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Gravitational Waves from Oscillating Black Holes

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ABSTRACT The generation and propagation of gravitational waves in dynamical black hole spacetimes is calculated by constructing numerical solutions to the Einstein equations for general relativity. Numerical relativity offers the only means by which the nonlinear, time dependent dynamics of strong gravitational fields can be calculated and a successful method for doing so is discussed in this article. Guidelines for using the code are provided.

1.1 Introduction

Black holes represent some of the most exotic objects in the Universe and have been used to explain a number of astrophysical observations (Blandford, 1987). Supergiant ($10^6 M_\odot$) black holes are expected to occupy the centers of galaxies including our own Milky Way and they may also be responsible for some observed gravitational lenses. Smaller black holes are thought to be one of the constituents of binary X-ray sources while the accretion of gasses around black holes may create highly collimated relativistic jets of matter. Astrophysical black holes can be created by the collapse of a stellar core having a mass greater than a few solar masses or the collapse of relativistic star clusters. Some may even be primordial remnants of the Big Bang.

In spite of these theoretical predictions, clear evidence of the existence of black holes has yet to be confirmed and more observations need to be made. Nevertheless calculations that predict astrophysical phenomena driven by black holes are being made in order to direct further observational efforts.

The computation of observable effects for which black holes may be responsible can be divided into two classes. The first class assumes that the

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presence of matter or nongravitational fields near a black hole does not perturb the structure of the black hole itself. In such cases one can solve the relativistic equations of motion for the matter (or fields), using the fixed background gravitational field provided by the black hole. In the second and more difficult case the black hole is itself dynamic. Its evolution is determined by the Einstein field equations of general relativity and the black hole interacts with matter and nongravitational fields in a nonlinear manner. It is with the behavior of dynamical black holes that this article is concerned.

Besides the existence of black holes, general relativity also predicts that disturbances of the gravitational field should propagate with the speed of light. Gravitational radiation is generated by the motions of massive objects and is detected directly by observing the accelerations of test masses in the field of a gravitational wave. Indirect evidence of gravitational waves comes from data concerning the decay of the period of the binary pulsar PSR 1913+16. The measurements are consistent with the loss of energy and angular momentum due to the emission of gravitational waves (Taylor, 1992). The waves themselves have not yet been detected directly but highly sensitive devices are being constructed that should soon be capable of measuring gravitational waves originating from astrophysical systems (Abramovici, 1992).

The strongest sources of detectable gravitational waves will likely be highly compact objects that interact nonlinearly with the waves they generate (e.g. coalescing neutron star binaries, black hole collisions, and possibly supernova explosions). Such systems cannot be described easily through a perturbative analysis of the Einstein equations. Analytic methods for solving the Einstein equations are limited to dealing with idealized cases and numerical methods will become increasingly important for the construction of physically realistic solutions involving dynamic strong gravity sources.

Gravitational radiation can provide direct evidence for the existence of black holes since the wave forms emitted by them are uniquely determined by the few parameters needed to describe black holes. Intense sources can also act as standard candles by which one can make cosmological distance measurements (Schutz, 1986). Because of their weakly interacting nature, gravitational waves can probe events that occurred in the early universe before the moment of decoupling between matter and electromagnetic radiation. Once constructed, gravitational wave detectors should provide astronomers with a new window through which to view the Universe.

While a catalogue of expected gravitational wave forms is one of the goals of numerical relativity, the computational requirements for calculating a well resolved, fully relativistic, 3D time dependent spacetime is beyond the capabilities of current technology. Until computers are developed that can regularly sustain Teraflop rates, today's research in numerical relativity remains focused on (a) experimenting with new numerical techniques, (b) developing highly accurate and efficient 2D codes that can calculate the

gravitational wave forms from realistic sources, or (c) attempting to develop full 3D codes in preparation for more powerful machines.

This article will focus on an example of a 2D code designed to calculate gravitational waves emitted by oscillating axisymmetric black holes. Since it is unlikely that a stellar collapse will be spherically symmetric, the geometry of a newly formed black hole will deviate significantly from sphericity. Therefore a black hole will exist in a transient, nonequilibrium state before settling down to a static or stationary configuration. An advantage of 2D systems over their 3D counterparts is that such systems are relatively simple from a computational perspective.

Numerical relativity need not be restricted to black hole physics. It is a rapidly growing endeavor and research in cosmology, stellar core collapse, gravitational wave generation and even the mathematical structure of the Einstein equations is being aided through computer based calculations. Those interested in the descriptions of some projects designed to calculate 2D and even 3D relativistic systems may find them in edited volumes specifically concerned with numerical relativity (Centrella, 1986, Evans, *et al.*, 1989).

1.2 The geometrodynamics of the vacuum

In the absence of matter or nongravitational fields the self-interaction of the vacuum gravitational field is sufficient for producing black holes and gravitational waves. The numerical algorithm used to compute such interactions is based upon a formalism known as "geometrodynamics" which combines the geometric description of gravity provided by the general theory of relativity with a form of the Einstein equations that resembles the equations of other classical field theories such as electrodynamics or hydrodynamics. In this section we summarize the essential ideas of this formalism and introduce the notation needed to understand the structure of the black hole code. A more pedagogical treatment may be found in (Arnowitt, *et al.*, 1962).

The theory of general relativity begins by defining the structure of a Riemannian spacetime geometry in terms of a 4×4 symmetric metric tensor $g_{\mu\nu}$ that determines the distances between the infinitesimally separated points (called "events") in the four-dimensional spacetime. For notational purposes Greek indices run from 0 to 3, with $x^0 = t$ representing a time coordinate. Latin indices run from 1 to 3 and label the spatial coordinates $[x^1, x^2, x^3]$ (e.g. $[x, y, z]$ in Cartesian, or $[r, \theta, \phi]$ in spherical polar coordinates). The Einstein summation convention is used where a product with repeated indices indicates summation over the range of those indices (i.e. $p^\mu p_\mu = \sum_{\mu=1}^{4} p^\mu p_\mu$). Unless indicated otherwise geometrical units will be used where the speed of light, c, and the Newtonian gravitational constant,

G, are set to unity ($c = G = 1$).

The standard method of splitting spacetime into "space plus time" is to slice the four-dimensional spacetime into three-dimensional spatial hypersurfaces (3D volumes) threaded together by a congruence of timelike curves whose tangent vectors determine a 'time' vector (Arnowitt, *et al.*, 1962, York, 1979). This is called the "ADM" or "3+1" formalism. The 4-metric $g_{\mu\nu}$ is then determined by the three-dimensional sub-geometry and a choice of time coordinate. The infinitesimal spacetime distance, ds, between two spacetime events (separated in time by dt and in space by dx^i) is written in the generalized Pythagorean form

$$ds^2 \;=\; g_{\mu\nu}\, dx^\mu\, dx^\nu \;=\; -(\alpha^2 \;-\; \beta^i\beta_i)dt^2 \;+\; 2\beta_i\, dx^i\, dt \;+\; \gamma_{ij}\, dx^i\, dx^j\,.$$

The scalar function α is called the lapse function and determines the amount of proper time (measured by a freely-falling observer), $d\tau = \alpha dt$ that has elapsed between points in the 3D volume labeled by t and the 3D volume labeled by $t+dt$. The shift vector β^i is the coordinate three-velocity that determines the relative position of the spatial coordinate x^i from one volume to the next. Finally the 3-metric determines the line element $d\ell$ that measures distances between points in the 3D spatial volumes defined by $t = const$: $d\ell^2 = \gamma_{ij}\, dx^i\, dx^j$.

The lapse function and shift vectors are not dynamic variables. The Einstein equations do not provide evolution equations for them. The freedom to choose them arbitrarily is often exploited to simplify the evolution equations, avoid spacetime singularities, decrease the amount of distortion in the coordinates, or stabilize a numerical method.

