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# Simplicial Quantum Gravity and Random Lattices

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## 1. Introduction

The purpose of these lectures is to present a short review of some recent developments of the lattice approach to the problem of the quantization of gravity. This approach goes back to the seminal paper by Regge (Regge (1961)), but many new and sometimes surprising results have been obtained during the last ten years, thanks both to the progresses of computers and of numerical techniques, and to new theoretical ideas, developed in particular for two dimensional ($2d$) theories of gravity and for string theories. These developments came also from a convergence of different fields of theoretical physics: quantum gravity, string and superstring theories, but also lattice gauge theories and statistical mechanics, in particular random surface models.

The construction of a theory of gravity which encompasses both General Relativity (GR), or some of its extensions, and the principles of Quantum Mechanics is probably one of the most difficult and exciting problems of theoretical physics. It is well-known that there are at least two basic difficulties. Firstly, the coupling constant for GR (Newton's constant) has the dimension of the inverse of a mass squared, and leads to a non-renormalizable quantum theory, with violations of unitarity at Planckian energies. Secondly, the Einstein equations (the evolution equations for the metric) are derived from an action principle, but the Einstein-Hilbert action is unbounded from below, and a naive path integral formulation of a quantum theory is expected to be mathematically ill defined. Other conceptual problems, such as the quantum effects at singularities, the problem of the vanishing of the cosmological constant, the meaning of measurements, are also far from being properly understood. Apart from the problem of the construction of a realistic model which could make contact with the low energy standard model and with cosmological observations, superstring theories are believed to solve the first problem, and to provide a consistent unitary quantum theory which contains gravity, at least in string perturbation theory. However, non-perturbative issues, such as the very existence of a vacuum state, the emergence of a realistic cosmology, the fate of black holes, etc., are still very open in these theories.

In the lattice formulations of quantum gravity, one tries to construct a discretization procedure which makes the functional integration over metrics (or over the other degrees of freedom which describe classical gravity) meaningful, by truncating this infinite dimensional integral into a sum over a finite number of variables. Then one should look for critical values of the parameters (the coupling constants) of the discretized theory where a scaling behavior occurs, so that the details of the discretization become irrelevant and a continuum theory can be constructed. This lattice approach has been much successful for studying ordinary relativistic quantum field theories, such as non-abelian gauge theories, and in some cases to construct them rigorously. When trying to apply these ideas to General Relativity one should keep in mind that one is making two basic assumptions:

▷ One hopes that a quantum theory can be obtained from a path integral formulation based on the Lagrangian formulation of General Relativity, namely from the Einstein-Hilbert action, which is a functional of the metric tensor $g$, taken as the dynamical variable. There are some lattice formulations based on the first order Palatini formalism of GR, where the dynamical variables are the metric and the connection (or more adequately the local frame and the spin connection). I do not feel qualified enough to discuss these aspects here. I refer in particular to the lectures by A. Ashtekar for a thorough presentation of the Hamiltonian formulation of General Relativity and its canonical quantization.

▷ One assumes that it is possible to recover Lorentzian gravity (where the metric has signature $(-+++)$) from Euclidean gravity (where the metric has signature $(++++)$), by some kind of Wick rotation (this is the essence of the proposal by Hartle and Hawking (1983) for the wave function of the universe).

Therefore, we are led to consider the functional integral over Euclidean metrics

$$\int \mathcal{D}[g]\ \mathrm{e}^{-S_{\mathrm{E}}[g]} \tag{1.1.1}$$

with the Euclidean Einstein-Hilbert action

$$S_{\mathrm{E}}[g] \;=\; \frac{1}{16\pi G_{\mathrm{N}}}\int \mathrm{d}^D x\,\sqrt{|g|}\,(-R+2\Lambda) \tag{1.1.2}$$

($D$ is the dimension of space-time, $G_{\mathrm{N}}$ Newton's constant and $\Lambda$ the cosmological constant. I have set $\hbar = c = 1$).

One should also keep in mind that some very important problems have to be solved, and are not cured automatically by a lattice discretization.

▷ Reparametrization invariance is the basic symmetry of GR. What does this symmetry become in a discretized theory, where space-time is not continuous? If it is broken by the discretization, how to ensure that it is restored (at least at large distances) in a continuum limit?

▷ Are topology changing quantum fluctuations important, and how to deal with such fluctuations in the functional integral?

▷ What about the unboundness of the Einstein action?

▷ There is an important difference between a discretization procedure, which consists in making the number of dynamical variables finite or at least discrete, and a regularization procedure, which consists in introducing a fundamental energy scale (a regulator) above which quantum fluctuations are suppressed, in order to make the theory finite. Some regularizations (for instance higher derivatives theories) are not discretizations, on the other hand some discretizations cannot be considered as true regularizations, if they do not incorporate a *non-zero minimal* short distance cut-off.

## 2. Piecewise flat manifolds, Regge calculus and dynamical triangulations

### *2.1. Simplicial manifolds and piecewise linear spaces*

Let me start with some geometry. I refer to Spivak (1970), Choquet-Bruhat et al. (1982) and to Dubrovin et al. (1984) for background in differential and Riemannian geometry. A simple and natural way to discretize space is to use lattices which are build out of *simplices*. I recall that a (oriented) $d$-dimensional *simplex* is made out of $d+1$ points $p_1, \dots p_{d+1}$, the vertices, modulo (positive signature) permutations. For instance, a 0-simplex is a point, a 1-simplex a pair of points, i.e. a line or an edge, a 2-simplex a triangle, or a face, a 3-simplex a tetrahedron, etc... A subsimplex of a given simplex is simply a subset of points of this simplex.

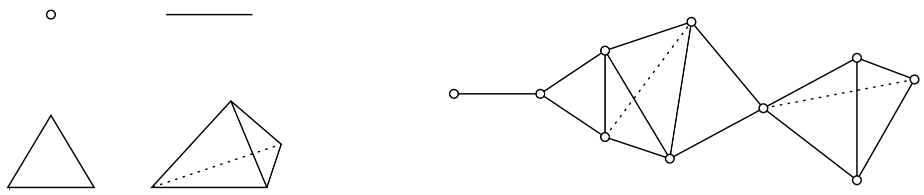


Fig. 1. 0–3 dimensional simplices, simplicial complex

Out of a collection of simplices one can construct a *simplicial complex*, by gluing simplices along some of their subsimplices. Simplicial complexes may seem quite trivial objects. However one can already define exterior calculus on them. A $p$-form $f$ is simply a function which associates to each $p$-simplex $\sigma$ some number $f(\sigma)$ (or more generally some element of an Abelian group), and such that, if $-\sigma$ is the orientation-reversed simplex, $f(-\sigma) = -f(\sigma)$. One can define exterior product of forms, and an exterior derivation operation d, which maps $p$-forms into $(p+1)$-forms, by

$$\mathrm{d}f((p+1)\text{-simplex}) \;=\; \sum_{\substack{p-\text{subsimplices}\\ \text{of the simplex}}} f(p\text{-simplex}) \tag{2.1.1}$$

with the correct orientation. For instance

$$\begin{aligned} \mathrm{d}f(p_1,p_2) &= f(p_2) - f(p_1) \\ \mathrm{d}f(p_1,p_2,p_3) &= f(p_1,p_2) + f(p_2,p_3) + f(p_3,p_1)\ . \end{aligned} \tag{2.1.2}$$

d satisfies the usual rule $\mathrm{d}^2 = 0$. Therefore one can define cohomology theory on a simplicial complex and apply Stokes theorem

$$\int_D \mathrm{d}f \;=\; \int_{\partial D} f \qquad . \tag{2.1.3}$$

From a physicist point of view, this means that one can define scalar fields (0-forms), gauge fields (1-forms), antisymmetric tensors, etc... on any simplicial complex (see for instance Sorkin (1975)). However, a simplicial complex does not look like a manifold with definite dimension, since it may be constructed from simplices with different dimensions.

A $d$-dimensional *simplicial manifold* (SM) is a simplicial complex such that the neighborhood of each point, i.e. the set of simplices which contain this point as a vertex, is homeomorphic to the $d$-dimensional ball $B_d$. We shall often denote SM as triangulations, even when they are not two dimensional.

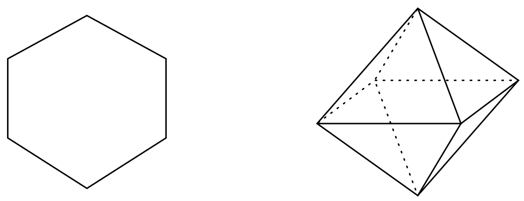


Fig. 2. One and two dimensional simplicial manifolds

A $d$-dimensional SM can be obtained by successively gluing pairs of $d$-simplices along some of their $(d-1)$-faces, until one gets a complex with no boundary. However, when performing such an operation, one does not obtain in general a SM, but only a so-called *pseudomanifold* (PM). Indeed, the complex constructed in that way are such that some vertices have neighborhood which are *not homeomorphic to* $B_d$. An example of 3-dimensional PM is depicted on Fig. 3.

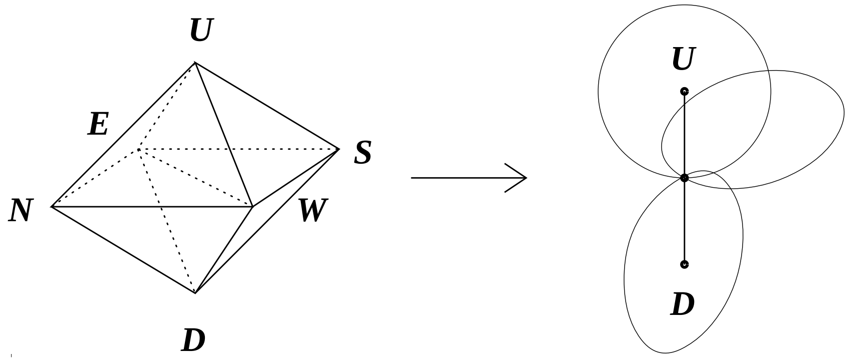



Fig. 3. An example of three dimensional pseudomanifold

It consists of 4 tetraedra, 8 triangles, 5 edges and 3 vertices, and is obtained from the "double pyramid" by identifying the opposite faces of each pyramid, i.e. $UNW \leftrightarrow UES$, $UWS \leftrightarrow UNE$, $DNW \leftrightarrow DES$ and $DWS \leftrightarrow DNE$, so that the "base" $NESW$ becomes a 2-torus. The boundary of the neighborhood of each summit (U or D) is made of the two base triangles, and is homeomorphic to the 2-torus $T_2$, not to the 2-sphere $S_2 = \partial B_3$ ! However, all one or two dimensional complexes obtained by gluing lines at end-points or triangles along edges are indeed SM. This fact makes discretized models for two dimensional gravity much easier to handle.

The simplicial manifold structure corresponds to a discretization of ordinary manifolds, but there are no notions of distance between points, nor of volume element, which would allow to define angles or scalar products between vector fields, as well as to integrate functions over the manifold. One can embody a SM with a Riemannian metric by specifying that:

▹ The metric is flat inside each d-simplex (i.e. the curvature vanishes).
▹ The metric is continuous when one crosses the faces ($(d-1)$-simplices).
▹ Each face of a simplex is a linear flat subspace of this simplex (i.e. its extrinsic curvature also vanishes).

One thus obtains a *piecewise linear space* (PL). The metric on a PL space is entirely specified by the distance elements between nearest neighbor vertices, i.e. by the squared length $l_{ij}^2$ of every edge $(i,j)$, of the corresponding SM. For instance, if we consider a triangle with vertices $(1,2,3)$ and with edges length $l_{12}$, $l_{23}$, $l_{31}$, in the coordinate system where the metric inside the triangle is constant and the coordinates of the vertices are respectively $(0,0)$, $(0,1)$ and $(1,0)$, the metric is

$$g_{\mu\nu} = \begin{pmatrix} l_{12}^2 & (l_{12}^2 + l_{31}^2 - l_{23}^2)/2 \\ (l_{12}^2 + l_{31}^2 - l_{23}^2)/2 & l_{31}^2 \end{pmatrix} \quad . \tag{2.1.4}$$

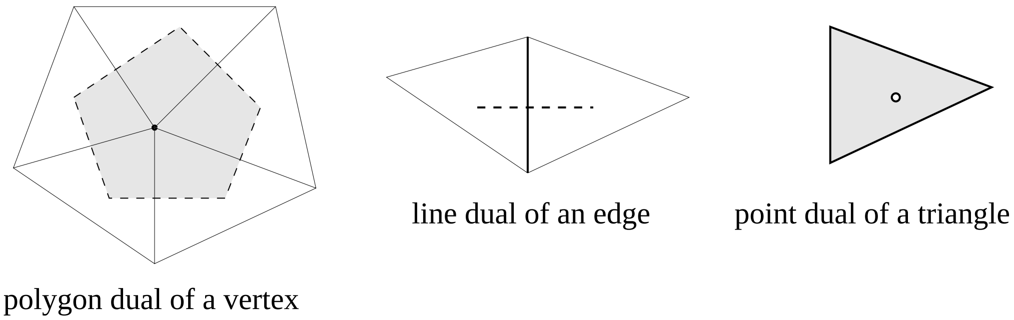
The number of edges of a simplex, $d(d+1)/2$, equals the number of independent components of the metric tensor. The constraint that the metric must be Riemannian means that the metric tensor $g_{\mu\nu}$ is a positive definite matrix. It implies "generalized triangle inequalities" between the $l_{ij}^2$. More generally if one wants a metric with signature $(p,q)$, some $l_{ij}^2$ may be negative so that $g_{\mu\nu}$ inside each simplex has $p$ positive and $q$ negative eigenvalues.

Finally let me recall that, while the metric is continuous when crossing the faces, it is discontinuous across the lower dimensional simplices. In particular, the curvature is concentrated along the $(d-2)$-dimensional "hinges" (Regge (1961), Misner et al. (1973)).

