92] This is a fully general relativistic code which uses a cartesian grid with zero shift vector and is not restricted in the choice of time slicings. For the matter terms, it can deal only with vacuum and scalar fields. A great feature of the code is that it solves the initial value problem. The metric is evolved using a staggered grid and a leapfrog scheme. The code requires 80 Words per zone; 24MW for a $64^3$ grid and takes around 5 hours to perform 3000 iterations on a CRAY Y-MP at an effective speed of 140 MFLOPs, that is, around 25 $\mu$s per zone-cycle. It has been applied to inflationary cosmological evolution with excellent results.[93,94]

Stark[95] has also performed some tests with a code for non-axisymmetric rotating gravitational collapse, studying the suitability of the radial gauge in the 3-D case. Bishop, Clarke and d'Inverno[96] have performed some preliminar work evaluating the use of transputer arrays in 3-D Numerical Relativity using the $2+2$ decomposition.[10]

In 3-D, one of the major difficulties resides in the an efficient solution of the initial value problem and the related problem of solving elliptic equations (also necessary for the evolution using maximal slicing or constant mean curvature slicings). Enormous work in the last years has been devoted to its study: initial data for black hole collisions,[97] multigrid techniques for elliptic equations,[98,99] the general solution of the IVP for N-black holes,[27,26] physical content in black hole initial configurations,[100] initial data for colliding neutron stars in cartesian coordinates,[101] Null Strut calculus,[102] finite element methods in the IVP,[103,104] multiquadrics,[105] parallelizable conjugate gradient methods,[106] initial data for the black hole collision in cartesian coordinates,[107] efficient elliptic solvers for 2-D and 3-D problems,[108] etc.

3-D Relativistic Hydrodynamics has also progressed in the last years. Some Newtonian codes have been developed to perform preliminary studies of the necessary numerical techniques. Smith, Centrella and Clancy[109] are developing a Newtonian code to study instabilities of rapidly rotating compact stars. Mezzacapa[110] has performed some work with neutrino radiation hydrodynamics. Nakamura and Oohara[75] developed a code which is giving its first post-newtonian results in the problem of coalescence of two neutron stars. Some initial results in 3-D have been attempted using smooth particle hydrodynamics.[111] Bonazzola and Marck are applying spectral methods to solve the hydrodynamic equations[112] and recently they have made progress in applying spectral methods to 1-D problems containing shocks. The work of Ibáñez *et al.*[113] is also directed towards shock capturing methods in multidimensions.

Finally, attempts with a fully relativistic hydrodynamic code have been recently done by Duncan, Sczaniecki and Shastri.[114] They have developed a code which uses an Eulerian grid in cartesian coordinates. It does not evolve the metric quantities, being limited to Minkowski spacetimes in the current version, which has only been applied to some standard shock tube tests and off-center collisions. It uses Wilson's approach[32] to evolve transport terms, Norman and Wilson "consistent advection" and operator splitting. The variables are centered and artificial viscosity is introduced in the source terms. The code requires 30 Words per zone, resulting in 51 MW for a $50^3$ grid, and it runs at 160 MFLOPs on a CRAY Y-MP, resulting in 15 $\mu$s per zone-cycle. It is interesting to note that some parallelization tests have been done, achieving 712 MFLOPs on a CRAY Y-MP 864.

#### *2.4.4 Shape of the things to come*

At the beginning of this review, we said that Numerical Relativity placed enormous demands of computer time and memory on even the largest and fastest supercomputers. We have seen that this is true for 3-D work. Progress in 3-D Numerical Relativity is closely related to computational advances and memory and speed is still insufficient for high resolution runs. This situation is rapidly changing. This year, machines with 16 GFLOPS (16 processor CRAY C-90) are already available. These advances in highly parallelizable supercomputers and the perspective of massive parallelizable machines – the CM-5 (Thinking Machines) at NCSA is now running at 64GFLOPs and it has 16GWords of main memory, and it will be upgraded soon[115] – suggests that in a couple of years we will be able to reach 1 TERAFLOP machines. The posibilities that this power opens for Numerical Relativity, and Computational Physics in general, are endless. Let us quote Smarr[116] and say that in ten years most of the relativistic community will become numerically oriented.