The *intrinsic* curvature of a 3D spatial volume is calculated in the usual manner established by Riemannian geometry and the 3D Ricci tensor is

$$R_{ij} \;=\; \partial_k\Gamma^k_{ij} \;-\; \partial_j\Gamma^k_{ik} \;+\; \Gamma^m_{ij}\Gamma^k_{mk} \;-\; \Gamma^k_{im}\Gamma^m_{jk}$$

where the 3-Christoffel symbols are determined from the 3-metric by

$$\Gamma^i_{jk} \;=\; \frac{1}{2}\gamma^{im}(\partial_k\gamma_{jm} \;+\; \partial_j\gamma_{km} \;-\; \partial_m\gamma_{jk}).$$

Here ∂_i represents the partial derivative with respect to the coordinate x^i.

The *extrinsic* curvature K_{ij} expresses how the 3D volumes are embedded in the full four-dimensional spacetime. Explicitly, it is related to the time derivative of the metric: $K_{ij} = \frac{1}{2\alpha}(\nabla_i\beta_j + \nabla_j\beta_i - \partial_t\gamma_{ij})$. Here ∇_i is the covariant derivative with respect to γ_{jk} on the three-surface.

Having established the kinematic and dynamical variables the vacuum Einstein equations break up into the following set: 1) the constraints which are dependent only upon quantities defined in the 3-dimensional volume

$$R - K_{ij}K^{ij} + K^2 = 0\,; \quad \textit{Hamiltonian constraint} \tag{1.1}$$

$$\nabla_j(K^j_i - \delta^j_i K) = 0\,; \quad \textit{momentum constraint} \tag{1.2}$$

and 2) the evolution equations for the 3-metric and extrinsic curvature

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \tag{1.3}$$

$$\partial_t K_{ij} = -\nabla_i \nabla_j \alpha + \alpha[R_{ij} + K_{ij} K - 2K_{ik} K^k_j] + \beta^k \nabla_k K_{ij} + K_{kj} \nabla_i \beta^k + K_{ik} \nabla_j \beta^k \tag{1.4}$$

where the scalar 3-curvature is $R = \gamma^{ij} R_{ij}$ and the trace of the extrinsic curvature is defined by $TrK \equiv K = \gamma^{ij} K_{ij}$. Thinking of R as a second order differential operator acting on γ_{ij}, the constraint equations form a coupled set of elliptic equations. Together with the hyperbolic evolution equations they govern the dynamics of the gravitational field.

The construction of a numerical solution to these equations begins by solving the constraint equations on the initial 3-volume for those components of the metric and extrinsic curvature that cannot be freely specified. The solutions to the constraints are then used to provide the right-hand-sides of the evolution equations which determine the metric and extrinsic curvature on the succeeding slice of spacetime. Once the metric and extrinsic curvature are determined the construction of all future slices is completed by the evolution equations.

Analytically, the Einstein equations obey certain differential identities (the Bianchi identities) that make the evolution equations consistent with the constraints and prevents the system from being overdetermined. However a numerical evolution needs to be continually monitored since finite differences in space and time do not always commute. Problems can arise if small numerical errors grow in a nonlinear manner such that the constraints are not obeyed by the evolution. In such cases spurious non-physical sources to the gravitational field can be introduced.

1.3 Modeling pure vacuum black holes

All black holes are surrounded by a closed two-dimensional surface called the "event horizon" at which the gravitational escape velocity is equal to the velocity of light. Any observer in the region exterior to this boundary is unable to receive any information about processes occurring inside the boundary. Since the exterior of a black hole is causally disconnected from the interior, the portion of the spacetime inside the black hole may be modeled by any solution satisfying the Einstein equations that can be joined continuously to the exterior solution.

While it is expected that matter will occupy at least some part of the interior of a black hole, particularly if the black hole is the product of gravitational collapse, the interior of a black hole can be modeled by a vacuum solution to the Einstein equations. The simulations discussed in this article model the interior of the black hole with an "Einstein-Rosen bridge". This

configuration provides well defined boundary conditions inside the black hole and one only needs to solve the vacuum Einstein equations for the black hole exterior and part of the interior. The remainder of the spacetime is *isometrically* related to the computed numerical solution (i.e., there is a coordinate transformation which maps the computed metric into an identical one in the remainder of the spacetime).

In order to understand what the Einstein-Rosen bridge represents consider the Schwarzschild spacetime in the usual Schwarzschild polar coordinates (t, r, θ, ϕ). The line element is given as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2. \qquad (1.5)$$

where the parameter M is the mass of the black hole. The event horizon is located at $r = 2M$ and although the metric is singular there that singularity is not physical. It is only an aspect of the choice of radial coordinate. A true singularity exists at $r = 0$ since it cannot be removed by any coordinate transformation.

The Schwarzschild radial coordinate is determined by defining the area of a 2-sphere at constant r to be $4\pi r^2$. Other conditions on the radial coordinate can be imposed. One particularly useful condition requires the 3-metric to be conformally flat, i.e. the spatial line element is of the form:

$$d\ell^2 = \psi^4 (d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta \, d\phi^2). \qquad (1.6)$$

Introducing the "isotropic" radial coordinate, $\bar{r}$, defined by its relation to the Schwarzschild coordinate r by

$$r = \bar{r}\left(1 + \frac{M}{2\bar{r}}\right)^2 \qquad (1.7)$$

the four-dimensional line element is written as

$$ds^2 = -\left(\frac{1 - M/2\bar{r}}{1 + M/2\bar{r}}\right)^2 dt^2 + \left(1 + \frac{M}{2\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta \, d\phi^2). \qquad (1.8)$$

Now the event horizon is located at $\bar{r} = M/2$ and the relation

$$r' = \frac{M^2}{4\bar{r}} \qquad (1.9)$$

is an isometry operation (i.e. the metric is unchanged under this transformation) at the event horizon $r' = \bar{r} = M/2$.

The new radial coordinate r' describes an asymptotically flat space time covering the interior region of the black hole. The physical singularity at $\bar{r} = 0$ is mapped into the sphere at $r' = \infty$. The two asymptotically flat spacetimes are connected at the event horizon representing the surface of isometry between the two metrics.

In more general situations involving time dependence and multiple black holes, the isometry between the interior and exterior portions of the black hole can be maintained (although the isometry surface, called the "throat", is located inside the event horizon rather than on it). Such spacetimes consisting of two causally disconnected asymptotically flat spacetimes can be used to represent black holes in the vacuum without the need to introduce material sources inside the event horizon.

1.4 Coordinate choices for numerical evolution

In spite of the general covariance of the Einstein equations, a numerical computation cannot proceed until the coordinates and the numerical grid upon which the computation takes place are defined. This section provides a brief description of the coordinates and gauge conditions used in developing the code for oscillating black holes.

1.4.1 The 3-metric of spatial hypersurfaces

The spacetimes we are interested in calculating are axially symmetric and non-rotating. Therefore the solution must be independent of a third coordinate defined in terms of the angular rotation about an axis of symmetry. The solution does depend upon a radial coordinate and a polar angle if spherical polar coordinates are used.

For numerical calculations it is useful to introduce a dimensionless radial coordinate η related to $\bar{r}$ by

$$\bar{r} = \frac{M}{2} e^{\eta}. \tag{1.10}$$

In the (t, η, θ, ϕ) coordinates the static metric Eq. 1.5 appears as

$$ds^2 = -\tanh^2(\eta/2) dt^2 + \left[\sqrt{2M}\ \cosh(\eta/2)\right]^4 (d\eta^2 + d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1.11}$$

The throat is located at $\eta = 0$ and the isometry operation is simply $\eta \rightarrow -\eta$ (Bernstein, *et al.*, 1989). Since η is a logarithmic coordinate the numerical grid can easily cover regions far from the black hole while providing good resolution near the horizon. The coordinates θ and ϕ will represent the standard angular coordinates on a 2-sphere.