### *2.2. The dual complex and volume elements*

On a PL space we can construct the so-called dual complex, which is a natural generalization of the Voronoï construction of the dual lattice for a random lattice in Euclidean space (see in particular Christ et al. (1982) and Friedberg et al. (1984)). An intuitive (but not fully rigorous) definition is as follows: To any given $p$-dimensional subsimplex $\sigma$ we associate the flat $(d-p)$-dimensional dual *polytope* $\sigma^\star$ that consists of all the points which are at equal distances of all the vertices of $\sigma$, but closer to these vertices than to the other vertices of the complex (in fact this definition has to be slightly extended to take into account "obtuse-angled" simplices, which give non-convex polytopes).

Examples in two dimensions are depicted on Fig. 4. The dual of a triangle is a point, the dual of an edge is a segment, and the dual of a vertex is a polygon. The dual polytopes of the simplicial complex $\mathcal{C}$ form a polytopial (but not simplicial) complex, $\mathcal{C}^\star$. Let $V(\sigma)$ be the $p$-dimensional volume (in the PL metric) of the simplex $\sigma$,



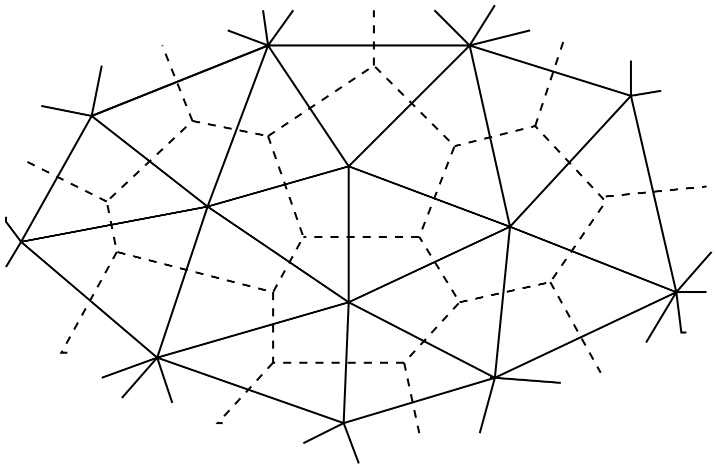



Fig. 4. simplicial manifold (filled) and the dual complex (dashed)

and $V(\sigma^\star)$ the algebraic $(d-p)$-dimensional volume of the dual polytope $\sigma^\star$. These volumes are function of the metric, i.e. of the edges lengths. The $d$-dimensional volume element associated both to $\sigma$ and to its dual $\sigma^\star$) is simply

$$A(\sigma) \;=\; A(\sigma^\star) \;=\; \frac{p!(d-p)!}{d!}\, V(\sigma)V(\sigma^\star) \qquad . \tag{2.2.1}$$

It is the volume of the set of points closer to $\sigma$ than to the others simplices of $\mathcal{C}$. These volume elements allow to define a global inner-product over $p$-forms (functions over $p$-simplices) by

$$[a,b] \;=\; \frac{p!(d-p)!}{d!} \sum_{\substack{p-\text{dimensional}\\ \text{simplices }\sigma}} \frac{V(\sigma^\star)}{V(\sigma)}\, a(\sigma)b(\sigma) \qquad , \tag{2.2.2}$$

and a $^\star$ operation (Poincaré duality) which maps $p$-forms $a$ on $\mathcal{C}$ onto $(d-p)$-forms $^\star a$ on the dual complex $\mathcal{C}^\star$ (i.e. functions over $(d-p)$-dimensional polytopes) by

$$^\star a(\sigma^\star) \;=\; \frac{V(\sigma^\star)}{V(\sigma)}\, a(\sigma) \qquad . \tag{2.2.3}$$

This allows to define in full generality actions for fields on a PL space. For instance, the massless free field action for a scalar field $\Phi$ (a 0-form) can be written as $[\mathrm{d}\Phi,\, \mathrm{d}\Phi]$ and becomes

$$\int \mathrm{d}x\,\sqrt{g}\,g^{\mu\nu}\,\partial_\mu\Phi\partial_\nu\Phi \quad\to\quad \sum_{\substack{\text{edges}\\ ij}} \frac{V(ij^\star)}{V(ij)}\,(\Phi(i)-\Phi(j))^2 \qquad . \tag{2.2.4}$$

It is important however to notice that the $^\star$ operation is not a duality, like in the smooth case, since the dual complex $\mathcal{C}^\star$ differs from $\mathcal{C}$.

*2.3. Spin connection, spinors*

In fact it is more natural and convenient to define matter fields on the dual complex than on the initial simplicial complex (see in particular Ren (1988)). This is needed for spinors fields, which require the introduction of local frames $\vec{e}_a = (e^\mu_a)$ which live on the tangent space, since there are no well-defined tangent and cotangent planes at the vertices of $\mathcal{C}$, and more generally at any point of the $(d-2)$-dimensional hinges. One can associate an orthonormal local frame $\vec{e}_a(\boldsymbol{i})$ $(a = 1, \ldots d)$ to each vertex $\boldsymbol{i}$ dual to a $d$-simplex, such that

$$\vec{e}_a(\boldsymbol{i})\vec{e}_b(\boldsymbol{i}) \;=\; e^\mu_a(\boldsymbol{i})e^\nu_b(\boldsymbol{i})g_{\mu\nu}(\boldsymbol{i}) \;=\; \delta_{ab} \qquad . \tag{2.3.1}$$

One can parallel transport these frames across the $(d-1)$-dimensional faces by simply translating them along the dual links. This allows to define the spin connection $\omega$ as the 2-form which to a dual link $(\boldsymbol{ij})$ associates the rotation matrix $\omega^a{}_b(\boldsymbol{ij}) = \vec{e}_a(\boldsymbol{i})\vec{e}_b(\boldsymbol{j})$ Then one can define covariant derivatives for spinor fields $\Psi^a(\boldsymbol{i})$ living on the sites of the dual complex, couple spinors to gauge fields living on dual links $(\boldsymbol{ij})$, etc...

*2.4. Curvature and the Regge action (classical Regge calculus)*

The curvature can be defined easily by performing parallel transport along a loop $\ell$ of the dual lattice which encircles a given $(d-2)$-dimensional hinge $h$. Taking the ordered product of the rotations $\omega(\boldsymbol{ij})$ we obtain a rotation matrix $\omega(\ell)$ which corresponds only to a rotation in the plane orthogonal to the hinge $h$, and thus is characterized only by the deficit angle $\delta_h$, which can be obtained as

$$\delta_h \;=\; 2\pi - \sum_\sigma \theta_{h,\sigma} \qquad , \tag{2.4.1}$$

where the sum runs over all the $d$-simplices meeting along $h$, and $\theta_{h,\sigma}$ is the dihedral angle of $\sigma$ along $h$. An explicit expression for $\theta_{h,\sigma}$ involves the volumes of $\sigma$, of the two $(d-1)$-faces of $\sigma$ which share $h$, and of $h$, that we denote respectively $V_d$, $V_{d-1}$ and $V'_{d-1}$, and $V_{d-2}$. It reads

$$\sin(\theta_{h,\sigma}) \;=\; \frac{d}{d-1}\,\frac{V_d V_{d-2}}{V_{d-1}V'_{d-1}} \qquad . \tag{2.4.2}$$

The Regge discretization of the Einstein action is (Regge (1961))

$$S_{\mathrm{E}} \;=\; \int \mathrm{d}^d x \,\sqrt{g}\, R \;\rightarrow\; S_{\mathrm{Regge}} \;=\; \sum_{\text{hinges } h} V(h)\,\delta_h \qquad , \tag{2.4.3}$$

and is, through the volumes of the simplices, faces and hinges, a function of the edge length variables $l_{ij}$. When varying the Regge action with respects to one edge length $l_e$, one obtains the Regge form of the Einstein equations (see Misner et al. (1973) for details)

$$\sum_{\substack{\text{hinges } h\\ \text{opposite to } e}} \delta_h \,\mathrm{cotan}(\theta_h) \;=\; 0 \qquad . \tag{2.4.4}$$

The cosmological constant term in the action is simply

$$\int \mathrm{d}^d x \,\sqrt{g} \;\rightarrow\; \sum_{\substack{d-\text{simplices}\\ \sigma}} V(\sigma) \qquad . \tag{2.4.5}$$

This discretized formulation of General Relativity, usually denoted Regge Calculus, is both elegant and very convenient for the study of various aspects of classical General Relativity. Indeed, when one approximates a smooth Riemannian space by a sequence of PL spaces obtained, for instance, by taking finer and finer triangulations of this space, with edge lengths given by the geodesic distance between the vertices, the sequences of metrics, curvature distribution, etc..., can be shown to converge, in a suitable sense, toward the smooth continuum ones. I refer to Christ et al. (1984), Cheeger et al. (1982) and (1984) and to Feinberg et al. (1984) for details on this classical continuum limit. Therefore, Regge calculus is an adequate framework to discuss classical gravitation (construction of classical solutions, discussion of gauge fixing). Small deformations of the edges length allow to deal with quantum fluctuations around classical solutions and to consider some semiclassical issues of quantum gravity. I refer to the recent and very complete review by Williams and Tuckey (1992) for references on the foundations of Regge calculus, recent advances and applications.

### *2.5. Topological invariants*

Before addressing the problems of quantization, let me discuss shortly the issue of topological invariants for PL spaces. It is easy to write the Euler characteristic $\chi$, which is a non-trivial topological invariant for even-dimensional spaces, as

$$\chi \;=\; \sum_{p=0}^{d} (-1)^p \; N_p \quad ; \quad N_i = \text{number of } p\text{-simplices} \qquad . \tag{2.5.1}$$

However, it is in general difficult to write simply other topological invariants, even when they can be written in terms of differential forms. For instance the Hirzebruch signature $\tau$ for 4-dimensional manifolds is given by

$$\tau \;=\; \frac{1}{96\pi^2}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{g}\epsilon^{\mu\nu\rho\sigma}R_{ab\mu\nu}R^{ab}{}_{\rho\sigma} \qquad , \tag{2.5.2}$$

but for a PL space this involves products of $\delta^{(2)}$ distributions with support on the hinges. There is no simple regularization of these products which respects the topological invariance of $\tau$. I refer to Cheeger et al. (1984) for a general and detailed discussion of the definition and properties of curvature and of the simplicial approximation of manifolds by PL manifolds.

### *2.6. Simplicial Euclidean quantum gravity (quantum Regge calculus)*

Can one use such discretization procedures to define the functional integral (1.1.1) over Euclidean metrics and to construct a quantum theory of Euclidean gravity? We have seen that it is possible to discretize the action. It is a more subtle issue to discretize the measure $\mathcal{D}_{\mathcal{M}}[g]$ over the so-called "superspace" of all Riemannian metrics over some manifold $\mathcal{M}$ . In fact, since the local symmetry of GR is reparametrization invariance, one has to gauge away the diffeomorphism group $\mathrm{Diff}_{\mathcal{M}}$, and to sum over orbits of $\mathrm{Diff}_{\mathcal{M}}$ in the superspace, namely to define $\int \mathcal{D}_{\mathcal{M}}[g]/\mathrm{Diff}_{\mathcal{M}}$. In the continuum a possibility is to start from the metric in superspace (see for instance DeWitt (1984)) given by the distance element

$$\|\delta g\|^2 \;=\; \int_{\mathcal{M}} \mathrm{d}^dx\,\sqrt{g}\,\delta g_{\mu\nu}\,\left[\alpha g^{\mu\rho}g^{\nu\sigma}+\beta g^{\mu\nu}g^{\rho\sigma}\right]\delta g_{\rho\sigma} \quad , \tag{2.6.1}$$

which defines a measure over metrics which is local and invariant under the action of $\mathrm{Diff}_{\mathcal{M}}$, then to gauge away $\mathrm{Diff}_{\mathcal{M}}$ by using one's favorite gauge. Formally, this measure can be written as a product over all points of $\mathcal{M}$ (DeWitt (1967), Fujikawa (1983))

$$\mathcal{D}[g] \;=\; \text{“}\prod_x\text{”}\,|g|^{\frac{(d-4)(d+1)}{8}}\prod_{\mu\leq\nu}\mathrm{d}g_{\mu\nu} \quad , \qquad (2.6.2$$

but other choices of measure have been proposed, for instance with a different power for $|g|$ (Misner (1957), Leutwyler (1964), Fradkin and Vilkoviski (1973)).

In order to define a discretized integration prescription over metrics out of Regge calculus, a first approach is to choose a given simplicial manifold $\mathcal{C}$, and to consider all the admissible metrics (i.e. all the possible PL spaces) that one can construct out of $\mathcal{C}$ by varying the edge lengths $l_e$. The integration over the $l_e$'s will be restricted by the triangular inequalities which imply that the metric is positive definite on all the simplices. Then it is expected that reparametrization invariance will be restored in a continuum limit because one is summing over all the different piecewise flat metrics on $\mathcal{C}$ which are (at least approximately) equivalent by diffeomorphisms.

This approach, that we shall call here the fixed triangulation (FT) approach, has been mostly developed by Hamber and Williams (see Hamber (1986) for the basic theory, and Hamber and Williams (1984), and (1986)). In my opinion it has some appealing aspects but also some drawbacks.

In the FT approach, many choices of measures for the edge lengths are possible. For instance, one can use simply a factorized measure of the form

$$\prod_{\substack{\text{edges}\\ e}} \frac{\mathrm{d}l_e}{l_e^\beta} V(l_e^\star)^\gamma \quad , \tag{2.6.3}$$

where $V(l_e^\star)$ is the $(d-1)$-volume of the face dual to $l_e$, and $\beta$ and $\gamma$ are some powers. More elaborate choice are possible, which involve for instance the volumes of the d-simplices. Let me mention that the natural expression in terms of edge lengths of the measure derived from the supermetric (2.6.1) involves a very non-local determinant depending on all the edges of the lattice. Another problem is that there can be in general very singular simplices, such as the triangles depicted on Fig. 5, with very small or very long edges. One usually deals with that problem by putting upper and lower bounds on the length variables.