Unfortunately, algorithmic advances are not progressing at the same pace. The extended use of maximal slicing conditions causes that most of the computer time is spent in elliptic solvers. Towns[117] has made impressive advances to reduce the time spent in elliptic solvers from 95% to 80% of the total CPU time, but this is still a lot. One would like to get rid of maximal slicing. But not everything is a question of speed. Better algorithms may not only faster but also more accurate and stable. We have seen that the mathematical structure of Einstein Equations depends on the coordinate system that we chose. This makes difficult the election of appropriate numerical algorithms to solve them. Although Numerical Relativistic Hydrodynamics has already incorporated some modern numerical methods,[68] the same does not occur in the treatment of the Einstein Equations, where numerical methods that were introduced in the seventies or even before are still used. In contrast, Computational Fluid Dynamics is using state-of-the-art numerical techniques to solve the Navier-Stokes equations (see any issue of *Journal of Computational Physics*).

## 3 Hyperbolicity and coordinate conditions

As we saw in Section 1, the main problem in Numerical Relativity is to follow the spacetime evolution from given initial data. Obviously, this sets a strong connection with the General Relativistic Cauchy problem: we can expect to be able to develop consistent stable numerical algorithms only when the existence and uniqueness of the solution is ensured by a well posed initial value problem. As we saw, the Einstein field equations must be supplemented with some coordinate conditions and this determines the mathematical structure of the evolution system.

We will consider here a variant of the harmonic coordinates condition introduced by Choquet-Bruhat and Ruggeri,[47] by keeping only the harmonicity condition for the time coordinate:

$$\Gamma^0 = -\Box x^0 = 0 \tag{14}$$

and choosing the space coordinates such that the shift vector vanishes

$$g^{0i} = 0 \;\; (i = 1, 2, 3) \tag{15}$$

leading to "Eulerian observers". In that work, the term "algebraic gauge" was used. We will avoid using this term because it may be a little too general. A large number of algebraic gauges have already been introduced in Numerical Relativity. For this reason we have used the more specific term "harmonic slicing" to emphasize that the time coordinate is harmonic in that gauge.

### 3.1 The Harmonic Slicing

Allowing for (15), the line element can be decomposed in the form

$$ds^2 = -\alpha^2 dt^2 + g_{ij} dx^i dx^j \tag{16}$$

and this means that the time lines are orthogonal to the $t$ =constant slices. This ensures that the congruence of temporal lines will be as regular as the time slicing itself.

Note that the spacetime slicing is independent of the choice of space coordinates. It is governed just by the lapse function $\alpha$ in (16). This means that both the harmonic coordinates and harmonic slicing conditions lead to the same time slicing and, then, they have the same singularity avoidance properties.

An algebraic condition on the lapse is obtained writing equation(14) in the form:

$$\partial_t \frac{\alpha}{\sqrt{g}} = 0 \; , \tag{17}$$

so that

$$\alpha = C(x)\sqrt{g} \; , \tag{18}$$

where $g$ stands for the determinant of the three-dimensional metric $g_{ij}$ and $C(x)$ is an integration constant.

Equation (18) shows that the time evolution will slow down whenever the space volume element $\sqrt{g}$ decreases and this is the key for showing the singularity avoidance properties of the harmonic slicing, which is what we are going to do next.

### 3.2 Singularity avoidance

As we have restricted to Eulerian (zero shift vector) coordinate systems, the three-dimensional metric $g_{ij}$ induced on each slice coincides with the space (quotient) metric of the normal observers. Therefore, the singularities of the slicing must correspond to singularities in the congruence of time lines and vice versa.

Let us consider the time line with local equation

$$x^i = x^i_0 \tag{19}$$

and let us use the proper time $\tau$ to label the points along this line. A *"crushing singularity"* occurs when the three-dimensional volume element $\sqrt{g}$ vanishes at a given value $\tau_S$ of $\tau$:

$$\sqrt{g}(\tau_S) = 0\,. \tag{20}$$

We are supposing that the determinant does not vanish identically along the line as it would happen if the value $x_0^i$ corresponds to a singularity (center, polar line) of the spatial coordinate system on every slice. Then, Eq.(20) represents a singularity of both the time lines (the normal lines are focusing to a caustic) and the slicing, because the induced metric $g_{ij}$ passing on that point is not invertible there. As we saw in Section 1, this is fatal for the construction of the spacetime.