The most general 3-metric for axially symmetric systems depends upon four functions of η and θ ($\gamma_{\eta\phi} = \gamma_{\theta\phi} = 0$ since the metric must not change under the operation $\phi \rightarrow -\phi$). Adding an overall conformal factor $[\psi(\eta, \theta)]^4$ for reasons that are related to methods for solving the initial value problem, the general 3D line element is written in the form

$$d\ell^2 = \psi^4 \hat{\gamma}_{ij}\, dx^i dx^j = \psi^4 \left(A d\eta^2 + 2C d\eta d\theta + B d\theta^2 + D \sin^2\theta\, d\phi^2\right) \tag{1.12}$$

where A, B, C and D are functions of η, θ and t.

The extrinsic curvature tensor has a similar structure

$$K_{ij} = \psi^4 \hat{K}_{ij} = \begin{pmatrix} H_a \psi^4 & H_c \psi^4 & 0 \\ H_c \psi^4 & H_b \psi^4 & 0 \\ 0 & 0 & H_d \psi^4 \sin^2\theta \end{pmatrix}. \tag{1.13}$$

The conformal factor is introduced into the extrinsic curvature to simplify the evolution equations.

1.4.2 Choice of shift vector

Since there is no rotation and therefore no preferred direction for the ϕ coordinate, the metric coefficient $g_{t\phi} = \beta_\phi$ vanishes and the most general shift vector will have two components

$$\beta^i = (\beta^\eta, \beta^\theta, 0). \tag{1.14}$$

Now that the most general expressions for the 3-metric, extrinsic curvature and shift have been given one may explicitly write out the "3+1" equations. Unfortunately the full expressions are rather complicated and too long to reproduce in this article. However these equations are provided on the diskette in LaTeXformat in the file `adm.tex`.

The shift vector can be chosen arbitrarily and is often made to vanish so that the coordinates x^i represent the positions of Eulerian observers. In the simulations described here the shift vector is chosen to eliminate the function $\gamma_{\eta\theta} = \psi^4 C$. This condition simplifies the equations and provides long term stability on the axis of symmetry (Bernstein, *et al.*, 1993b, Bernstein, 1992).

The shift can be determined from the evolution equation for C:

$$C = \frac{\partial C}{\partial t} = -2\alpha H_c + B\frac{\partial \beta^\theta}{\partial \eta} + A\frac{\partial \beta^\eta}{\partial \theta} = 0, \tag{1.15}$$

which provides one equation for both components of β^i. By further assuming that the shift vector can be derived from a scalar function χ

$$\beta^\eta = \frac{\partial \chi}{\partial \theta}, \quad \beta^\theta = \frac{\partial \chi}{\partial \eta} \tag{1.16}$$

one obtains a Poisson-like equation for χ

$$B\frac{\partial^2 \chi}{\partial \eta^2} + A\frac{\partial^2 \chi}{\partial \theta^2} = 2\alpha H_c. \tag{1.17}$$

1.4.3 The lapse function

Like the shift vector, the function $\alpha(t, \vec{x})$ may be chosen arbitrarily subject to the requirement that $\alpha \geq 0$. Where $\alpha = 1$ the time coordinate

is the proper time for observers at fixed coordinate locations, and those grid points are in free-fall motion. Where $\alpha = 0$ the intervals of proper time vanish and there is no change in the values of physical quantities. A vanishing lapse can have certain advantages. Since well posed, perfectly regular initial conditions can lead to spacetime singularities (Hawking and Ellis, 1973) the vanishing of the lapse prior to the formation of a spacetime singularity can be used to prevent numerical floating-point overflows.

As an example, consider the case of $\alpha = 1$ everywhere and initial conditions that describe the spacetime outside a Schwarzschild black hole. All points in the spacetime are in free-fall and the observer initially located at the event horizon ($r = 2M$ or $\eta = 0$) hits the spacetime singularity in a time $\tau = \pi M$ (Misner, *et al.*, 1973). A numerical construction of the spacetime must then come to an end. Clearly this is not long enough for any information about the processes occurring near the event horizon to propagate out to a detector located at a reasonable distance from the black hole. In order to overcome this problem, one requires a lapse function that tends to zero inside the black hole and is close to unity far away from the black hole.

The maximal slicing condition, (obtained from maximizing the volume $V = \int dV = \int \sqrt{Det\gamma_{ij}} d^3x$ of the $t = const$ 3D hypersurfaces) can avoid singularities and leads to the relation $K = 0$. Taking the trace of the extrinsic curvature evolution equation one obtains

$$\partial_t K = -\nabla^i \nabla_i \alpha + \alpha \left[R + K^2 \right] + \beta^i \nabla_i K. \tag{1.18}$$

Imposing the maximal slicing condition $K = 0$ reduces this to:

$$\nabla^i \nabla_i \alpha = \alpha R. \tag{1.19}$$

Alternatively replacing the scalar curvature R with an expression involving only the extrinsic curvature from the Hamiltonian constraint yields

$$\nabla^i \nabla_i \alpha = \alpha K^{ij} K_{ij}. \tag{1.20}$$

This latter form tends to be better behaved numerically, and is generally used in the code. In addition, the term $K^{ij} K_{ij}$ is positive definite and theorems concerning the existence and uniqueness of solutions to the linear equation for α have been proven (Choquet-Bruhat and York, 1980).

Given an initial $K = 0$ slice, a subsequent one may be constructed by specifying appropriate boundary conditions on the lapse and solving Eq. 1.20. In this work we use the mixed Neumann/Dirichlet conditions

$$\left. \frac{\partial \alpha}{\partial \eta} \right|_{\eta=0} = 0, \quad \alpha|_{\eta=\eta_{max}} = \tanh(\eta_{max}/2) \tag{1.21}$$

which determine the isometry across the throat and gives a Schwarzschild behavior at large distances. The Schwarzschild lapse function is also a solution to the maximal slicing condition but with Dirichlet conditions

$$\alpha|_{\eta=0} = 0 \qquad \alpha|_{\eta=\infty} = 1. \tag{1.22}$$

The maximal slicing equation for α, like the equation for the shift "potential", is an elliptic equation and can be solved using the same numerical methods as those employed to solve for χ.

1.5 Boundary Conditions

Both the "3+1" evolution equations and the equations that determine the kinematic variables (i.e., the lapse α and the shift β) require conditions on the functions at both the spatial boundaries (i.e. grid edges) and temporal boundaries (i.e. Cauchy or initial data). The spatial boundary conditions will be determined by symmetry restrictions and the requirement that the spacetime approach the Minkowski space of special relativity far from the black hole. The initial conditions imposed upon the dynamical variables will establish the physical content of the spacetime and how it evolves into the future.

1.5.1 Spatial boundary conditions

The condition that the spacetime have axial symmetry as well as reflection symmetry about the equatorial plane provides a computational domain with boundaries at the axis of symmetry ($\theta = 0$) and the equator ($\theta = \pi/2$). Along the radial coordinate, the isometry surface at $\eta = 0$ and asymptotic flatness at large values η provide the remaining spatial boundaries of each $t = const$ slice.

A complete derivation of the spatial boundary conditions can be found in (Bernstein, *et al.*, 1993b, Bernstein, 1992). Essentially the isometry conditions at the axis and equator require that the metric functions ψ, A, B, and D all be symmetric functions across $\theta = 0$ and $\theta = \pi/2$, while the isometry condition at the throat requires that the metric functions be even functions of η at $\eta = 0$.