Fig. 5. Singular triangles

However, such configurations are unavoidable when on tries to obtain a metric with very high curvature out of the flat metric. This is an example of the occurrence of what is is my opinion the main problem in this approach. It is not clear whether one can explore by this process regions in the space of metrics where the metric is very different from the typical metric on the reference simplicial complex that we have chosen at the beginning.

These problems are related to the issue of gauge fixing and of the meaning of diffeomorphism invariance on simplicial spaces. This is discussed in particular in Fröhlich (1982), Roček and Williams (1984) and in Lehto et al. (1986).

### *2.7. Dynamical triangulations (DT)*

A possible solution to these problems is to extend the sum over metrics by summing also over all topologically equivalent simplicial manifolds (if one fixes the topology), or even by summing over all possible simplicial manifolds (if one wants to deal with arbitrary topologies). A drastic simplification is then to reduce the sum over admissible metrics on a given $\mathcal{C}$ to *only one representative metric*, for instance the metric such that all edge lengths are equal to some fundamental length $a$. This length $a$ will correspond to the ultra-violet cut-off of the theory. This approach has been first proposed for two dimensional gravity (see Fröhlich (1985), David (1985a) and Kazakov (1985)), and then extended to higher dimensions. For some previous related ideas, see Weingarten (1977) and (1982). In the DT approach, one replaces the infinite sum over the infinite dimensional space of metrics by a *discrete* sum over simplicial manifolds, which is mathematically more manageable.

$$\int \mathcal{D}_{\mathcal{M}}[g]/\mathrm{Diff}_{\mathcal{M}} \qquad \longrightarrow \qquad \sum_{\mathrm{SM}} \quad . \tag{2.20}$$

For any "regular enough" Riemannian space, one can construct a sequence of PL spaces by taking a regularly distributed set of points, constructing a lattice out of these points and then taking as edge length the geodesic distance between nearest neighbors. Thus we can expect that the discrete set of PL space that we are considering is "regularly" distributed in the space of all metrics (Römer and Zähringer (1986)), while FT are localized in a neighborhood of smooth metrics on the regular lattice that has been chosen. Moreover we have introduced a natural UV cut-off in this approach.

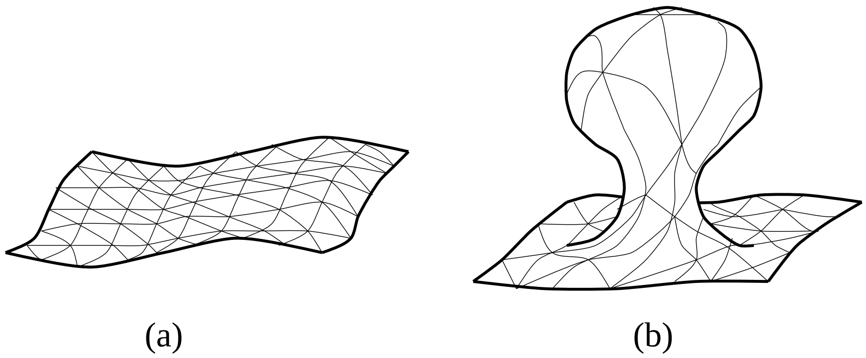



Fig. 6. Fixed triangulations with varying edges lengths are good approximations of metrics close to the flat one (a), but not of some other metrics, even when the curvature is bounded by the short distance cut-off (b)

On the other hand we have completely lost diffeomorphism invariance, since one cannot deform smoothly (even approximately) a simplicial lattice into another one. One has to check if diffeomorphism invariance is recovered in the continuum limit, if it exists. Finally, it is not clear how to recover a classical continuum limit in this approach.

Another problem, both conceptual and practical, for instance in numerical simulation, is how to construct topologically equivalent triangulations. In fact such a notion of topological equivalence is a non-trivial one, and corresponds to (at least) two different concepts of equivalence. Two simplicial manifolds are homeomorphically equivalent if they are equivalent by a (in general non-smooth) diffeomorphism. They are combinatorially equivalent if the two triangulations can be subdivised into the same finer triangulation. For two and three dimensions these two concepts are equivalent, but not in dimension $d > 4$ ! However for smooth triangulations they are always equivalent. I refer to Gross and Varsted (1992) for more details.

It is better to consider only combinatorially equivalent triangulations, because they can be constructed by the so-called Alexander moves (Alexander (1930)), that I describe now. Given a triangulation, take an edge $e = (ij)$, then add a point $k$ on $e$ and subdivise the simplices which have $e$ as edge into two simplices with edges $(ik)$ and $(kj)$ respectively. It is proven that one can always pass from one triangulation to another combinatorially equivalent triangulation by a finite series of such moves, with their inverses.

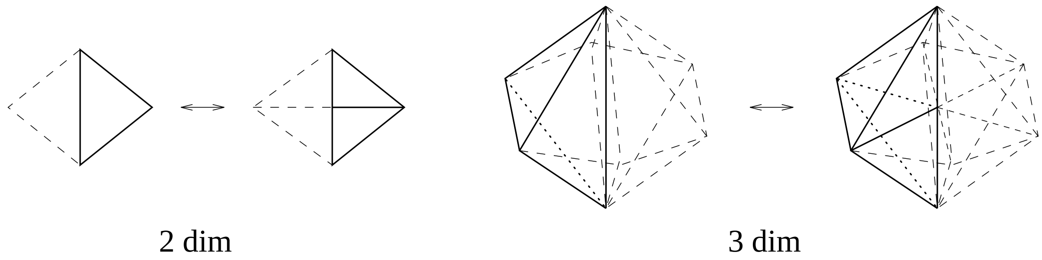



Fig. 7. Alexander moves in 2 and 3 dimensions.

In practice another set of moves is often used, called $(p, q)$ moves in Gross and Varsted (1992). Let $\mathcal{C}$ be a $d$-dimensional triangulation. The boundary (i.e. the set of faces) of the $(d+1)$ dimensional simplex is a $d$-dimensional complex, $\Sigma_d$ , which is a triangulation of the $d$-dimensional sphere $S_d$. Take a subset $S$ of $p$ $d$-dimensional simplices of $\mathcal{C}$ which can be mapped into a part of this complex $\Sigma_d$ and replace it by its complement $\bar{S}$ (it consists of $q = d + 2 - p$ simplices) in $\Sigma_d$.

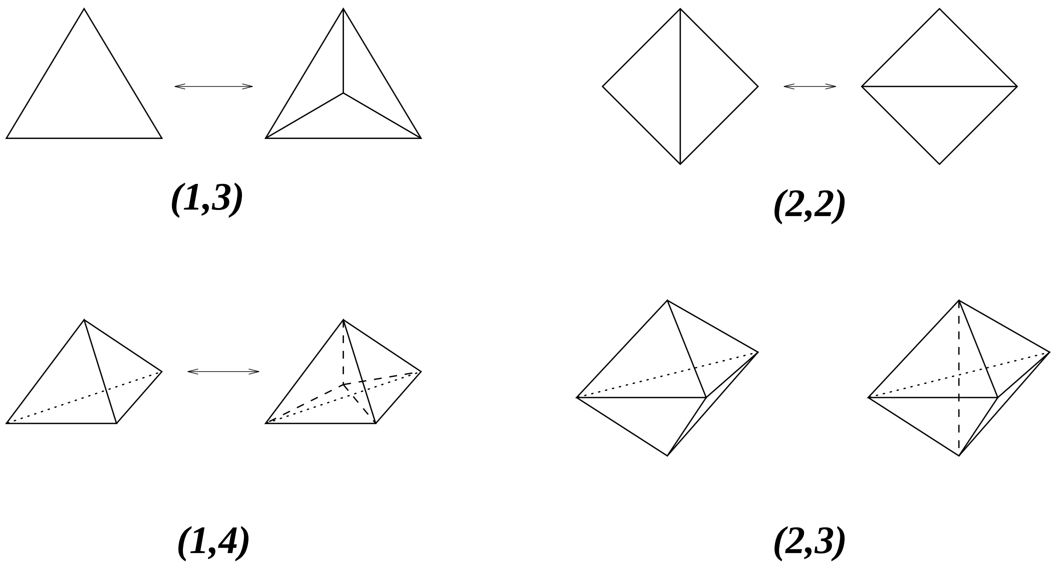



Fig. 8. (p,q) moves in 2 and 3 dimensions.

It is known that such moves are equivalent to Alexander moves in 2, 3 and 4 dimensions. Recently a general argument in any dimensions has been given by Pachner (Pachner (1991)).

## 3. Two dimensional gravity, dynamical triangulations and matrix models

In this second lecture I shall discuss various formulations of Euclidean two dimensional gravity. Besides their interest as toy models for quantum gravity in higher dimensions, some models of two dimensional gravity, when coupled to two dimensional field theories, are in fact equivalent to models of string theories moving in some special backgrounds. This provides a strong motivation for their study, and explains some of their fascinating mathematical properties. I refer to the lectures by L. Alvarez-Gaumé for more details.

### *3.1. Continuum formulation of 2D gravity*

Classically, Einstein theory is meaningless in two dimensions, since the Einstein equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$ are automatically satisfied. Indeed, the Einstein-Hilbert action is a topological invariant, since for closed orientable $2d$ manifolds we have the Gauss-Bonnet theorem

$$\int_{\mathcal{M}} \mathrm{d}^2 x\, R \;=\; 4\pi\,\chi \;=\; 8\pi\,(1-h) \quad , \tag{3.1.1}$$

where $\chi$ is the Euler characteristic (2.5.1) and $h$ the genus of $\mathcal{M}$ (number of handles). Nevertheless, some arguments indicate that a non-trivial quantum theory does exist.

- ▷ Early renormalization group calculations by Weinberg (1979), following Christensen and Duff (1978) and Gastmans et al. (1978), for the theory in $2+\epsilon$ dimensions (this makes sense by dimensional regularization, at least in perturbation theory) show that the scale invariance (and in fact the whole conformal invariance) of two dimensional gravity is spoiled by quantum anomalies. But if the quantum effective action is not scale invariant it cannot be a topological invariant !
- ▷ Polyakov (1981) stressed the crucial importance of conformal anomalies for string theories and 2d gravity. Starting from the functional integral (1.1.1) with the classical action (1.1.2) (with a cosmological constant), he showed that in the conformal gauge, i.e. in isothermal coordinates where

$$g_{\mu\nu}(x) \;=\; \delta_{\mu\nu}\; \mathrm{e}^{\Phi_L(x)} \quad , \tag{3.1.2}$$

  the quantum action for the so-called "Liouville field" $\Phi_L(x)$, obtained by integrating out the other matter fields and by taking into account the gauge fixing Fadeev-Popov determinant, can be calculated explicitly from the conformal anomaly and is still local in $\Phi_L$. It is the celebrated Liouville action

$$S_{\mathrm{L}}[\Phi_L] \;=\; \frac{26-c}{48\pi} \int \mathrm{d}^2 x \left[\frac{1}{2}\,(\partial\Phi_L)^2 \;+\; \Lambda_R\; \mathrm{e}^{\Phi_L}\right] \quad , \tag{3.1.3}$$

  where the number $c$ is the conformal charge of the matter sector (it measures the amount of violation of scale invariance induced by the matter fields). $\Lambda_R$ is the renormalized cosmological constant, obtained from the bare one, $\Lambda$, by some UV divergent vacuum energy subtraction.

▷ Finally, as discussed by Teitelboim (1983), the following classical equation for the metric (constant curvature equation)

$$R + \Lambda_R = 0 \tag{3.1.4}$$

makes sense in two dimensions, and can be used as a starting point for quantization. It is nothing but the classical Liouville equation,

$$-\Delta\Phi_L + \Lambda_R \, \mathrm{e}^{\Phi_L} = 0 \tag{3.1.5}$$

(hence the name Liouville field for $\Phi_L$) obtained by the action principle from the Liouville action, rewritten in arbitrary coordinates.

The conformal gauge is a very convenient framework to quantize 2d gravity, although other choices, such as the light-cone gauge, have some advantages (see Polyakov (1987) and Knizhnik et al. (1988)), in particular to bring forward the underlying current algebra of the model. The classical Liouville equation is integrable, and the program of the quantization of the Liouville theory has been initiated by Curtright and Thorn (1982) and by Gervais and Neveu (1982), and pursued actively by these last two authors and their collaborators (Gervais and Neveu (1983–85), Gervais (1991)). Many interesting results can already be drawn from the massless Liouville theory at $\Lambda_R = 0$, which is a free field theory, but is modified by boundary terms that I have not written in (3.1.3), which correspond to adding a "background charge" at infinity. I refer to the lectures by G. Zuckermann and L. Alvarez-Gaumé for more details.