To study the harmonic slicing singularity avoidance properties, we will restrict ourselves to "focusing singularities". The line (19) has a focusing singularity at $\tau_S$ when (20) holds and the proper-time derivative of the volume element remains bounded at this point, so that there exists a constant $B$ such that:

$$|(\partial_\tau \sqrt{g})(\tau_S)| < B \tag{21}$$

We adopt this condition to exclude from our consideration singularities accompanied with sudden variations of the spatial volume element.

So, let us consider a harmonic slicing starting from a regular set of initial data on the initial slice with coordinate time $t = t_0$. Equation (18) is gives that

$$C(x^i) = \frac{\alpha_0}{\sqrt{g}} \tag{22}$$

If we choose any normal line (19) parametrized by its own proper time $\tau$ (by choosing the $t = t_0$ slice as the origin), we will find along such a line that

$$\alpha(\tau, x_0^i) = C(x_0^i)\sqrt{g} \tag{23}$$

and then

$$\partial_\tau \alpha = C(x_0^i)\partial_\tau \sqrt{g}\,. \tag{24}$$

Allowing for (23) and (24), we see that we can replace $\sqrt{g}$ by $\alpha$ in Eqs.(20) and (21) defining a focusing singularity. This can be reformulated as

$$\lim_{\tau \to \tau_S} \frac{\tau - \tau_S}{\alpha} \neq 0\ , \tag{25}$$

where the limit is taken along the line (19).

The coordinate time interval $\Delta t$ elapsed along that line between the points where $\tau = 0$ (that is, $t = t_0$) and $\tau = \tau_S$ is given by the improper integral

$$\Delta t = \int_0^{\tau_S} \frac{1}{\alpha(\tau, x_0^i)}\, d\tau \tag{26}$$

and it follows from (25) that this integral cannot converge. This means that focusing singularities cannot be reached by harmonic slicings in a finite number of timesteps. The same thing is known to be true for maximal slicing, but then, the assumption (21) is not necessary. Maximal slicings can avoid stronger singularities than harmonic ones. For a comparison, see Bona and Massó.[118,119]

Note that singularity avoidance does not mean absence of coordinate singularities, as it is shown by the following example:

$$ds^2 = e^{-2t}(-dt^2 + dx^2) + dy^2 + dz^2\ , \tag{27}$$

which is a kasner-like form of the Minkowski metric; a coordinate singularity appears at a finite value of the proper time which is avoided by the harmonic slicing (it does not appear at any finite value of the coordinate time). The unphysical slicing is produced by an unphysical choice of initial conditions (the time derivative of the vacuum metric is chosen to be spatially constant).

### 3.3 A hyperbolic evolution system

It is well known that Einstein Equations lead to a hyperbolic evolution system in harmonic coordinates (see, for instance, York[24]). The same thing is true for the harmonic slicing.[47] We have shown its singularity avoidance properties and we will derive now a first order hyperbolic system suitable for Numerical Relativity.

In order to obtain a first order system, we will introduce the auxiliary quantities $Q^{ij}$, $D_k^{ij}$ and $L_i$, defined as follows:

$$Q^{ij} \equiv \partial_t g^{ij} \quad , \quad D_k^{ij} \equiv \partial_k g^{ij} \quad , \quad L^i \equiv \partial^i \ln \alpha . \tag{28}$$

We can perform the $3+1$ decomposition of the vacuum Einstein field equations (8). Allowing for the coordinate condition (14), the time component can be written as:

$$\partial_t \left[ \frac{\sqrt{g}}{\alpha} Q \right] + \partial_k \left[ 2\alpha\sqrt{g} L^k \right] \;=\; \frac{\sqrt{g}}{2\alpha} \left[ Q^{ij} Q_{ij} - Q^2 \right] \;, \tag{29}$$

where we have noted $Q \equiv Q_k^k$. Allowing for the *energy constraint*, (29) could also be obtained by taking the second time derivative of the harmonic slicing condition (17).