Furthermore requiring that the spatial boundary conditions on the extrinsic curvature components H_a, H_b and H_d to be the same as those imposed upon the associated metric functions provides the following conditions on the lapse function

$$\partial_\theta \alpha|_{\theta=0} = \partial_\theta \alpha|_{\theta=\pi/2} = \partial_\eta \alpha|_{\eta=0} = 0. \tag{1.23}$$

The shift vector is the only quantity that does not require a specific symmetry condition at the throat. Since it represents the freedom to make spatial coordinate transformations, it merely relabels points on a hypersurface without changing its geometry. In practice it is convenient to set symmetry conditions on the components of β^i for the purpose of simplifying the geometrodynamics code. If the spatial coordinates do not cross the

axis of symmetry or the equator during the evolution one has

$$\partial_\theta \beta^\eta|_{\theta=0} = \partial_\theta \beta^\eta|_{\theta=\pi/2} = \beta^\theta\big|_{\theta=0} = \beta^\theta\big|_{\theta=\pi/2} = 0. \tag{1.24}$$

Imposing the condition that the isometry operation ($\eta \rightarrow -\eta$) not change throughout the evolution (i.e. the coordinates do not move through the throat) provides the remaining conditions

$$\beta^\eta|_{\eta=0} = \partial_\eta \beta^\theta\big|_{\eta=0} = 0. \tag{1.25}$$

At the outer boundary, assuming that it is far from the black hole's surface, one would like to have an outgoing wave condition. If the boundary is so far away that the initial conditions do not contain any significant amounts of incoming radiation and that all departures away from a Schwarzschild spacetime are concentrated in a compact region close to the black hole, then no changes are expected to occur at the outer boundary. Therefore the spacetime is considered to be Schwarzschild at the outer boundary. The metric functions at η_{max} are then $A = B = D = 1$, $\psi = \sqrt{2M}\cosh(\eta_{max}/2)$, $\alpha = \tanh(\eta_{max}/2)$ and all components of the shift vector and extrinsic curvature tensor vanish.

1.5.2 Initial data

The initial data for the black hole plus gravitational wave spacetime has been described extensively in (Bernstein, *et al.*, 1993a, Bernstein, 1992). The essential feature of the initial data is that it consists of a time symmetric gravitational wave overlaid on an Einstein-Rosen bridge topology. This configuration represents a superposition of two well studied initial value data sets: the Brill wave solution (Brill, 1959, Wheeler, 1964, Eppley, 1979) and the Schwarzschild spacetime. The presence of the gravitational wave distorts the black hole which subsequently sheds those distortions by emitting gravitational radiation.

The initial value problem consists of finding a 3–metric and extrinsic curvature satisfying the constraint equations [Eqs. 1.1 and 1.2]. Time symmetric initial data has zero momentum and vanishing extrinsic curvature ($K_{ij} = 0$). This simplifies the problem since the momentum constraints vanish identically. One only needs to solve the reduced Hamiltonian constraint, $R = 0$, for γ_{ij}.

Using the conformal decomposition technique of York (York, 1979) and writing γ_{ij} in conformal form $\gamma_{ij} = \psi^4 \hat{\gamma}_{ij}$ the scalar curvature becomes

$$R = \psi^{-4}\hat{R} - 8\psi^{-5}\hat{\nabla}^2\psi, \tag{1.26}$$

where $\hat{R}$ and $\hat{\nabla}^2$ are the scalar curvature and laplacian associated with the conformal metric $\hat{\gamma}_{ij}$. For this choice of the conformal factor the Hamiltonian constraint is the *linear* elliptic equation

$$\hat{\nabla}^2\psi = \frac{1}{8}\psi\hat{R}. \tag{1.27}$$

In many problems $\hat{\gamma}_{ij}$ is chosen to be the flat Euclidean metric and in that case $\hat{\nabla}^2$ is the standard laplacian operator. The Schwarzschild conformal factor $\psi = \sqrt{2M}\cosh(\eta/2)$ is the monopole solution.

In his studies of the initial value problem for gravitational waves Brill (Brill, 1959) began by assuming a three metric of the form

$$ds^2 = \psi^4 \left[e^{2q}(d\rho^2 + \rho^2 d\theta^2) + \rho^2 \sin^2\theta \, d\phi^2\right]. \tag{1.28}$$

It was shown that in order for the mass of the hypersurface to be well defined, q must vanish on the axis ($\theta = 0$) and decrease at least as fast as ρ^{-2} in the radial direction.

In the (η, θ, ϕ) coordinates, with $\rho = (M/2)e^{\eta}$, the spatial line element is

$$d\ell^2 = \psi^4 \left[e^{2q}(d\eta^2 + d\theta^2) + \sin^2\theta \, d\phi^2\right] \tag{1.29}$$

and the Hamiltonian constraint becomes a linear second-order elliptic equation for $\psi(\eta, \theta)$

$$\frac{\partial^2\psi}{\partial\eta^2} + \frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial\psi}{\partial\theta}\cot\theta = -\frac{1}{4}\psi\left(\frac{\partial^2 q}{\partial\eta^2} + \frac{\partial^2 q}{\partial\theta^2} - 1\right). \tag{1.30}$$

The boundary conditions on q obey the same conditions as the metric functions. In this work q has the form

$$q = a_0\left(e^{-(g_+)^2} + e^{-(g_-)^2}\right)\sin^n\theta, \tag{1.31}$$

with

$$g_+ = \frac{\eta + \eta_0}{\sigma}, \quad and \quad g_- = \frac{\eta - \eta_0}{\sigma}, \tag{1.32}$$

and n is an even integer. The function q has three independent parameters a_0, η_0, and σ which specify its amplitude, range, and width. Note that it vanishes faster than $e^{-2\eta}$, as required and the angular function vanishes at, and is symmetrical through $\theta = 0$. The numerical solution to the Hamiltonian constraint is achieved by introducing a second-order finite differenced approximation to the partial derivatives and solving the resulting linear system. When $a_0 = 0$ the numerical solution for ψ is the Schwarzschild conformal factor and this provides a check on the accuracy of the elliptic equation solver.

1.6 The Numerical Algorithms

A number of different numerical techniques and code tests have been applied to the 2D oscillating black hole code and have been described extensively elsewhere (Bernstein, *et al.*, 1989, Bernstein, *et al.*, 1993b, Bernstein, 1992). This section will discuss only those algorithms that provided the

highest accuracy, the most stable evolution and the most efficient use of computational resources.

The discussion of numerical algorithms begins with the establishment of a spacetime grid of discrete points in both space and time. Spatially the grid covers the region $0 \leq \theta \leq \pi/2$ and $0 \leq \eta \leq \eta_{max}$. One of the advantages of using the η coordinate is that it allows one to use constant values of $\Delta\eta$ and $\Delta\theta$ and maintain grid zones that are as square as possible throughout the spatial domain. The coordinate η also provides fewer grid zones than would be needed for a linear radial coordinate in regions far from the black hole where linear perturbation theory is expected to provide an adequate description of the spacetime structure.

A disadvantage of the radial coordinate η becomes apparent in the region where the gravitational waves freely propagate on an almost flat background. If $\eta_{max} = 6.00$ with $\Delta\eta = 0.03$ (or 200 radial zones) then the maximum value of the Schwarzschild radius at η_{max} is $r_{max} = 202M$ and $\Delta r_{max} = 5.96M$ due to the logarithmic nature of the coordinate η. The wavelength of gravitational waves from a Schwarzschild black hole is expected to be $\lambda \sim 17M$ and one needs approximately 25 zones per wavelength to accurately propagate a wave. In such cases either a smaller value of η_{max} is required or a smaller value of $\Delta\eta$ (i.e. more grid zones) or both.

We will not discuss code tests here but will refer the interested reader to published references (Bernstein, *et al.*, 1989, Bernstein, *et al.*, 1993b, Bernstein, 1992, Abrahams, *et al.*, 1992) that describe them in detail.