An important result concerns the scaling properties for conformal field theories coupled to 2d gravity, and that of the partition function itself (Knizhnik et al. (1988), David (1988), Distler and Kawai (1989)). Let us consider the functional integral (1.1.1) for 2d gravity, with the action (1.1.2), where the metric $g_{\mu\nu}$ is now coupled to some scale invariant field theories, described by some fields $\Psi$, that we call matter, with an action $S_{\mathrm{m}}[\Psi, g_{\mu\nu}]$. The generating functional for field operators $\mathcal{O}_i(x)$ integrated over the whole space, $\mathcal{O}_i = \int \mathrm{d}^2x\,\sqrt{g}\,\mathcal{O}_i(x)$, is obtained by adding source terms of the form $J_i\mathcal{O}_i$ to the action,

$$Z[G_N, \Lambda, J_i] = \int \mathcal{D}[g]\,\mathcal{D}[\Psi]\;\mathrm{e}^{-\left(S_{\mathrm{E}}[g] + S_{\mathrm{m}}[\Psi,g] + \sum_i J_i\mathcal{O}_i\right)} \quad . \tag{3.1.6}$$

The cosmological constant $\Lambda$ is proportional to the source $J_0$ for the operator unity $\mathcal{O}_0 = \int \mathrm{d}^2x\,\sqrt{g}\,\mathbf{1}$, and we recast it in the $J$'s. From (3.1.1) $Z$ can be rewritten as a sum over the contribution of manifolds with fixed topology (given by the genus $h$) as

$$Z[G_{\mathrm{n}}, J_i] = \sum_{h=0}^{\infty} \left(\mathrm{e}^{\frac{1}{2G_{\mathrm{N}}}}\right)^{1-h} Z_h[J_i] \quad . \tag{3.1.7}$$

By taking derivatives w.r.t. the $J$'s we define correlation functions (i.e. vacuum expectation values) for local operators averaged on a 2d space with fixed genus $h$, but with fluctuating metric, by

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_p \rangle_h \;=\; \frac{\partial}{\partial J_1} \cdots \frac{\partial}{\partial J_p} Z_h[J_i]\bigg|_{J_i=0} \quad . \tag{3.1.8}$$

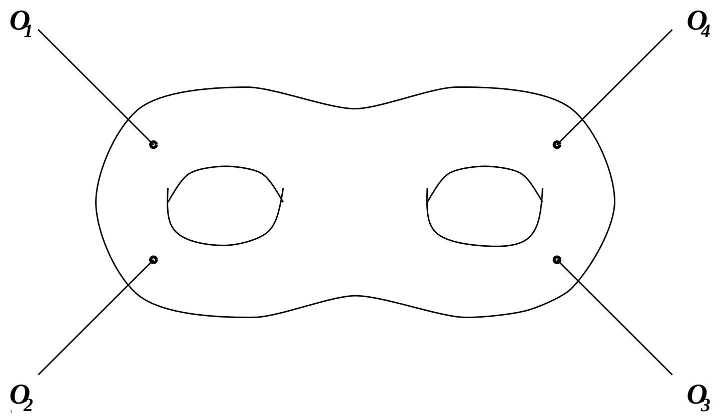



Fig. 9. Correlation function for (microscopic) local operators

If the metric $g_{\mu\nu}$ did not fluctuate, under a global dilation $g_{\mu\nu} \to \lambda g_{\mu\nu}$ these correlators would scale with the total area $A = \int \mathrm{d}^2\sqrt{g}$ as

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_p \rangle \;\propto\; A^{\sum\limits_{i=1}^{p}(1+\Delta_i^0)} \quad , \tag{3.1.9}$$

where $\Delta_i^0$ is the scaling dimension (or conformal weight) of the local operator $\mathcal{O}_i(x)$. When the metric is quantized, the area $A$ is no more fixed, but one can show that its average value $\langle A \rangle$ scales with the renormalized cosmological constant (see eq. (3.1.3)) as $\langle A \rangle \propto 1/\Lambda_R$. Then the scaling law given by eq. (3.1.9) is modified into

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_p \rangle_h \;\propto\; \Lambda_R^{-\sum\limits_{i=1}^{p}(1+\Delta_i)+(2-\gamma_s)(1-h)} \quad , \tag{3.1.10}$$

where $\Delta_i$ is now the scaling dimension of the operator $\mathcal{O}_i$ "dressed" by gravity. The additional term in the exponent, proportional to the Euler characteristic $\chi = 2(1-h)$, can be viewed as a topology dependent divergent part of the vacuum energy caused by the quantum fluctuations of the metric. The exponent $\gamma_s$, often denoted the *string exponent*, depends only on the matter central charge $c$ through the equation

$$1 - \gamma_s \;+\; \frac{1}{1-\gamma_s} \;=\; \frac{13-c}{6} \tag{3.1.11}$$

(valid for unitary theories) and is given by

$$\gamma_s = \frac{1}{12}\left(c - 1 - \sqrt{(25-c)(1-c)}\right) \quad . \tag{3.1.12}$$

The dressed scaling dimensions $\Delta_i$ are related to the bare dimensions $\Delta_i^0$ by

$$\Delta_i - \frac{\Delta_i(1-\Delta_i)}{1-\gamma_s} = \Delta_i^0 \quad . \tag{3.1.13}$$

These scaling laws are often referred to as KPZ scaling, and can be derived easily from the Liouville theory. In the conformal gauge, one can show that the renormalized Liouville action for $\Phi_L$ that has to be used in the functional integral given by eq. (3.1.6) is (taking into account quantum corrections to eq. (3.1.3))

$$S[\Phi_L] = \frac{25-c}{48\pi}\int \mathrm{d}^2x\sqrt{\hat{g}}\,\left(\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\Phi_L\partial_\nu\Phi_L + \hat{R}\Phi_L\right) \quad , \tag{3.1.14}$$

where $\hat{g}_{\mu\nu}$ is a classical background metric and $\hat{R}$ the associated background curvature. The local operators $\mathcal{O}_i$ are "dressed" by gravity and become

$$\mathcal{O}_i = \int \mathrm{d}^2x\,\sqrt{\hat{g}}\,\mathcal{O}_i(x)\,\mathrm{e}^{\alpha_i\Phi_L(x)} \quad . \tag{3.1.15}$$

The consistency requirement that the physics should not depend on the background classical metric $\hat{g}$ fixes the Liouville coupling constant $25-c$ in eq. (3.1.14) and the renormalization factors $\alpha_i$ to be

$$\alpha_i = \frac{1}{12}\left[(25-c) - \sqrt{(25-c)(1-c+24\Delta_i^0)}\right] \quad . \tag{3.1.16}$$

The dressed scaling dimensions $\Delta_i$ are given by the following formula, equivalent to eq. (3.1.13)

$$\Delta_i = \frac{\alpha_i}{\alpha_0} \quad , \tag{3.1.17}$$

where $\alpha_0$ is the renormalization factor for the operator with lowest dimension $\Delta^0$ coupled to the area, which is generically the dimension 0 operator $\mathcal{O}_0 = \mathbf{1}$.

It is much more difficult to compute explicitly the full correlation functions. Explicit results are known for genus $h = 0$ and 1 (Gupta et al. (1990), Goulian and Li (1991), Di Francesco and Kutasov (1991), Dotsenko (1991). In addition to the correlations functions for local operators (sometimes denoted *microscopic operators*, one can also consider correlation functions for *macroscopic operators* $W(\ell)$. Such an operator creates a loop with length $\ell$. In the presence of matter one must specify also the state $|\Psi\rangle$ of the matter along the loop. The corresponding macroscopic loop correlation functions $\langle W(\ell_1)\dots W(\ell_p)\rangle_h$ are obtained by summing over all metrics on a surface with $h$ handles and $p$ boundaries with *fixed lengths* $\ell_1, \dots, \ell_p$. Such objects are much more difficult to compute in the framework of the Liouville theory.

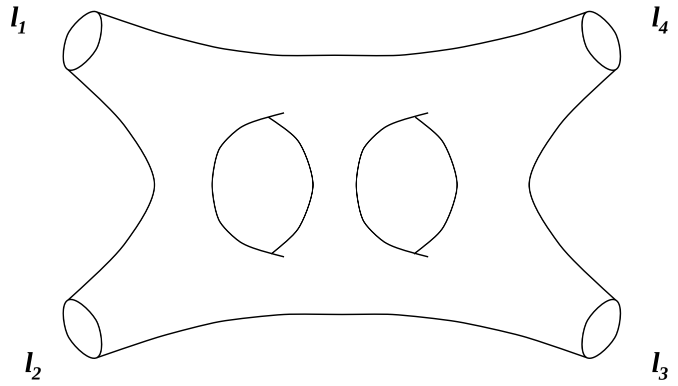



Fig. 10. Correlation function for (macroscopic) loop operators

*3.2. Dynamical triangulations and continuum limit*

Following the ideas of dynamical triangulations developed in the first lecture (sect. 2), it is possible to define regularized models of two dimensional gravity by approximating the integration over Riemann surfaces by a sum over random triangulations (Fröhlich (1985), David (1985), Kazakov (1985)). Restricting oneself at the moment to *oriented* triangulations, one defines the partition function of the model as

$$Z \;=\; \sum_{\substack{\text{oriented}\\ \text{triangulations}}} \frac{1}{s(\mathcal{T})}\, \mathrm{e}^{a_0 N_0 + a_1 N_1 + a_2 N_2} \quad , \tag{3.2.1}$$

where $N_0$, $N_1$ and $N_2$ are respectively the number of vertices, links and triangles of $\mathcal{T}$, and $s(\mathcal{T})$ is the order of the symmetry group of the triangulation $\mathcal{T}$. $a_0$, $a_1$ and $a_2$ are free parameters. The symmetry factor $s(\mathcal{T})$ is natural for the following reason. There are many different way to construct the same triangulation. We can start from sets of *labeled* vertices $\{v_1,\ldots,v_{N_0}\}$, links $\{l_1,\ldots,l_{N_1}\}$ and triangles $\{t_1,\ldots,t_{N_2}\}$, and glue them to get a *labeled* triangulation, and define the sum in eq. (3.2.1) as the sum over labeled triangulations, divided by $N_0!N_1!N_2!$ which represents the number of labelings. Many labeled triangulations can be mapped one into another by a relabeling (i.e. a permutation) of their vertices, links and triangles, and therefore correspond to the same unlabeled triangulation. There are in general $N_0!N_1!N_2!$ labeled triangulations for a unlabeled one, but for symmetric triangulations this number is smaller, and equals $N_0!N_1!N_2!/s(\mathcal{T})$ , so that

$$\sum_{\substack{\text{labeled}\\ \text{triangulations}}} \frac{1}{N_0!N_1!N_2!} \;=\; \sum_{\substack{\text{unlabeled}\\ \text{triangulations}}} \frac{1}{s(\mathcal{T})} \quad . \tag{3.2.2}$$

For instance, the order of the triangulation made out of two triangles glued along their edges is $s=6$, the order of the tetrahedron is $s=12$. Since relabeling of elements of $\mathcal{T}$ can be thought as discrete diffeomorphisms, the factor $1/s(\mathcal{T})$ can be viewed as what remains in the discrete case of the factor $1/\mathrm{Diff}_{\mathcal{M}}$.

From the topological relation $\chi = N_0 - N_1 + N_2$ and the relation $2N_1 = 3N_2$ specific to triangulations, one sees that the discrete action in eq. (3.2.1) can be rewritten as

$$
\begin{aligned}
a_0 N_0 + a_1 N_1 + a_2 N_2 &= -\lambda N_2 + \gamma\chi \quad , \\
\lambda &= -\left(\frac{a_0}{2} + \frac{3a_1}{2} + a_2\right) \quad , \quad \gamma = a_0 \quad ,
\end{aligned} \tag{3.2.3}
$$

and that the action depends only on two independent parameters: $\lambda$, that we identify with the cosmological constant $\Lambda$; and $\gamma$, that we identify with $1/G_{\mathrm{N}}$. Similarly we can define correlators for *discrete loop operators* $W_k$, $\langle W_{k_1} \dots W_{k_p}\rangle$, by the sum over triangulations with $p$ boundaries, which are made out of $k_1, \dots, k_p$ links respectively.

How to construct a continuum theory out of this discrete model? Let us first work with triangulations with fixed topology. The partition function $Z_h$ depends only on $\lambda$. Then we introduce a fundamental length $a$ (which corresponds to a short distance cut-off) that measures the size of the triangles. Up to inessential numerical factors, the dimensionful "physical quantities" are

$$
\begin{aligned}
\text{Area of the surface} \quad A &= a^2\, N_2 \quad , \\
\text{Length of loop} \quad L_i &= a\, k_i \quad , \\
\text{Cosmological constant} \quad \Lambda &= \lambda/a^2 \quad .
\end{aligned} \tag{3.2.4}
$$

To construct a continuum theory, we must find a critical point $\lambda_c$ where the partition function is singular and the average number of triangles $\langle N_2\rangle$ diverges. Indeed, let us assume that at $\lambda_c$, $Z_h$ behaves as

$$
Z_h(\lambda) \propto (\lambda - \lambda_c)^{-\alpha} \quad . \tag{3.2.5}
$$

Then

$$
\langle N_2\rangle_h = -\frac{\partial}{\partial\lambda} \ln Z_h = \frac{\alpha}{\lambda - \lambda_c} + \text{less singular terms} \quad . \tag{3.2.6}
$$

If we define the renormalized cosmological constant $\Lambda_R$ as

$$
\Lambda_R = a^{-2}(\lambda - \lambda_c) \quad , \tag{3.2.7}
$$

we can define a *continuum limit*, where the short distance cut-off goes to zero, while physical quantities are kept finite

$$
a \to 0 \quad , \quad \lambda \to \lambda_c \quad , \quad \Lambda_R \text{ fixed} \quad . \tag{3.2.8}
$$

Indeed, in this limit, the average physical area $\langle A\rangle$, as well as its moments $\langle A^n\rangle$, are finite

$$
\langle A\rangle = a^2\langle N_2\rangle \propto \frac{1}{\Lambda_R} \quad . \tag{3.2.9}
$$

Similarly, one expects that the loop correlation functions, when expressed in terms of the physical lengths $L_i$, are finite (up to possible wave function renormalizations).

Of course at that point we have not said anything specific about the discrete model. In order to test these ideas one must

▷ Solve analytically or numerically the model, and look for its critical point(s).

▷ Check that the observables of the model are singular, and obey scaling relations which allow to define a continuum limit. In particular one should test *universality*, i.e. the irrelevance of the details of the regularization (one can use different lattices with various triangles, squares, add higher curvature terms to the action, etc...) in the continuum limit.

▷ Finally, one should test the predictions of the continuum theory, in particular KPZ scaling, eq. (3.1.11–13), in order to check that the continuum theory that we have constructed is indeed two dimensional quantum gravity. In particular the exponent $\alpha$ in eq. (3.2.5) should be equal to $(\gamma_s - 2)(1 - h)$.