The mixed spacetime components of (8) can be written in our gauge:

$$\partial_t \Gamma^i = Q L^i - 2 Q_j^i L^j + \Gamma^i_{j\,k} Q^{j\,k} \;. \tag{30}$$

Allowing for the *momentum constraint*, (30) could also be obtained from the definition of $\Gamma^i$,

$$\Gamma^i \equiv -\Box x^i = \frac{1}{2} g_{j\,k} D^{ij\,k} - D_k^{ki} - L^i \;. \tag{31}$$

The space components of (8) can be written as

$$\begin{aligned} \partial_t \left[ \frac{\sqrt{g}}{\alpha} Q^{ij} \right] - \partial_k \left[ \alpha\sqrt{g} ( D^{kij} + g^{ki}\Gamma^j + g^{kj}\Gamma^i ) \right] &= \\ = \frac{\sqrt{g}}{\alpha} Q^{ik} Q_k^j - 2\alpha\sqrt{g} \left[ \Gamma^{ikl}\Gamma^j_{kl} + L^i L^j - \Gamma^i\Gamma^j \right] &\;. \end{aligned} \tag{32}$$

In order to complete the evolution system, we will provide evolution equations for the remaining quantities (28), namely

$$\begin{aligned} \partial_t \left[ D_k^{ij} \right] - \partial_k \left[ Q^{ij} \right] &= 0 \;, \\ \partial_t L_i + \frac{1}{2} \partial_i Q_k^k &= 0 \;. \end{aligned} \tag{33}$$

The complete evolution system formed by (32,17,33) has the balance law structure:

$$\partial_t D(u) + \partial_k F^k(u) = S(u) \;, \tag{34}$$

where the densities $D$, fluxes $F^k$ and sources $S$ are vector valued functions of the set of variables

$$u = \left( \alpha, L^i, g^{ij}, Q^{ij}, D_k^{ij} \right) \;. \tag{35}$$

The balance law form (34) of the evolution system is well known in hydrodynamics and standard numerical methods for this kind of systems have been developed. The consistency and stability of these methods have been established only for hyperbolic systems, that is, systems in which the characteristic matrix (the Jacobi matrix of the projection of the Fluxes along a given direction) has real eigenvalues and a full set of independent eigenvectors. The most powerful methods actually use the spectral decomposition of the characteristic matrix to obtain a finite difference version of the equations. But the complete evolution system for (35) is not hyperbolic. A hyperbolic system could be obtained by taking the time derivative of (32) and using (30) to eliminate the $\Gamma^i$ terms in the principal part of the derived system.[47]

We shall follow here a much simpler way[120] replacing $L^i$ by $\Gamma^i$ in the list of basic quantities (35)

$$u = \left(\alpha, \Gamma^i, g^{ij}, Q^{ij}, D_k^{ij}\right) \tag{36}$$

and using (30) instead of the second set of equations in (33) to complete the system (note that $L^i$ is now to be obtained algebraically from $\Gamma^i$ by using (31). The resulting system in the quantities (36) is hyperbolic as we will show in what follows.

#### *3.3.1 Hyperbolicity*

To study the hyperbolicity of a system of the form (34) we need to consider only the principal part (densities and fluxes). This is closely related to the "operator splitting" approach. The time evolution is decomposed into a combination of two kinds of processes, namely:

1. transport, governed by the flux-conservative law

$$\partial_t D(u) + \partial_k F^k(u) = 0 \ . \tag{37}$$

2. evolution, driven by the source terms only.

The hyperbolicity of the system can only ensure the stability of the numerical methods used in the transport step. The stability of the whole scheme would require in addition that of the system of nonlinear ordinary differential equations given by the source terms. This is beyond our goal here, where we will be concerned only with the transport step.