1.6.1 Hyperbolic equations

The evolution equations for the Einstein equations in 3+1 form were discussed in detail in an earlier section. For brevity we will write them here in schematic form

$$\partial_t \gamma = -2\alpha K \tag{1.33}$$

and

$$\partial_t K = \alpha(K^2 + R) + \Delta\alpha. \tag{1.34}$$

The symbols γ, K and R are symbolic of the metric quantities γ_{ab}, K_{ab} and the Ricci tensor respectively. Finally, $\Delta\alpha$ will represent the covariant Hessian of the lapse.

The numerical grid has fixed step sizes in both temporal and spatial zoning. In the finite differenced form of the evolution equations to follow, the spatial steps will be labeled by the letter 'i', while the time steps will be labeled by 'n'. All partial derivatives will be approximated by second-order centered finite differences of the form: $\partial f/\partial\eta = (f^n_{i+1,k} - f^n_{i-1,k})/(2\Delta\eta)$ and $\partial^2 f/\partial\eta^2 = (f^n_{i+1,k} - 2f^n_{i,k} + f^n_{i-1,k})/\Delta\eta^2$.

The standard (space centered, time centered) leapfrog method is used to evolve the hyperbolic system of equations. This method has "momentum-like" quantities (in this case K_{ab}) offset from "field-like" quantities (here,

the 3-metric γ_{ab}) by 1/2 time step so that the time derivatives are properly centered. Thus, one can write the evolution system as

$$\gamma_i^{n+1/2} = \gamma_i^{n-1/2} - 2\alpha_i^n K_i^n \Delta t \tag{1.35}$$

and

$$K_i^{n+1} = K_i^n + \{\alpha_i^n[(K_i^n)^2 + R_i^{n+1/2}] - (\nabla\nabla\alpha)_i^n\}\Delta t. \tag{1.36}$$

However, this introduces first order errors due to the nonlinear term $(K_i^n)^2$. This problem can be cured by extrapolating the quantity K_i to the time slice labeled by $n + 1/2$ using the second order accurate formula

$$K_i^{n+1/2} = \frac{3}{2}K_i^n - \frac{1}{2}K_i^{n-1}. \tag{1.37}$$

A fully second order accurate method can then be written as

$$\gamma_i^{n+1/2} = \gamma_i^{n-1/2} - 2\alpha_i^n K_i^n \Delta t \tag{1.38}$$

and

$$K_i^{n+1} = K_i^n + \{\hat{\alpha}_i^{n+1/2}[(K_i^{n+1/2})^2 + R_i^{n+1/2}] - (\hat{\nabla\nabla}\alpha)_i^{n+1/2}\}\Delta t. \tag{1.39}$$

This method is not necessarily the most accurate method for evolving the Schwarzschild solution alone (Bernstein, *et al.*, 1989), but it does accurately propagate the gravitational waves with less dissipation and dispersion than other methods that have been tried (Abrahams, *et al.*, 1992.)

The numerical methods discussed above can be easily vectorized or split up amongst parallel tasks since there are no data dependencies. An updated time step depends only upon the values of the previous one or two time steps. This part of the simulation requires very little CPU time.

1.6.2 Elliptic equations

In order to start the evolution, one must first solve the constraint equations. In two or more spatial dimensions they are second order elliptic partial differential equations. In addition, two scalar functions (the lapse and potential for the shift vector) are determined by linear second order elliptic equations on each time step.

The general form applicable to all of the elliptic equations appearing in the code can be represented as:

$$a_{11}\frac{\partial^2 u}{\partial\eta^2} + a_{12}\frac{\partial^2 u}{\partial\eta\partial\theta} + a_{22}\frac{\partial^2 u}{\partial\theta^2} + a_{01}\frac{\partial u}{\partial\eta} + a_{02}\frac{\partial u}{\partial\theta} + a_{00}u = b(\eta,\theta)$$

for a scalar function $u(\eta,\theta)$. The coefficients a_{11}, a_{12}, etc. are functions of both η and θ and are determined by the three-metric and various spatial derivatives of the three-metric. Replacing the partial derivatives with

second order space centered finite differences over the entire spatial grid results in a matrix equation of the form

$$A \cdot u = b, \tag{1.40}$$

where A is a sparse square matrix whose entries are obtained from the coefficients associated with the elliptic differential operator, u is the vector whose solution is desired and the input vector b will contain the source terms and some terms resulting from the boundary conditions.

If the 2D grid has I and J zones in the two different coordinate directions then the length of the vectors is $N = I \cdot J$ and the coefficient matrix is $N \times N$ in size. In our calculations the sizes run from a minimum of $N = 2,626$ to a maximum of $N = 46,917$. Of course the matrix is not dense and most of the coefficients vanish identically. For a second order accurate, 2D discretization of a second order elliptic equation, with no mixed partial derivatives of the form $\partial^2/\partial x\,\partial y$, the matrix will be penta-diagonal. For our system the coefficient matrix is not constant since the values of the metric components and their partial derivatives change in both space and time.

A number of methods have been developed to solve such equations and this is a subject of ongoing research. Direct inversion methods such as Gaussian elimination require order N^3 operations and are clearly unsuitable for a simulation where two matrix equations need to be solved on the order of 10,000 times each during the evolution. The most recent advances have been made in pre-conditioned conjugate gradient methods and multi-grid methods. These are iterative techniques that approximate a solution to the matrix equation to within some specified tolerance and decrease the number of operations to $O(NlogN)$ or $O(N)$.

Essentially the conjugate gradient method minimizes a quadratic function of the form

$$f(u) = \frac{1}{2} u^T \cdot A \cdot u - u^T \cdot b$$

by calculating the gradient along a certain direction and then minimizing this quantity. The process can be sped up by "preconditioning" the matrix **A** to have a certain form before proceeding with the minimization process. Multi-grid methods speed the iterative process by solving the matrix equation with simple methods (Gauss-Seidel, Jacobi methods, etc.) on grids of differing resolution. Solutions found on course grids are then refined on the finest grids with less computational effort. Multi-grid methods require fewer operations in general and therefore provide the greatest theoretical speed-up but not necessarily the greatest hardware performance.

Many different methods have been implemented for this part of the code. A complete report is provided in (Bernstein, *et al.*, 1992b, Towns and Seidel, 1992) where various comparisons are made. The code provided with this article contains a matrix solver based on the BiCGstab algorithm (Van der Vorst, 1992) for non-symmetric matrices. It is possible to write the various

matrices appearing in our problem in a symmetric form, in which case the standard conjugate gradient method would apply. However the more general algorithm can be used on a broader class of problems, and works well here. Presently no preconditioner is used with BiCGstab. Convergence rates could probably be increased if an effective preconditioner were used.

The limiting factor in the performance of the black hole codes is determined primarily by the speed of the elliptic solver since the solution to the hyperbolic system can be constructed at nearly the peak speed of a vector processor. The computational demands of the elliptic solvers are large and about 90% of the CPU time is utilized in performing the matrix multiplications involved in the iterative procedure.

1.7 Tools for examining black hole spacetimes

The previous sections have been concerned with the formalism associated with numerical relativity and effective numerical methods for evolving black hole spacetimes. In the following sections we discuss a number of techniques used to extract physical and geometric information from the computed spacetimes. All of these techniques have been implemented in the code. Space limitations prevent a complete description of all the methods developed to analyze the numerical data but references to detailed discussions are provided.