### *3.3. The one-matrix model*

We have seen that to every triangulation $\mathcal{T}$ corresponds a dual simplex $\mathcal{T}^\star$, which in the two dimensional case is made of polygons sharing trivalent vertices. Thus $\mathcal{T}^\star$ can be viewed as a Feynman diagram of a $\Phi^3$-like theory. The basic idea of the matrix model formulation is to construct precisely a field theory whose Feynman amplitudes are the contributions of the triangulations in eq. (3.2.1). It is inspired from the ideas of 't Hooft (1974) about the large $N_\mathrm{c}$ (number of colors) limit and the topological expansion for Quantum Chromo-Dynamics, and of subsequent works on planar diagrams and the planar limit (Brézin et al. (1978)).

Let us consider the following matrix integral over $N \times N$ Hermitian matrices $\Phi$ ($\Phi = \Phi^\dagger$)

$$Z(g) \;=\; \int \mathrm{d}^{N^2}\Phi \; \mathrm{e}^{-N\mathrm{Tr}(V(\Phi))} \quad , \tag{3.3.1}$$

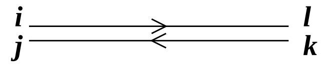
with the polynomial "potential"

$$V(\Phi) \;=\; \frac{1}{2}\,\Phi^2 \,-\, \frac{g}{3}\,\Phi^3 \quad , \tag{3.3.2}$$

and the flat measure

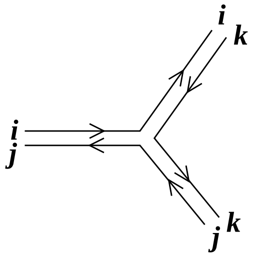
$$\mathrm{d}\Phi \;=\; \boldsymbol{C}_N \prod_i \,\mathrm{d}\Phi_{ii} \prod_{i<j} \,\mathrm{dRe}(\Phi_{ij})\;\mathrm{dIm}(\Phi_{ij}) \quad , \tag{3.3.3}$$

normalized by the constant $\boldsymbol{C}_N$ so that $Z(0) = 1$. This integral is not convergent, and at this stage we just consider it as a compact representation for its series expansion in powers of $g$, whose terms are well-defined. The term of order $g^k$ is given by the average $\langle[\mathrm{Tr}(\Phi^3)]^k\rangle$ w.r.t. the Gaussian measure $d\Phi\,\exp(-\frac{N}{2}\mathrm{Tr}(\Phi^2))$, and can be expressed in terms of Feynman diagrams by applying Wick's theorem. The propagator can be represented in terms of a double arrow (each arrow flowing from line to column indices) as

$$\langle \Phi_{ij}\, \Phi_{kl} \rangle \;=\; \frac{1}{N}\, \delta_{il}\, \delta_{jk}$$ 

. (3.3.4)

There is no momentum flowing through the propagator because we are dealing with a simple integral, i.e. a "zero dimensional field theory". The cubic interaction term is

$$g\, N\, \Phi_{ij}\, \Phi_{jk}\, \Phi_{ki}$$ 

, (3.3.5)

and the Feynman diagrams are trivalent "fat graphs" such as the ones depicted on Fig. 11.

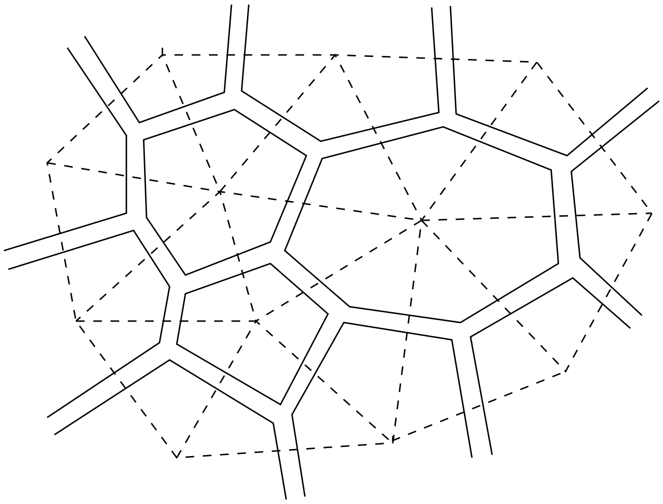


Fig. 11. a diagram and the dual triangulation

The generating functional for the connected vacuum diagrams is $F = \ln(Z)$, and the contribution of a diagram $\mathcal{G}$ is, counting the factors $N$ coming from the propagators, the vertices, and the sums over indices which flow along each closed oriented loop,

$$\frac{1}{s(\mathcal{G})}\, g^{n_v}\, N^{n_v - n_p + n_l} \quad , \tag{3.3.6}$$

where $n_v$, $n_p$ and $n_l$ are respectively the number of vertices, of propagators and of loops of the graph $\mathcal{G}$, while $s(\mathcal{G})$ is its symmetry factor. It is easy to see that any connected Feynman diagram (fat graph) generated by $F$ is the dual $\mathcal{T}^\star$ of some oriented triangulation $\mathcal{T}$ of the discrete model of subsec. 3.2. Since we have $n_v = n_2$, $n_p = n_1$, $n_l = n_0$ and $s(\mathcal{G}) = s(\mathcal{T})$, we can identify the partition function $Z_{\mathrm{DT}}$ of the DT model (eq. (3.2.1)) with the generating functional $F$ for connected diagrams of the matrix model

$$Z_{\mathrm{DT}}(\lambda,\gamma) \;=\; \ln\big(Z(g,N)\big) \quad ; \quad g \;=\; \mathrm{e}^{-\lambda}\,,\; N \;=\; \mathrm{e}^{-\gamma} \quad . \tag{3.3.7}$$

Similarly, one can rewrite the correlators for the loop operators $W_k$ as the connected correlation functions for the observables $\mathrm{Tr}(\Phi^k)$

$$\langle W_{k_1}\cdots W_{k_p}\rangle \;=\; \langle \mathrm{Tr}(\Phi^{k_1})\cdots \mathrm{Tr}(\Phi^{k_p})\rangle_{\mathrm{conn.}} \quad . \tag{3.3.8}$$

From eq. (3.3.6) a diagram with genus $h$ contributes with a factor $N^{2(h-1)}$. Therefore the topological expansion (eq. (3.1.7)) for two dimensional gravity corresponds to the $1/N^2$ expansion for the matrix model. The so-called planar limit, where only the lowest genus contribution is retained, is obtained by taking the limit $N \to \infty$.

The matrix models provide a surprisingly powerful method to study two dimensional gravity and string theories. Many matrix models are exactly solvable, and share very interesting mathematical properties with other fields of mathematics and physics (topology, integrable systems, etc...). They are discussed in details in the lectures by L. Alvarez-Gaumé and E. Brézin. I shall discuss only a few simple points here. Some more elaborate discussion about matrix models and two dimensional gravity beyond the topological expansion and non-perturbative effects will be given in sect. 5.

The most powerful method to discuss the matrix models is to use the invariance of the matrix integral under unitary transformations

$$\Phi \;\to\; U\,\Phi\,U^\dagger \qquad ; \qquad U\,U^\dagger \;=\; 1 \quad , \tag{3.3.9}$$

to diagonalize the matrix, and to rewrite the integral over the $N^2$ matrix elements as an integral over the $N$ real eigenvalues $\lambda_i$ of $\Phi$. There is another method to solve the model in the planar limit, which is also of some interest. It relies on the Schwinger-Dyson equations for the matrix models (called the loop equations), and on the factorization properties of the observables in the large $N$ limit (Wadia (1981), Migdal (1983)). If one performs the infinitesimal analytic change of variable $\Phi \to \Phi + \epsilon f(\Phi)$ in the integral given by eq. (3.3.1), one obtains the identity

$$N\,\langle \mathrm{Tr}[f(\Phi)V'(\Phi)]\rangle \;=\; \oint \frac{\mathrm{d}z}{2i\pi}\, f(z)\,\langle \mathrm{Tr}\big(\frac{1}{z-\Phi}\big)\,\mathrm{Tr}\big(\frac{1}{z-\Phi}\big)\rangle \;. \tag{3.3.10}$$

The l.h.s. comes from the variation in the potential $V(\Phi)$, and the r.h.s. from the variation in the measure $d\Phi$. Rather than using the loop operators $W_k = \mathrm{Tr}(\Phi^k)$, it is more convenient (but mathematically equivalent) to consider the operator

$$W(L) \;=\; \mathrm{Tr}(\mathrm{e}^{L\Phi}) \tag{3.3.11}$$

($L$ is a continuous "pseudo-length"). Indeed, if one takes $f(z) = \mathrm{e}^{Lz}$, eq. (3.3.10) becomes the loop equation

$$N\,V'(\frac{\partial}{\partial L})\,\langle W(L)\rangle \;=\; \int_0^L \mathrm{d}L'\,\langle W(L')\,W(L-L')\rangle \quad . \tag{3.3.12}$$

Since the topological expansion of *connected* correlation functions of $p$ loop operators involves surfaces with Euler characteristics $\chi \leq 2-p$, they behave as $N^{2-p}$. It is a general feature of the limit $N \to \infty$ that *non-connected* correlations functions factorize

$$\langle W(L_1)\cdots W(L_p)\rangle \;\simeq\; \langle W(L_1)\rangle\cdots\langle W(L_p)\rangle \quad . \tag{3.3.13}$$

Thus in the planar limit the loop equation (3.3.12) becomes a simple integro-differential equation for the one-loop correlator. The loop equations can be also used to compute multi-loops correlators, by adding to the potential $V(\Phi)$ a source term $J(L)$ for the loops

$$V(\Phi) \;\longrightarrow\; V(\Phi) - \int_0^\infty \mathrm{d}L\,J(L)\,\mathrm{e}^{L\Phi} \quad . \tag{3.3.14}$$

One obtains

$$\begin{aligned} N\,V'(\frac{\partial}{\partial L})\,\langle W(L)\rangle_J \;=\; &\int_0^L \mathrm{d}L'\,\langle W(L')\,W(L-L')\rangle_J \\ &+ \int_0^\infty \mathrm{d}L''\,L''\,J(L'')\,\langle W(L+L'')\rangle_J \quad . \end{aligned} \tag{3.3.15}$$

By taking derivatives w.r.t. $J(L)$ one obtains multi-loops correlators. The first term corresponds to the splitting of the loop with "pseudo-length" $L$ into two loops, while the second term corresponds to the fusion of the loop $L$ with a loop $L''$. Thus the loop equations have a simple geometric interpretation in terms of evolution in loop space (see Fig. 12). They correspond to constraints for the partition function which form a Virasoro algebra.

The solution of the matrix model in the planar limit ($N \to \infty$), obtained either by the integration over eigenvalues or by the loop equation, shows that the discretized model of dynamical triangulations has indeed a critical point, where one can construct a continuum limit. This continuum limit corresponds to "pure gravity". In particular the exponent $\alpha$ obtained through the matrix model is equal to $\alpha = -5/2$, which agrees with the KPZ scaling prediction for pure gravity ($c = 0$).

### *3.4. Various matrix models*

Let me give some examples of matrix models which describe more complicated geometries, or gravity coupled to various kind of matter fields.

*General potential*

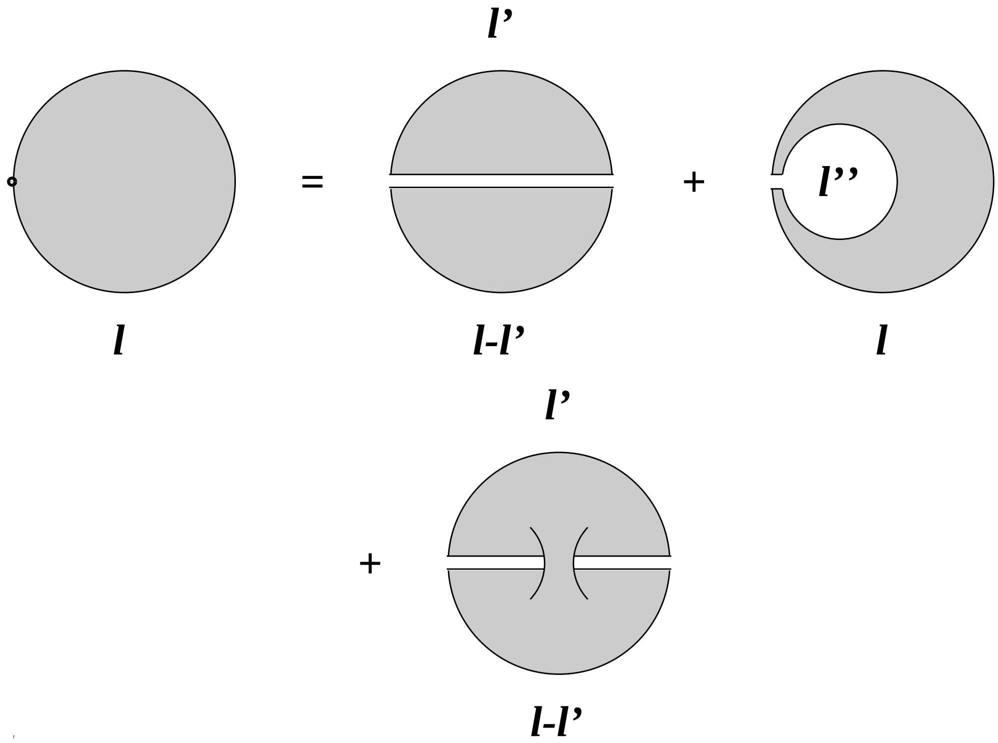



Fig. 12. Loop equation for the Hermitian matrix model

The one-matrix model can be solved for a general potential $V$ of the form

$$V(\Phi) = \frac{1}{2}\,\Phi^2 - \frac{g_3}{3}\,\Phi^3 - \frac{g_4}{4}\,\Phi^4 - \frac{g_5}{5}\,\Phi^5 - \ldots \tag{3.4.1}$$

This corresponds in the random lattice formulation of gravity to sum not only over triangulations, but over lattices made of triangles, squares, pentagons, etc... (Kazakov (1989)). For positive coefficients $g_n$, one can show that the continuum limit is the same for all these models, and still describes pure gravity.