Going back to our evolution system, let us note that the ten quantities $\alpha, \Gamma^i, g^{ij}$ are constant in the transport step (they do not have flux terms) and that the evolution of the remaining twenty-four quantities can be written then

$$\begin{aligned} \partial_t \left[\frac{\sqrt{g}}{\alpha} Q^{ij}\right] \quad &- \quad \partial_k \left[\alpha\sqrt{g}(g^{kl}\lambda_l^{ij})\right] = 0 \ , \\ \partial_t \left[\lambda_k^{ij}\right] \quad &- \quad \partial_k \left[Q^{ij}\right] = 0 \ , \end{aligned} \tag{38}$$

where we have noted

$$\lambda_k^{ij} \equiv D_k^{ij} + \delta_k^i \Gamma^j + \delta_k^j \Gamma^i \ . \tag{39}$$

The interest of this change of variables (39) resides in the fact that the resulting system (38) consists of six decoupled identical subsystems of four linear equations in $Q^{ij}, \lambda_k^{ij}$ (with $i, j$ fixed). This is important if we are planning to apply either spectral or implicit finite difference methods in the transport step, because it greatly reduces the dimensions of the matrices involved.

But the main point is that every subsystem in (38) has the structure of the *wave equation* in first order form. This shows that the evolution system is hyperbolic with characteristic surfaces given by the light cones (plus the time lines if one takes into account the quantities which are constant in the transport step), providing a good starting point for applying standard existence and uniqueness theorems. This also means that the arsenal of numerical methods designed and tested for the three-dimensional wave equations is at our disposal: this is a fast way of coming in on 3-D Numerical Relativity.

### 3.4 Other hyperbolic evolution systems

Let us note that the use of the harmonic slicing in our derivation of the hyperbolic system has been crucial. This is a point in favour of its usage, given that the evolution system with other coordinates conditions (as in the maximal slicing case) is not hyperbolic and, to our knowledge, existence and uniqueness theorems are yet to be proven in other gauges.

But it is also crucial the precise choice of the subset of the Einstein equations to use for evolving the system. To illustrate this point, let us write down the principal part of (29) and the second set of equations in (33),

$$\partial_t \left[\frac{\sqrt{g}}{\alpha} Q\right] + \partial_k \left[2\alpha\sqrt{g} L^k\right] \;=\; 0 \;,$$
$$\partial_t L_i + \frac{1}{2}\partial_i Q^k_k = 0 \;. \tag{40}$$

The system (40) has the wave equation structure and it was not used to form the evolution system (38).

We can actually combine (40) with the wave equation subsystems corresponding to the different $(i,j)$ components in (38) to get

$$\begin{aligned} \partial_t \left[\frac{\sqrt{g}}{\alpha}(Q^{ij} + n(i,j) g^{ij})\right] \;\; &- \;\; \partial_k \left[\alpha\sqrt{g} g^{kl}(\lambda_l^{ij} - 2n(i,j) L_l g^{ij})\right] = 0 \;, \\ \partial_t \left[\lambda_k^{ij} - 2n(i,j) L_k g^{ij}\right] \;\; &- \;\; \partial_k \left[Q^{ij} + n(i,j) g^{ij}\right] = 0 \;, \end{aligned} \tag{41}$$

which give us a different hyperbolic system for every choice of the symmetric array of arbitrary parameters $n(i,j)$, with a different space of solutions. Only the physical solutions (the ones obtained from initial data verifying the constraint equations) must coincide.

Note that the combinations appearing in (41) verify

$$Q^{ij} + n(i,j) g^{ij} \equiv \alpha^{2n(i,j)} \partial_t(\alpha^{-2n(i,j)} g^{ij}) \;, \tag{42}$$

so that the parameters $n(i,j)$ allow us to scale every metric component with a given power of the lapse function. This can be useful in numerical applications. Also, the particular choice $n(i,j) = 1$ can be suitable for Astrophysical applications involving small velocities and weak gravitational fields because, for a perfect fluid matter content, it cancels out the leading order (Newtonian) source terms.

Currently, we are developing a fully 3-D code which numerically implements the above system of equations using advanced numerical methods. We have performed a preliminar test in a Kasner-like spacetime for the case $n(i,j) = 0$ with excellent results.[121]

## 4 Conclusions

To finish this review, let us point out that the numerical techniques so far used in General Relativity have been developed in a somewhat *ad hoc* manner simply because the Einstein equations were not put in the form of hyperbolic balance law equations. Therefore, a large body of numerical methods useful for such equations were not applicable to General Relativity. The results presented here may help to change that situation and hopefully will stimulate rapid progress in the development of numerical simulations of relativistic astrophysical systems.

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