1.8 Gravitational Waveforms

The main purpose of the numerical relativity code described above is to accurately predict the waveforms one might expect to observe with a gravitational wave detector. One of the major difficulties with numerically constructed spacetimes is the problem of how to extract information about gravitational radiation and separate the physical signals from gauge or coordinate effects. This problem can be overcome by noting that waves will be detected far from the sources in a region where the spacetime is nearly flat. A further more accurate assumption would be that if the spacetime is nonrotating then the detector is located in the far-field of a Schwarzschild-like source. In that case, one can formally extract waveforms using perturbative methods that are well established (Thorne, *et al.*, 1986, Nakamura, *et al.*, 1987, Futterman, *et al.*, 1988). In the following subsections we present several independent treatments of this problem.

1.8.1 Zerilli Function

In the region where the curvature is dominated by the mass content of a small compact region the spacetime metric can be split into a spherically symmetric (static) background and a dynamic perturbation

$$g_{\alpha\beta} = \overset{o}{g}_{\alpha\beta} + h_{\alpha\beta} \tag{1.41}$$

where the background metric, written in Schwarzschild coordinates, is

$$\overset{o}{g}_{\alpha\beta} = \begin{bmatrix} -(1-\frac{2M}{r}) & 0 & 0 & 0 \\ 0 & (1-\frac{2M}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{bmatrix} \tag{1.42}$$

and for axisymmetric, non-rotating, systems the metric perturbation tensor is

$$h_{\alpha\beta} = \begin{bmatrix} (1-\frac{2M}{r})H_0 Y_{\ell 0} & H_1 Y_{\ell 0} & h_0 \frac{\partial}{\partial\theta} Y_{\ell 0} & 0 \\ H_1 Y_{\ell 0} & (1-\frac{2M}{r})^{-1} H_2 Y_{\ell 0} & h_1 \frac{\partial}{\partial\theta} Y_{\ell 0} & 0 \\ h_0 \frac{\partial}{\partial\theta} Y_{\ell 0} & h_1 \frac{\partial}{\partial\theta} Y_{\ell 0} & r^2 K Y_{\ell 0} + r^2 G \frac{\partial^2}{\partial\theta^2} Y_{\ell 0} & 0 \\ 0 & 0 & 0 & r^2 \left(K sin^2\theta + G sin\theta cos\theta \frac{\partial}{\partial\theta} \right) Y_{\ell 0} \end{bmatrix} \tag{1.43}$$

where the Regge-Wheeler perturbation functions $(H_0, H_1, H_2, h_0, h_1, K, G)$ depend only on (r,t) and the angular dependence is expanded in terms of the $(m=0)$ spherical harmonics $Y_{\ell 0}(\theta)$.

The Regge-Wheeler perturbation functions can be extracted by making use of the orthogonality conditions for the spherical harmonics and then performing appropriate integrations of the metric over a sphere (Abrahams and Evans, 1988, Abrahams, *et al.*, 1992). This must be done numerically since the spacetime metric is simply an array of numbers. Following upon the work of Moncrief (Moncrief, 1974), a gauge invariant perturbation function Ψ can be created from the Regge-Wheeler functions. The function Ψ obeys a simple wave equation first derived by Zerilli (Zerilli, 1970):

$$\frac{\partial^2}{\partial t^2}\Psi - \frac{\partial^2}{\partial r_*^2}\Psi + V_Z(r)\Psi = 0 \tag{1.44}$$

where a new radial coordinate r_* known as the "tortoise coordinate " is defined by

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right). \tag{1.45}$$

The Zerilli potential V_Z is determined by the ℓ-mode of the wave and the mass of the black hole. For $\ell = 2$ (quadrupolar mode) the potential is

$$V_Z(r) = \left(1 - \frac{2M}{r}\right)\left(\left(1 + \frac{3M}{2r}\right)^{-2}\left(\frac{9M^3}{2r^5} - \frac{3M}{r^3}\left(1 - \frac{3M}{r}\right)\right) + \frac{6}{r^2\left(1 + \frac{3M}{2r}\right)}\right). \tag{1.46}$$

The asymptotic energy flux carried by gravitational waves can be computed from the Zerilli function by

$$\frac{dE}{dt} = \frac{1}{32\pi}\left(\frac{\partial\Psi}{\partial t}\right)^2 \tag{1.47}$$

for all ℓ modes. For a more precise description of the extraction method and the normalization procedure employed one can consult (Abrahams, *et al.*, 1992).

In the case where the Brill wave amplitude is small, the entire spacetime can be considered as a black hole with a perturbation. In that case the difference between the numerical values of the metric functions A, B, and D for the single static black hole spacetime and the black hole plus Brill wave spacetime are virtually indistinguishable. The extraction of Ψ however does make the distinction between the two spacetimes and this can be used to determine the accuracy with which the code calculates the generation and propagation of gravitational waves.

The code in its present form is capable of calculating both the $\ell = 2$ and the $\ell = 4$ waveforms. From the analytic perturbation theory, it is found that the waves generated by a perturbed black hole can be classified by their quasi-normal modes. These waves are essentially damped oscillations that radiate away the perturbations only to leave behind a static Schwarzschild black hole. The characteristic damping time and the frequency of the waves are determined by the value of ℓ and the mass, M of the black hole. A table of the complex frequencies associated with the lowest quasi-normal modes can be found in the review by Chandrasekhar (Chandrasekhar, 1983).

Setting "detectors" at positions of $r = 15M$, $30M$ and $60M$ the numerical values of Ψ are extracted for the evolution. The results have been compared to the analytic perturbative solutions and it has been demonstrated that the numerical code is capable of reproducing the known analytic results to within a few percent (Abrahams, *et al.*, 1992).

1.8.2 Newman-Penrose Scalars

Another approach that can be used to treat the problem of radiation extraction is due to Newman and Penrose (Newman and Penrose, 1962) who

introduced a formalism that allows one to derive a compact set of equations characterizing the ten independent components of the four dimensional Riemann tensor in vacuum by five complex scalars. These scalars are essentially projections of the Riemann tensor onto a null tetrad basis set. Two of these scalars

$$\Psi_0 = R_{\alpha\beta\gamma\delta} k^\alpha m^\beta k^\gamma m^\delta \tag{1.48}$$

$$\Psi_4 = R_{\alpha\beta\gamma\delta} l^\alpha \overline{m}^\beta k^\gamma \overline{m}^\delta \tag{1.49}$$

are of particular importance in describing gravitational waves. The vectors k and l are real vectors that obey certain orthogonality conditions and are constructed by adding and subtracting a spacelike unit vector with a unit timelike vector. The vector m and its complex conjugate $\overline{m}$ are orthogonal null vectors tangent to the surface of a 2-sphere representing the wavefront of an outgoing shell of radiation.

The basis set we have chosen to work with is the following (Bernstein, 1992)

$$k_\mu = \frac{1}{\sqrt{2}} \left[-\alpha + \psi^2 \sqrt{A} \beta^\eta, \psi^2 \sqrt{A}, 0, 0 \right], \tag{1.50}$$

$$l_\mu = \frac{1}{\sqrt{2}} \left[-\alpha - \psi^2 \sqrt{A} \beta^\eta, -\psi^2 \sqrt{A}, 0, 0 \right], \tag{1.51}$$

$$m_\mu = \frac{1}{\sqrt{2}} \left[\psi^2 \sqrt{B} \beta^\theta, 0, \psi^2 \sqrt{B}, i\psi^2 \sqrt{D \sin^2 \theta} \right], \tag{1.52}$$

$$\overline{m}_\mu = \frac{1}{\sqrt{2}} \left[\psi^2 \sqrt{B} \beta^\theta, 0, \psi^2 \sqrt{B}, -i\psi^2 \sqrt{D \sin^2 \theta} \right]. \tag{1.53}$$

In general, Ψ_0 and Ψ_4 are not gauge invariant. However, in the asymptotic far field regime they are invariant to highest perturbative order.