When some of the coefficients $g_n$ are negative, one can obtain different continuum limits (Kazakov (1989)). It has been shown that they correspond to coupling gravity to non-unitary matter fields (Staudacher (1990)). For instance, the simplest non-trivial model corresponds to the CFT which describes the Lee-Yang singularity for the Ising model in imaginary magnetic field (i$\phi^3$ model).

*Symmetric matrix model*

Another interesting model consists in replacing in eq. (3.3.1) Hermitian complex matrices by *symmetric real matrices.* In that case the sum over orientable triangulations is simply replaced by the sum over all triangulations, including non-orientable ones (Brézin et al. (1978)). This can be seen explicitly in the Feynman diagrams of the model, or in the loop equations. For instance, eq. (3.3.12) is replaced by

$$N\,V'(\frac{\partial}{\partial L})\,\langle W(L)\rangle \;= \frac{1}{2}\int_0^L dL'\,\langle W(L')\,W(L-L')\rangle$$

$$+ \frac{1}{2} L \langle W(L) \rangle \qquad . \tag{3.4.2}$$

The first term, as previously, represents cutting a loop into two loops. The new term, which is of order $N^{-1}$, is an orientation changing term representing the cutting of a loop bounding a Moebius strip (see Fig. 13).

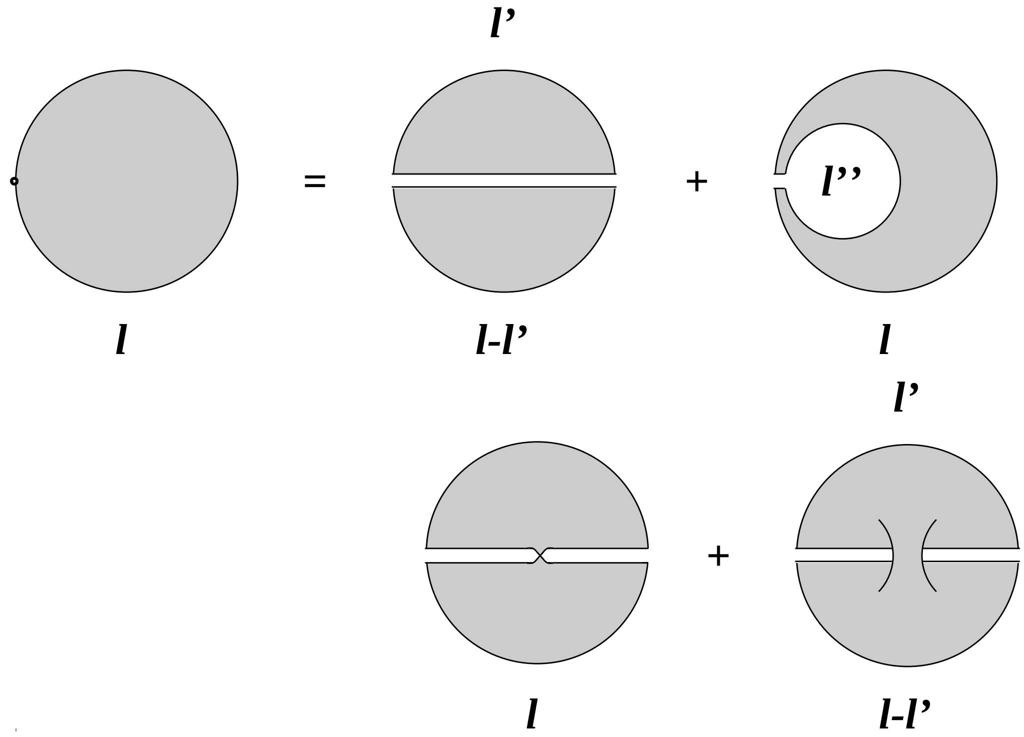



Fig. 13. Loop equations for the symmetric matrix model

*Multi-matrix models*

One can generalize this discussion to multi-matrix models. The simplest example is the two-matrix model (Kazakov (1986), Boulatov and Kazakov (1987)). It corresponds to an Ising model living on the random triangulation, i.e. to Ising spins $\sigma$ (taking values $\pm 1$) on each triangle (each vertex of the dual Feynman diagram), with nearest neighbors coupling. One can implement this system by considering two matrices $\Phi_+$ and $\Phi_-$ and a potential of the form

$$V(\Phi_+, \Phi_-) \;=\; \alpha(\Phi_+^2 + \Phi_-^2) \,+\, 2\beta\Phi_+\Phi_- \,-\, g\left(\mathrm{e}^h\ \Phi_+^3 \,+\, \mathrm{e}^{-h}\ \Phi_-^3\right) \ , \tag{3.4.3}$$

where $h$ represents the external magnetic field, while $\alpha$ and $\beta$ are related to the coupling $J$ (in the interaction $-J\sigma\sigma'$ between spins on nearest neighbors) by

$$\alpha \;=\; \frac{\mathrm{e}^J}{2\cosh(2J)} \qquad ; \qquad \beta \;=\; -\,\frac{\mathrm{e}^{-J}}{2\cosh(2J)} \quad . \tag{3.4.4}$$

In flat two dimensional space, the Ising model at its critical point is equivalent to a theory of free fermions, with central charge $c = 1/2$.

The two-matrix model given by eq. (3.4.3) can be solved by the orthogonal polynomial methods (see the lectures by E. Brézin), as well as a more general class of multi-matrix models, which includes chains of matrices (Kostov (1989b). This class contains models of two dimensional gravity coupled to unitary rational conformal field theories, which have central charge $c < 1$.

Another class of model which can be solved exactly (to a somewhat lesser extent) is provided by the model with one-matrix $\Phi$ and $n$ matrices $\Psi^a$, $a = 1, n$, with the following action (Duplantier and Kostov (1988) and (1990), Kostov (1988a)).

$$S \;=\; \mathrm{Tr}\left[\frac{1}{2}\Phi^2 \;-\; \frac{g}{3}\Phi^3 \;+\; \sum_{a=1}^{n}\frac{1}{2}\Psi^a(1-2\lambda\Phi)\Psi^a\right] \quad . \tag{3.4.5}$$

Integrating over the $\Psi$'s, it reduces to a one-matrix model with a more complicated interaction,

$$S[\Phi] \;=\; \mathrm{Tr}_N\left(\frac{1}{2}\Phi^2 - \frac{g}{3}\Phi^3\right) \;-\; \frac{n}{2}\,\mathbf{Tr}_{N^2}\left[\mathbf{Ln}\left[1\otimes 1 - \lambda(1\otimes\Phi + \Phi\otimes 1)\right]\right] \tag{3.4.6}$$

which describes a gas of self-avoiding loops (with line tension $\lambda$ and fugacity $n$) on random lattices. For fugacity $-2 \leq n \leq 2$ this system can be critical, and corresponds to the so-called O($n$) models, which include the self-avoiding walk ($n = 0$), the Ising model ($n = 1$), and the rotator model which describes the Kosterlitz-Thouless transition ($n = 2$).

*d-dimensional bosonic string*

Although one can obtain exact results for large classes of matrix models, many models have not yet been solved. This is the case of the $d$-dimensional matrix model, where $\Phi(X)$ depends on the space coordinate $X$

$$S \;=\; N\int \mathrm{d}^d X\,\mathrm{Tr}\left[\frac{1}{2}\frac{\partial\Phi}{\partial X}\frac{\partial\Phi}{\partial X} \;+\; \frac{1}{2}\,\Phi^2 \;-\; \frac{g}{3}\,\Phi^3\right] \quad . \tag{3.4.7}$$

This model should describe a Polyakov bosonic string in $d$-dimensional Euclidean space (Ambjørn et al. (1985), David (1985b), Kazakov et al. (1985)). It can be solved in the large $N$ limit only in the special cases $d = 0$ (the ordinary one-matrix model), $d = -2$ (it can be mapped onto the O($n = -2$) model described above, see Kazakov et al. (1985) and David (1985c)), and $d = 1$, which is the continuous limit of an infinite chain of matrices. The solution of the $d = 1$ model is quite interesting, since it can be interpreted as a string theory in a $D = 1 + 1$ dimensional background. The additional coordinate is provided by the Liouville field $\Phi_L$ of the effective string action (3.1.3). If one changes the kinetic term in eq. (3.4.7) so that the usual propagator becomes a Gaussian one

$$\frac{1}{K^2+1} \;\longrightarrow\; \mathrm{e}^{-K^2} \tag{3.4.8}$$

(this is expected not to change the critical properties of the model), this amounts to changing the weight of each triangulation $\mathcal{T}$ in the partition function $Z$ given by eq. (3.2.1) by a factor

$$w_d(\mathcal{T}) \;=\; \left[\frac{1}{N_2}\,\det{}'(-\boldsymbol{C}_{\mathcal{T}})\right]^{-d/2} \quad , \tag{3.4.9}$$

where $\boldsymbol{C}_{\mathcal{T}}$ is the $N_0 \times N_0$ connection matrix of the triangulation $\mathcal{T}$ (it corresponds almost to the discretized scalar Laplacian operator $\Delta = \mathrm{d}^\star\,\mathrm{d}$ discussed in subsec. 2.2), and $\det'$ means the product of the $N_0 - 1$ non-zero eigenvalues of the matrix $-\boldsymbol{C}$. Moreover one has an explicit expression for the determinant in eq. (3.4.9) in terms of the number of *spanning trees* of $\mathcal{T}$, i.e. of connected subgraphs of the triangulation which contain all the vertices of $\mathcal{T}$, but have no internal loops

$$\frac{1}{N_2}\,\det{}'(-\boldsymbol{C}_{\mathcal{T}}) \;=\; \#\ \text{of spanning trees of}\ \mathcal{T} \quad . \tag{3.4.10}$$

This weight corresponds to integrate over all embedding of the triangulation $\mathcal{T}$ in "physical" $d$-dimensional space. One assigns a position vector $\vec{X}(v)$ to each vertex $v$, the discretized action for the embedding is

$$S[\vec{X}] \;=\; \sum_{\substack{\text{edges}\\ vv'}} (\vec{X}(v) - \vec{X}(v'))^2 \quad , \tag{3.4.11}$$

while from the rules of subsec. 2.2, the measure over $\vec{X}$ becomes

$$\mathcal{D}[\vec{X}] \;\to\; \prod_{\text{vertices}} \mathrm{d}^d\vec{X}(v)\,c(v)^{d/2} \quad , \tag{3.4.12}$$

where $c(v)$ is the coordination number of the vertex $v$ (Exercise: show how this term comes from the rules of subsec. 2.2 ). One has finally

$$\int \mathcal{D}[\vec{X}]\ \mathrm{e}^{-S[\vec{X}]} \;\propto\; \prod_v c(v)^{d/2}\,w_d(\mathcal{T}) \quad . \tag{3.4.13}$$

### *3.5. Numerical studies of 2D gravity*

Besides the exact results that can be obtained through the matrix models, one must rely mostly on approximate numerical schemes to study the dynamical triangulation models. However rigorous estimates have also been obtained, and some will be discussed in due time. As discussed in subsec. 3.2, one is interested in the structure of the phase diagram and the possible critical points for the DT model defined through the partition functions of the form (3.2.1). The sum over triangulations makes sense for fixed topology, and if not otherwise specified, we shall deal with planar triangulations (genus $h = 0$). The partition function is of the form

$$Z(g) \;=\; \sum_{\substack{\text{planar}\\ \text{triangulations}}} \frac{1}{s(\mathcal{T})}\, w(\mathcal{T})\, g^{N_2} \quad , \tag{3.5.1}$$

where $N_2$ is the area of the triangulation $\mathcal{T}$ and the weight $w$ depends on the matter fields which live on the triangulation. The matrix model results show that the number of triangulations with fixed topology grow with the area as $(constant)^{N_2}$. Moreover the *constant* does not depend on the topology (genus) of the triangulation, but depends on the details of the discretization (Rivasseau (1991a)). This implies that eq. (3.5.1) is a convergent series for $|g|$ small enough, which defines $Z(g)$ as an analytic function around the origin. This function will have singularities in the complex plane. We are usually interested in the smallest singularity $g_0$ on the positive real axis. This singularity will allow to get a continuum theory, along the lines of subsec. 3.2 (see Fig. 14). I now present some of the numerical approaches to study the critical behavior.