The scalar Ψ_4 (which falls off as r^{-1} in the wave zone) can be used to approximate the energy loss (Newman and Unti, 1962):

$$\frac{dE}{dt} = \frac{1}{4\pi} \oint \left[\int_0^t dt' \Psi_4 \right]^2 r^2 d\Omega, \tag{1.54}$$

where the integration is over a 2-sphere (with an area of $4\pi r^2$ and a surface element $d\Omega$) lying in a spacelike hypersurface surrounding the radiating system. Far from the source Ψ_4 represents an outwardly propagating wave and is therefore naturally normal to a 2-sphere of constant r.

1.8.3 Bel-Robinson Vector

Due to the formal similarities between the so-called Bel-Robinson vector and the Poynting vector of electrodynamics a number of researchers have been motivated to investigate some of its properties (Smarr, 1979). Extracting the "electric" ($E_{\alpha\beta} = n^\gamma R_{\gamma\alpha\beta\delta} n^\delta$) and "magnetic" ($B_{\alpha\beta} =$

$\frac{1}{2}\varepsilon_{\alpha}^{\gamma\mu}n^{\epsilon}\gamma_{\beta}^{\nu}R_{\epsilon\nu\gamma\mu}$) components of the four-dimensional Riemann tensor, the Bel-Robinson vector can be written in the form:

$$p^{\gamma} = E_{\alpha\beta}\epsilon^{\beta\gamma\delta}B_{\delta}^{\alpha}. \tag{1.55}$$

Here $\varepsilon_{\alpha}^{\gamma\mu} = \varepsilon_{\alpha\beta}^{\gamma\mu}n^{\beta}$ is the 3-dimensional Levi-Civita permutation tensor, and $\gamma_{\beta}^{\nu} = g_{\beta}^{\nu} + n_{\beta}n^{\nu}$ where n^{α} is as the unit vector normal to the spacelike $t = const.$ hypersurfaces.

The above expression is the analog of the electromagnetic $(\vec{E}\times\vec{B})$ vector. It is important to note that p^{γ} is not a true momentum vector for gravitational waves, since energy-momentum cannot be localized. It is not gauge invariant and this makes it difficult to interpret results in a quantitative way. Nevertheless, p^{γ} has proven to be effective in qualitatively tracking gravitational radiation.

Since p^{γ} is dimensionally a flux vector quantity, the radiative energy loss can be approximated by integrating the *radial* energy flux p^{r} over a closed 2-sphere as described above for Ψ_4

$$\frac{dE}{dt} = \frac{1}{4\pi}\oint\left[\int_0^t dt'\sqrt{\frac{|p^{\gamma}r_{\gamma}|}{2}}sign(\Psi_4)\right]^2 r^2 d\Omega, \tag{1.56}$$

where $sign(\Psi_4)$ is $+1$ if $\Psi_4 \geq 0$ and -1 if $\Psi_4 < 0$. This construction yields results that are consistent with the Newman-Penrose and Zerilli formulations when the 2-sphere is placed in the asymptotic far field spacetime $(r \to \infty)$.

1.8.4 Riemann Invariants

The Riemann tensor may be used to construct spacetime scalars that are invariant with respect to coordinate transformations. In the general vacuum spacetime, the number of independent invariant scalars that can be constructed is four. They are generally represented by two complex variables

$$I = 4(E_{\alpha\beta}E^{\alpha\beta} - B_{\alpha\beta}B^{\alpha\beta}) + 8iE_{\alpha\beta}B^{\alpha\beta} \tag{1.57}$$

and

$$J = 8(E_{\alpha\beta}E_{\gamma}^{\beta}E^{\gamma\alpha} - 3B_{\alpha\beta}E_{\gamma}^{\beta}B^{\gamma\alpha}) - 8i(B_{\alpha\beta}B_{\gamma}^{\beta}B^{\gamma\alpha} - 3E_{\alpha\beta}B_{\gamma}^{\beta}E^{\gamma\alpha}). \tag{1.58}$$

For our class of spacetimes, the imaginary parts of I and J vanish and we are left with two fully gauge invariant quantities that can be used to describe real or "physical" geometric behavior.

Since the outgoing null component of the Riemann tensor (basically Ψ_4) falls off as r^{-1} and I is quadratic in the Riemann tensor, one can expect to observe gravitational waves at large distances from the source by looking at, for example, $r^2 I$ on spacelike hypersurfaces. Such constructions would be unambiguously gauge invariant even in fully nonlinear near field regions of spacetime where the notion of gravitational waves is not clearly defined.

1.9 Apparent Horizons

Having constructed at least part of the spacetime of interest, one may determine the location of the boundary that separates the interior from the exterior of the black hole. This boundary does not coincide with the boundaries used in the numerical construction of the spacetime and must be found from a knowledge of its properties.

The event horizon of a black hole is defined as the last 2-dimensional surface from which light will not reach infinity along lightlike directions. This boundary is defined globally. In order to determine its position all light rays must be followed for a long enough time to determine whether or not they escape to infinity. Since the spacetime is constructed spatial slice by spatial slice, the entire spacetime is not available to determine the position of the event horizon.

Locally one can define a marginally trapped surface as the surface at which the expansion of the outgoing null geodesics vanishes. This means that at such a surface the light rays are trapped to such an extent that they can never diverge from each other. Being a more extreme condition it can be shown that the outermost marginally trapped surface (called the apparent horizon) lies inside the event horizon. In the Schwarzschild case the apparent horizon coincides with the event horizon.

The apparent horizon is found by locating the points where a spacelike outward pointing unit vector, s_i normal to the surface obeys the following equation (York, 1989)

$$\nabla_i s^i - K + K_{ij}\, s^i s^j = 0. \tag{1.59}$$

Since the normals to the apparent horizon are surface forming, one may introduce a scalar function, $\Sigma(\eta, \theta)$ whose gradient defines the unit normals.

$$s_i = \frac{1}{s}\nabla_i \Sigma \qquad s = [(\nabla^i \Sigma)(\nabla_i \Sigma)]^{1/2}. \tag{1.60}$$

The level surface $\Sigma = \Sigma_0$ defines the apparent horizon. The apparent horizon is a two-dimensional surface but the axial symmetry condition allows one to define the position of the apparent horizon solely in terms of the value of the radius vector $\Sigma = \eta - h(\theta)$ that locates the level surfaces. If the apparent horizon is defined by the surface $\Sigma_0 = 0$, then $\eta = h(\theta)$ and the differential equation for the unit vectors s^i becomes a nonlinear second-order equation for h. Details of the equations and the numerical techniques (basically Newton's method) for solving them can be found in (Cook and York, 1990, Bernstein, 1992).

Once the apparent horizon is located, its two dimensional geometry can be visualized by removing it from the curved three dimensional space and embedding it into a flat three dimensional space. We accomplish this by identifying the general 3D line element (1.12) (with $C = 0$) to the flat space

metric

$$ds^2 = dz^2 + d\rho^2 + \rho^2 d\phi^2, \tag{1.61}$$

and constrain the embedding to be axisymmetric. This procedure gives for the variables ρ and z

$$\rho = \psi^2 \sqrt{D} \sin\theta, \tag{1.62}$$

$$\frac{dz}{d\theta} = \pm \left[\psi^4 \left(B + A \left(\frac{\partial \eta}{\partial \theta} \right)^2 \right) - \left(\psi^2 \sqrt{D} \cos\theta + \frac{\left(\psi^4 D\right)_{,\theta} \sin\theta}{2\psi^2 \sqrt{D}} \right)^2 \right]^{1/2}, \tag{1.63}$$

where the θ derivatives are evaluated along the curve that is the apparent horizon. The embedding is viewed by plotting $z(\rho)$ at specific values of θ.

1.10 Mass Measures

Probably the most important quantities associated with asymptotically flat radiative spacetimes are those that measure the "energy content" of the spacetime at different locations and at different times. While it is impossible to localize energy in general relativity, there are certain quantities that provide some information about the total energy content of the spacetime and how much energy is radiated away by an isolated system. This section discusses those quantities, their inter-relations, and their applications to numerical relativity.

1.10.1 ADM Mass

One important measure of the total energy content of the spacetime is the so-called ADM mass. This quantity is defined on a $t = const$ hypersurface and is the mass associated with that hypersurface as measured at spatial infinity ($r \to \infty$). This quantity is determined from the surface integral

$$M_{ADM} = \frac{1}{16\pi} \oint_S (\partial_j g_{ij} - \partial_i g_{jj}) dS^i \tag{1.64}$$

where the metric is evaluated in Cartesian coordinates (Arnowitt, *et al.*, 1962). Further work by Brill (Brill, 1959) and Eppley (Eppley, 1979) has shown that on the initial hypersurface at $t = 0$ this reduces to

$$M_{ADM} = -\frac{1}{2\pi} \oint_S (\nabla_a \psi) dS^a \tag{1.65}$$

when the metric is given in the form Eq. 1.12 and the function q asymptotically falls off as at least as fast as r^{-2} in order that the integral remains bounded when the surface S is located at spatial infinity.

This mass measure is inclusive. All of the mass associated with compact objects located deep within the spacetime, any binding energies between constituents, the energy associated with gravitational radiation, etc., contribute to the ADM mass. Since this quantity is measured at spatial infinity, the ADM mass is constant during the evolution. Any radiation in the spacetime will only reach spatial infinity in an infinite time interval.

In a numerical evolution, the outer boundary is not located at spatial infinity but at some large distance from the compact source. Therefore the ADM mass cannot be calculated from the formal definition but must be approximated by a surface integral taken at the maximum radius allowed by the numerical grid. As long as the gravitational waves do not propagate beyond the outer boundary, this "quasi-local ADM mass" should remain constant and provide a measure of the *total* mass of the spacetime.

1.10.2 Black Hole Mass Measures

In the absence of any other object or source of energy, a single Schwarzschild black hole has a uniquely defined mass, namely the parameter M that appears in the metric. This is consistent with the ADM mass definition as well and can be found by applying Eq. 1.65 to our metric 1.12 in the absence of any perturbations.

When the spacetime includes other sources of energy other than the black hole mass, methods of isolating the black hole contribution to the total (or ADM) mass must be introduced. The area of a Schwarzschild black hole's event horizon A_{eh} is given by the formula (Hawking and Ellis, 1973)

$$A_{eh} = 16\pi M^2 \tag{1.66}$$

where the value of M is the mass of the black hole. In general the apparent horizon lies just inside of the event horizon so that if one calculates the area of the apparent horizon that value should provide a lower limit on the mass of the black hole. Henceforth the black hole mass will refer to the expression

$$M_{BH} = \frac{1}{4}\left(\frac{A_{ah}}{\pi}\right)^{1/2} \tag{1.67}$$

where A_{ah} represents the area of the apparent horizon found by the methods discussed in the previous section.

The conservation of energy requires that the energy loss due to gravitational radiation should represent the difference between the total energy of the system and the mass of the black hole left behind:

$$M_{ADM} = M_{BH} + M_{rad} \tag{1.68}$$

Where M_{rad} represents the time integrated value of the energy loss rate determined from one of Eq. 1.47, Eq. 1.54 or Eq. 1.56 which should all be equivalent. This provides yet another test of the accuracy of the numerical code.

1.11 Code Description

We conclude this report with a brief description of the code and instructions on how to run it. The code is basically a collection of subroutines (written in FORTRAN-77) that calculate the initial data, lapse function, shift vectors, Zerilli function, Newman-Penrose scalars, Riemann invariants, Bel-Robinson vectors, apparent horizons (location, mass and embedding) and radiative losses for a single axisymmetric black hole spacetime distorted by the presence of an initially time symmetric gravitational wave. These routines are driven by a main program that performs the actual time integration of the metric and momentum variables in addition to controlling all I/O operations.

The driver acts upon the directives defined in the parameter file called `param.h` which supplies the code with all the information it requires (time step size, spatial grid size, Brill wave parameters, printing options/intervals and switches to compute various physical quantities). In this way, the user does not need to interact with or alter the source code in any way. One can run different problems by simply changing the appropriate parameter values in the file `param.h` which contains extensive comments documenting parameter choices and their meaning. Then the code must be recompiled and run. The parameter values included on the diskette are set for a moderately distorted black hole, corresponding to "run 1e" in (Abrahams, *et al.*, 1992). (Note: as documented in the subroutine `zerilli`, the normalization and time scale of the gauge-invariant waveform have been rescaled by the ADM Mass. This must be accounted for when comparing to those published results.) A computer animation of data provided by this run has also been produced, and can be obtained in video format from NCSA by sending an electronic mail request to the Internet address media@ncsa.uiuc.edu or by writing to Documentation Services, National Center for Supercomputing Applications, 605 E. Springfield Avenue, Champaign, IL 61820 USA.

The flow of the program execution is outlined below. The computational (e.g. grid) and physical (e.g. Brill wave amplitude) parameters contained in the `param.h` file are read in at compile time. Upon execution the subroutine `initdat` is called. This routine initializes the 3-metric γ_{ab} and extrinsic curvature K_{ab} and solves the hamiltonian constraint for the conformal factor ψ. Then the initial data are evolved `nmax` steps into the future using the extrapolated leapfrog algorithm described above. At prechosen time intervals determined by parameters in the `param.h` file, various quantities described in section 1.7 are computed and output for analysis. The structure of the code is as follows:

```
INPUT param.h
SET t=0
CALL initdat
     INITIALIZE γ_ab and K_ab based on Brill wave parameters
```

```
    CALL conform
        SOLVE hamiltonian constraint for ψ
FOR n = 1 to nmax
    COMPUTE γ_ab on time step t^{n-1/2}
    COMPUTE K_ab on time step t^n
    CALL maximal
        COMPUTE α
    CALL shiftv
        COMPUTE β^a
    SET t=(n)*Δ t
    IF desired
        OUTPUT metric and momentum data
    ENDIF
    IF desired
        COMPUTE radiation and/or horizon data
        OUTPUT radiation and/or horizon data
    ENDIF
CONTINUE
COMPUTE total energy radiated
STOP
```

The version of the code provided here outputs data files in the NCSA HDF format (hierarchical data format). Therefore, if the code is to be run in its present form, it must be linked to an HDF library. If HDF libraries are not available on your computer, it is a simple matter to modify the source to provide output files in the format required for local graphics programs.

To compile and link the code, choose parameters in the `param.h` file and execute the commands in the provided makefile which is written to support the Cray, Convex, RS/6000, and Sun systems. The makefile commands will create two subdirectories; *exe* and *obj* containing the executable and object files respectively. To run the code, it is important to first delete any unwanted existing HDF files from the directory *exe* (the output procedures we use will append data to existing files) and then execute the binary file called `onebh` in the *exe* directory. The source code (with the exception of requiring HDF libraries) is completely self-contained and is written such that it should be compatible with any standard FORTRAN-77 compiler and therefore should be completely portable.

A graphical user interface is under development that will allow complete control of input parameters, code execution, visualization, and symbolic analysis using MACSYMA or Mathematica. This "relativity workbench" will operate under X-Windows and should make numerical relativity accessible to a variety of researchers. This software will be made available when it is ready.

Finally, we have provided the source code for MACSYMA programs used to derive and code the equations. The file `adm.1BH` can be loaded into MAC-

SYMA by issuing the command `batch(''adm.1BH'');` from within MACSYMA, and the equations will be generated automatically. Note that this file, which is customized to treat the distorted black hole system described in this article, will call another file called `adm.mac` that contains general expressions for the ADM form of the Einstein equations. Users familiar with MACSYMA will be able to modify this code for their own purposes.

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