*Series expansions*

Usually $g_0$ is the closest singularity from the origin, and the critical behavior of $Z(g)$ at $g_0$

$$Z(g) \;\simeq\; z_c\,(g_0 - g)^{2-\gamma_s} \;+\; \text{regular or less singular terms} \tag{3.5.2}$$

is therefore related to the large order behavior of its series expansion

$$Z(g) \;=\; \sum_{k=0}^{\infty} z_k\, g^k \qquad ; \qquad z_k \;=\; z_c\, k^{\gamma_s - 3}\, {g_0}^{-k} \left(1 + o(k^{-1})\right) \tag{3.5.3}$$

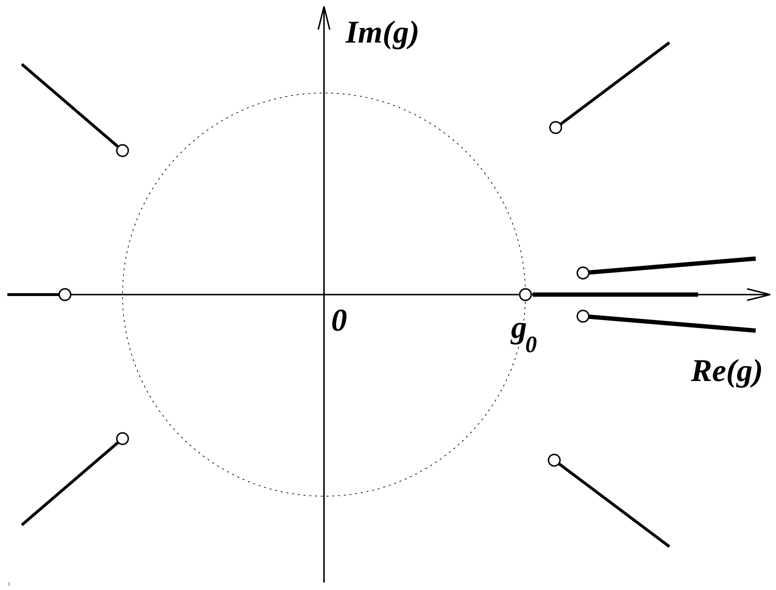



Fig. 14. Singularities of the partition function in the complex plane

One can compute the first terms $z_k$ of the expansion by constructing all the triangulations with a fixed number of vertices (this can be done with the loop equations), and by estimating their weight factor $w$. By such methods one can typically compute the first $10 \sim 15$ terms of the expansion (David (1985b), David et al. (1987), Ambjørn et al. (1986b)). Then one obtains estimates for the value of the critical point $g_0$ and the critical exponent $\gamma_s$. However in general this is made difficult because the subdominant terms in eq. (3.5.3) are not small. One reason for this is that there might be other singularities in the complex plane, beyond $g_0$, but close to it. Various recipes exist to deal with these problems. One can take adequate combinations of the $z_k$ which depend weakly of these terms (ratio methods). One can also approximate $Z(g)$, or rather some derivative of its logarithm $\ln(Z(g))$, by rational fractions (d-log Padé methods). These kinds of analysis deal less easily with subleading singularities (such as $(g_0 - g)^{2-\gamma'}$, $\gamma' > \gamma_s$) which occurs at the same critical point (confluent singularities). These singularities are expected to occur for DT describing some matter fields coupled to gravity, since they will correspond to perturbation by matter operators less relevant than the identity operator $\mathbf{1}$ coupled to the area via the cosmological constant.

*Monte-Carlo simulations*

Instead of enumerating exactly all the triangulations with a given number of triangles, one can explore the space of triangulations by Monte-Carlo simulations (Kazakov et al. (1986)). One constructs a stochastic process, which is defined by its transition probabilities: if at "time" $t$ the system is in a configuration $\mathcal{T}$, it has a probability $P(\mathcal{T} \mapsto \mathcal{T}')$ to move to configuration $\mathcal{T}'$ at time $t+1$. To reconstruct the sum over triangulations given by eq. (3.5.1), one must ensure the fact that in the large $t$ limit, the probability to visit a configuration $\mathcal{T}$ is proportional to its weight $w(\mathcal{T})$ (we have recast the factor $g^{N_2}$ in $w$) in the partition function. This will be the case if:

▹ the process is ergodic, i.e. all configurations can be reached starting from an arbitrary initial configuration;

▹ the probabilities $P$ satisfy the detailed balance equations

$$W(\mathcal{T}) \sum_{\mathcal{T}'} P(\mathcal{T} \mapsto \mathcal{T}') = \sum_{\mathcal{T}'} W(\mathcal{T}') \, P(\mathcal{T}' \mapsto \mathcal{T}) \quad . \tag{3.5.4}$$

We have discussed in subsect. 2.7 examples of moves (Alexander moves and $(p, q)$ moves) which are ergodic in the space of (combinatorially) equivalent triangulations. In two dimensions the situation is simpler, since the "flip" (the $(2, 2)$ move) can be shown to be ergodic in the subspace of triangulations with fixed number of triangles. Therefore one can perform M.C. simulations in the two following ensembles:

▹ By using only the flip, one deals with the "canonical ensemble", where the number of triangles $N_2$ (the volume of the manifold) is fixed. The limit $N_2 \to \infty$ corresponds simply to take the critical limit $\Lambda_R \to 0$.

▹ By using the $(1, 3)$ and $(3, 1)$ moves (insertion and deletion of a vertex inside a triangle) one deals with the "grand-canonical ensemble", where the volume varies.

One has to fine tune the coupling constant $g$ (the bare cosmological constant) to reach the critical point.

When matter fields are present (for instance the Ising spins of the model described by eq. (3.4.3), or the $d$ Gaussian fields $\vec{X}(v) = (X^i(v);\ i = 1, d)$ which describe the position of the vertices $v$ of the triangulation in the external space for the model given by eq. (3.4.7-9)), the average over these fields can be performed independently by standard Monte-Carlo methods.

The canonical ensemble does not allow in principle to estimate the susceptibility exponent $\gamma_s$. In practice, the use of ensembles interpolating between the canonical and the grand-canonical ensemble is quite convenient. For instance one can allow the volume of the system to vary between two bounds $N_{\rm min}$ and $N_{\rm max}$, and extract the value of the critical coupling and that of $\gamma_s$ from the study of the average distribution of $N_2$ between these two bounds, either by direct fits, or by using a maximum likelihood principle.

Up to now the most thoroughly studied model (both by series expansions and by M.C. simulations) is the $d$-dimensional string model corresponding to the Gaussian weight of eq. (3.4.9). I refer to Kazakov et al. (1986), Boulatov et al. (1986), Ambjørn et al. (1986), Billoire and David (1986), Jurkiewicz et al. (1986a–b), David et al. (1987), Ambjørn et al. (1987), and to the more recent reviews by Ambjørn (1991) and Kawai (1992) for details.

Other models have also been studied recently, in particular systems of $n$ Ising models coupled to a random triangulation (only the case $n = 1$ is exactly solvable) (Brézin and Hikami (1992)).

*Sampling methods using exact results*

In the special case $c = 0$, the weight factor of a triangulation is basically 1 (up to symmetry factor), and the number of open planar triangulations (with the topology of a disc) with fixed area (number of triangles) and fixed boundary length (number of edges) is known exactly (from the matrix model exact solution). This allowed Agishtein and Migdal (1991a) to construct samplings of triangulations via a biased random process, with the correct probability distribution, and to go to very large triangulations (up to $10^7$ triangles, i.e. two orders of magnitude larger than with ordinary M.C. Methods). Of course this is not useful for evaluating the critical exponent $\gamma_s$, which is exactly know, but this allows to study the "internal geometry" of $2d$ quantum gravity, and to measure quantities such as the "internal Hausdorff dimension" of space, which are not (yet ?) accessible through exact solutions. This method has been cleverly extended to the $c = -2$ case (Kawamoto et al. (1992)). I shall discuss briefly some issues about the internal geometry of $2d$ gravity in subsec. 3.7.

*$2d$ quantum Regge calculus*

For a recent review on numerical simulations of $2d$ gravity using quantum Regge calculus (fixed triangulation and varying edge lengths) discussed in subsec. 2.6, I refer to (Hamber (1991a) and to Gross and Hamber (1991)). Numerical simulations have been performed with or without a higher derivative term (the square of the deficit angle) which amounts to add a $a\boldsymbol{R}^2$ term in the action. Estimates for the $\gamma_s$ exponent for lattices with the topology of the torus (genus $h = 1$) give $\gamma_s \simeq 2.025 \pm 0.025$, in agreement with KPZ scaling. For the genus 0 case, the results of Gross and Hamber (1991), although less accurate, agree also with KPZ scaling.

As discussed above, I think that the problems of the restoration of coordinate invariance, and how FT take into account large fluctuations of the metric away from the regular flat metric are not fully clarified. In particular the result $\gamma_s = 2$ is quite universal for genus $h = 1$ surfaces, since it does not depend on the matter central charge $c$, and is even valid when the metric does not fluctuate ($c = -\infty$). More numerical studies for genus $h = 0$ lattices are in my opinion needed to really understand the model.

Studies of the phase diagram of a model of FT in $d$-dimensional space, as a function of the $\boldsymbol{R}^2$ coupling $a$, of the form of the measure (see eq. (2.6.3)), and of the dimension $d$, have also been performed. They indicate a transition between a "smooth phase" for positive $a$ and a "rough phase" for negative $a$.

The simulations of Ising model on $2d$ Regge lattice indicate no drastic modifications of the critical exponents of the Ising transition when compared to the Ising model on a regular flat lattice (Gross and Hamber (1991). This is at variance with the predictions of KPZ scaling, and the results of the Ising model on dynamical triangulations, and is an argument for the fact that these models do not belong to the same universality class than the DT models.

### *3.6. The $c = 1$ barrier*

The numerical methods described previously give excellent agreement with the exact solutions for DT models coupled to statistical models corresponding to $c < 1$ matter. I shall mostly discuss here the numerical results for systems where no exact solutions are available.

It is clear from the KPZ scaling formulae (eq. (3.1.12–13)) that the continuum formulation of $2d$ gravity makes sense in the weak coupling phase $c \leq 1$, but that something happens for $c > 1$, since the critical exponent $\gamma_s$ becomes complex for $1 < c < 25$. In fact one expects also that the theory will be ill-defined for $c > 25$, since the kinetic term for the Liouville field becomes negative (the $26-c$ in eq. (3.1.3) is renormalized into $25-c$ by quantum corrections). Before discussing numerical studies of this regime, let me first explain what is the present theoretical understanding of this question.

#### *Surfaces versus branched polymers*

A general argument given by Durhuus, Fröhlich and Jónsson (Durhuus et al. (1984)) suggests that, if the string exponent for spherical topologies, $\gamma_s$, is positive, branched-polymer-like configurations should dominate the functional integral over $2d$ metrics. In fact their argument is rigorous (up to very plausible scaling hypothesis) only for a specific class of random surface models made of plaquettes on a lattice in bulk space, but heuristically it sounds quite reasonable for the random triangulation models.

A simple model of abstract branched polymer can be constructed out of links which can join themselves at vertices in all possible ways to form a connected tree. To each vertex we assign a statistical weight of the form

$$g \cdot f(c) \quad ; \quad c = \text{number of links attached at the vertex.} \tag{3.6.1}$$

$g$ is a "vertex fugacity", and $f(c)$ some connectivity dependent factor. The coordination number $c$ can go from 1 (end-point) to $\infty$. The partition function of the model is simply

$$Z(g) \ = \ \sum_{\text{trees}} g^{\#\ \text{of vertices}} \ \prod_{\text{vertices}} f(c) \quad . \tag{3.6.2}$$

$Z(g)$ becomes singular at a critical point $g_c$, where the average number of vertices diverges. Generically, if $f(c)$ decreases fast enough as $c \to \infty$, this critical point describes branched polymers; the average coordination number stays finite and the critical exponent $\gamma_s$ (such that $Z(g) \propto (g_c - g)^{2-\gamma_s}$) is that of branched polymers

$$\gamma_s \ = \ \frac{1}{2} \quad . \tag{3.6.3}$$

The basic idea of Durhuus et al. (1984) is that if one decomposes planar surfaces into "blobs" connected by small "bottlenecks" (such configurations are always present in the integral over metrics), one can show that if $\gamma_s > 0$, the contribution of each blob to the sum over metrics grows as $g \to g_c$ in such a way that the total number of blobs and bottlenecks will diverge before the average size of each blob becomes large. It follows that, close to the critical point, i.e. for surfaces with large area, the configuration space is dominated by branched polymers made out of blobs (with a finite area), rather than by $2d$ surfaces. In fact for the dynamical triangulation model discussed in subsec. 3.4, it is shown rigorously that if $\gamma_s > 0$, then $\gamma_s \leq 1/2$, so that the true branched polymer exponent $1/2$ represents an upper bound.

A simpler, but heuristic argument (David (1992)), consists in comparing the area dependence of the partition function for one surface

$$Z(A) \ \propto \ A^{\gamma_s - 3} \tag{3.6.4}$$

with that of a pinched surface of the same area, made by gluing two marked surfaces of area $A'$ and $A - A'$ at the marked point. Since the partition function of a marked surface should scale as

$$Z_{\text{marked}}(A) \ \propto \ A \cdot A^{\gamma_s - 3} \ = \ A^{\gamma_s - 2} \quad , \tag{3.6.5}$$

the partition function for the pinched surface scales as

$$Z_{\text{pinched}}(A) \ \propto \ \int \mathrm{d}A' \, A'^{\gamma_s - 2} \, (A - A')^{\gamma_s - 2} \ \simeq \ A^{2\gamma_s - 3} \quad . \tag{3.6.6}$$

Comparing eq. (3.6.4) and eq. (3.6.6) one sees that pinches are entropically favorable if $\gamma_s > 0$. For $\gamma_s < 0$ ($c < 1$) such pinches are not favorable. However, the statistics of pinches which create small "baby universes" with area $A' \ll A$ leads to interesting questions, and can be studied via matrix models and the Liouville theory (see Jain et al. (1992)).

*Tachyons for the bosonic string*

From eq. (3.1.13) $\gamma_s$ is negative for $c < 1$ and vanishes at $c = 1$. This can be related to the problem of the tachyon in bosonic string theories (see in particular Seiberg (1991)). The $d$-dimensional non-critical bosonic string described by the continuous action (3.4.10) (with $c = d$) corresponds to a string in $D = d + 1$ dimensions. Indeed, if the Liouville field $\Phi_L$ is identified with an additional coordinate $X^0$ by $\Phi_L = \sqrt{\frac{12}{25-d}}X^0$, so that $\mathbf{X} = (X^0, \vec{X})$, the Liouville+matter action becomes

$$\begin{aligned} &\frac{1}{4\pi}\int \mathrm{d}^2x\,\sqrt{\hat{g}}\,\left(\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\mathbf{X}\partial_\mu\mathbf{X} \;+\; \hat{R}\,D(X^0)\right) \quad , \\ &D(X^0) \;=\; \sqrt{\frac{25-d}{12}}X^0 \quad , \end{aligned} \tag{3.6.7}$$

where $D(X^0)$ is a linear background dilaton field. The operator $\mathrm{e}^{i\vec{P}\vec{X}}$ dressed by gravity (see eq. (3.1.15)) can be identified with a tachyon vertex operator (i.e. an operator which creates a spin 0 state of the string) in $D = d + 1$ dimension with momentum $\mathbf{P} = (P_0, \vec{P})$ by

$$\mathrm{e}^{i\vec{P}\vec{X}+\alpha_{\vec{P}}\Phi_L} \;=\; \mathrm{e}^{D(X^0)}\;\mathrm{e}^{i\vec{P}\vec{X}+P_0X^0} \quad . \tag{3.6.8}$$

The component $P_0$ of the momentum $\mathbf{P}$ is

$$P_0 \;=\; \sqrt{M^2+(\vec{P})^2} \quad ; \qquad M^2 \;=\; \frac{1-d}{12} \quad . \tag{3.6.9}$$

For $d \leq 1$, $P_0$ is real. This suggests to consider $X^0$ as a time-like coordinate, $P_0$ as the associated energy-like momentum component, and $M$ as the mass of the spin 0 ground state of the string. For $d < 1$ the ground state of the string is massive, for $d = 1$ it is massless, while for $d > 1$ it is really tachyonic. Thus the $c = 1$ barrier is nothing but the $D > 2$ tachyonic barrier of the critical bosonic string. Seiberg (Seiberg (1991)) has related the existence of tachyons for $d > 1$ to the so-called "macroscopic states": for $d > 1$ the insertion of the puncture operator $\mathbf{1}$ (i.e. the tachyon vertex operator at zero momentum $\vec{P} = 0$) does not create a small, local, disturbance of the metric, but a large, macroscopic disturbance, because the distances do not become small when one approaches the insertion point. A related problem is that for $c < 1$ there is only a finite number of states for $2d$ gravity coupled to matter (this can be viewed as the reason why the $c < 1$ models are solvable), while for $c > 1$ there is an infinite number of states.

*Spikes/wormholes picture*

Finally another point of view (but somehow related to the previous one), consists in looking at the importance of singular configurations with conical curvature singularities (spikes) in the functional integral over metrics (Förster (1987), Cates (1988)). In the Liouville formulation of $2d$ gravity such defects appear as point like charges, with a charge $q$ proportional to the defect angle $\Delta\theta$ at the conical singularity. A simple argument, similar to the well-known Kosterlitz-Thouless argument for the $2d$ Coulomb gas model, shows that curvature defects such that $|\Delta\theta|$ is smaller then some critical angle $\Delta(c)$ are liberated. Explicit estimates indicate that as long as $c<1$, $\Delta(c)<2\pi$ and this gas of curvature defects just contributes to the fluctuations of the metric, but for $c\geq 1$, $\Delta(c)\geq 2\pi$ and singular spikes (infinitely long and thin cylinders) are liberated (see Cates (1988), Krzywicki (1990) for details).

*Numerical studies of the $c>1$ phase*

I now briefly summarize the numerical results obtained so far. As already mentioned, numerical studies agree quite well with exact solutions which corresponds to $-2\leq c\leq 1$, i.e. $-1\leq\gamma_s\leq 0$. The "strong coupling regime" $c>1$ is much more delicate to explore. Here I shall describe mainly the results of Ambjørn et al. (1986a,b), Jurkiewicz et al. (1986a,b) and of David et al. (1987) on the $d$-dimensional DT model described in subsec. 3.4, which corresponds to a non-critical string in $d$-dimensional Euclidean space. To probe the effects of higher order curvature terms in the action, as well as the importance of the form of the measure given by eq. (3.4.11), one can add a term of the form $\prod\limits_{\text{vertices}} c_i^\alpha$, where $c_i$ is the coordination number of the vertex $i$, so that the weight of a triangulation $\mathcal{T}$ becomes

$$w_{d,\alpha}(\mathcal{T}) \;=\; \prod_{\text{vertices}} c_i^{\alpha+\frac{d}{2}} \left(\#\,{}^{\text{of spanning}}_{\text{trees on }\mathcal{T}}\right)^{-d/2} \quad . \tag{3.6.10}$$

Expanding $c_i$ around its average value $\langle c_i\rangle = 6$, and using Euler theorem, one sees that

$$\prod_i c_i^\alpha \;\simeq\; \mathrm{e}^{-\frac{\alpha}{2}\sum_i\left(\frac{c_i-6}{6}\right)^2+\dots} \quad , \tag{3.6.11}$$

so that $\alpha>0$ will decrease the average squared curvature $R^2$ and flatten the triangulation (at least at short scales), while $\alpha<0$ will increase $R^2$ and make the triangulation more "crumply".

This is indeed what is observed, both in MC simulations and series analysis. For $c<1$ and $\alpha>0$ one is in a "Liouville phase" where KPZ scaling holds and where $\gamma_s$ is negative and independent of $\alpha$. For $\alpha<0$ and large enough one enters a new "collapsed phase" where $\gamma_s$ becomes very negative and $\alpha$ dependent. The existence of two different regimes is also observed for $c>1$. For $\alpha>0$ and large, one is in a "branched phase" with $\gamma_s\simeq .5$, in qualitative agreement with branched polymer models. For $\alpha<0$ and large, one is still in the collapsed phase.

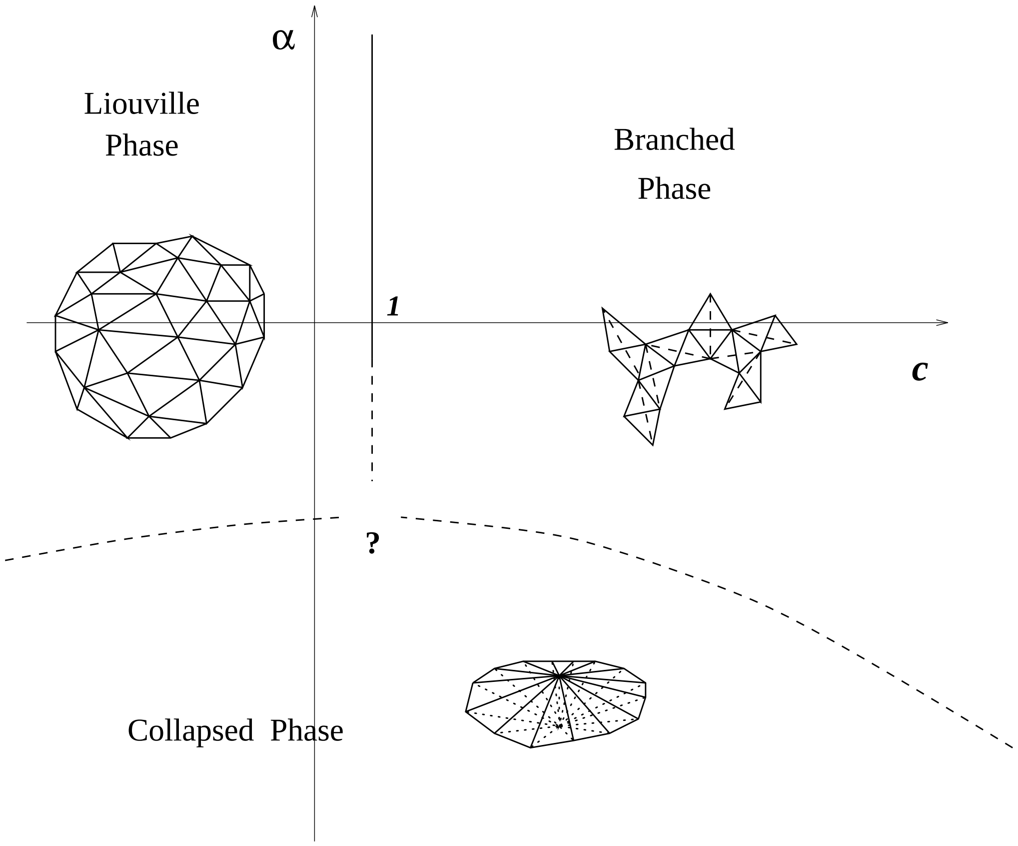



Fig. 15. Qualitative picture of the phase diagram for the $d$-dimensional DT model

The nature of the branched and collapsed phases may be understood qualitatively when $c$ is large by analytical estimates (Ambjørn et al. (1986a,b)). One can show that in this limit, triangulations obtained by gluing together tetraedra along edges dominate the partition function, since they minimize $\det'(-C_{\mathcal{T}})$ in eq. (3.4.9). Such configurations correspond to branched polymers. Depending on whether $\alpha \gg -d/2$, or $\alpha \ll -d/2$, two very different kinds of BP on maximize the weight $w_{d,\alpha}$.

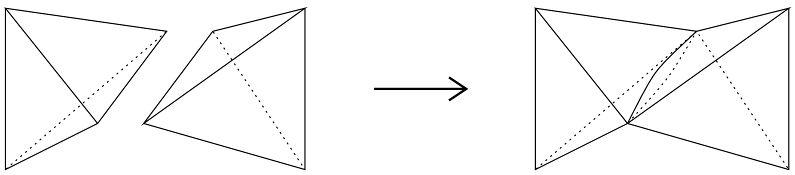


Fig. 16. Gluing two tetraedra

If $\alpha \ll -d/2$, it has been shown that very singular configurations, where all the tetraedra are glued along a single link, are dominant. This explains the collapsed phase. The $R^2$ coupling is so large and negative that the metric collapses into a few very singular regions where the curvature is very large and negative, which correspond to the points with coordination number of the same order of magnitude than the total number of points of the triangulation. The susceptibility exponent is found to vary linearly with $\alpha$ as $\gamma_s \simeq 2\alpha + d/2$. Such a phase has no continuum limit interpretation, since all points are at a finite distance from each others, and the Hausdorff dimension of such a space is infinite.

On the other hand, if $\alpha \gg -d/2$ (this includes the "physical situation" $\alpha = 0$), it is ordinary BP, with no vertices with large coordination number, which dominates, and $\gamma_s = 1/2$. This corresponds to the "branched phase".

Unfortunately, this picture is only valid for large $d$, and checked numerically in a convincing way for typically $d > 10$. A qualitative understanding of the $c > 1$ phase for $c$ close to 1 is still lacking. Numerical estimates for $\gamma_s$ show a smooth increase from 0 to .5 as $c$ goes from 1 to $\infty$, however finite size effects seem important, even for the largest triangulations used up to now in MC simulations. The same conclusion can be drawn from series analysis from the $d$-dimensional bosonic string model. Recent series analysis of Ising models coupled to DT by Brézin and Hikami (1992) corroborate these conclusions. Therefore it is not clear yet whether for $c > 1$, $2d$ gravity is in a branched polymer phase, with $\gamma_s = 1/2$, as long as it is not in a singular, collapsed phase, or if there might exist some intermediate non-trivial phases with $0 < \gamma_s < 1/2$. Such phases would be extremely interesting to characterize. It would also be worth investigating the nature of the transition between the branched and collapsed phases.

### *3.7. Intrinsic geometry of 2D gravity*

Let me discuss briefly numerical studies of the intrinsic geometry of $2d$ gravity. In addition to the global observables, obtained by averaging over $2d$ space local (microscopic) operators or macroscopic loop operators, one may consider correlations involving the *geodesic distance* between points. The simplest of such observables is the volume $V(R)$ of the ball (in $2d$ the disk) with radius $R$, defined as the set of points $x$ at geodesic distance $d_{\mathrm{g}}(x, x_0) \leq R$ from the center $x_0$. This allows to define the *internal Hausdorff dimension* $d_H$ as

$$\langle V(R)\rangle \underset{R\to\infty}{\longrightarrow} R^{d_H} \quad . \tag{3.7.1}$$

In a non flat $2d$ geometry the disk is connected but does not need to be simply connected, so that the number of connected components of its boundary, $n$, might be larger than 1 (see fig. 17). If this number scales with the radius $R$ of the disc, this defines a new *branching dimension* $\delta_B$ by

$$\langle n(R)\rangle \underset{R\to\infty}{\longrightarrow} R^{\delta_B} \quad . \tag{3.7.2}$$

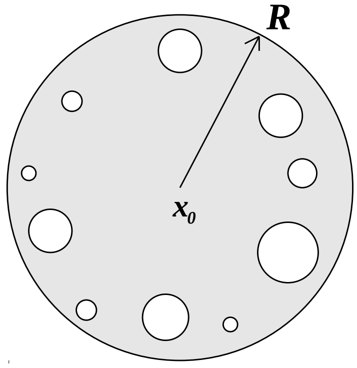



Fig. 17. A non-simply-connected disk

For flat geometry one has obviously $d_H = 2$ and $\delta_B = 0$. For DT these quantities cannot be calculated exactly from the matrix models. One has to use to very large triangulations ($\sim 10^6$ triangles !) in numerical simulations to see some scaling behavior, and some qualitative estimates have only been obtained for the cases $c = 0$ (Agishtein and Migdal (1991a)) and $c = -2$ (Kawamoto et al. (1992), by using the sampling methods described in subsec. (3.5). For $c = -2$ the Hausdorff dimension $d_H$ seems much larger than 2 ($d_H \simeq 3.5$), and the branching dimension is positive $\delta_B > 1$. For $c = 0$, $d_H$ and $\delta_B$ are even larger, but the numerical evidences for scaling are less convincing. This shows that, at the scales considered in numerical simulations, the $2d$ geometry for DT is already very "spiky" in the phase $c < 1$. However, one sees that, from these numerical results, DT seem more branched than "ordinary" branched polymers, which are characterized by $d_H = 2$ and $\delta_B = 1$ (see David (1992)), and more similar to infinitely branched trees with very few end-points, such as the Cayley tree.

Let me mention also that analytical estimates of these quantities through Liouville theory is a difficult problem (see David (1992)). Indeed, KPZ scaling holds for field theories close to their critical points coupled to gravity, and therefore can be viewed as involving *massless* particles, or at least particles with masses $m \propto L^{-1} \ll a^{-1}$ of the order of the IR cut-off ($L \sim \sqrt{A}$ is the size of the manifold), and much smaller than the UV regulator ($a$ is the UV cut-off). On the contrary, the geodesic distance can be expressed in terms of the propagation of *very massive* particles ($L^{-1} \ll m < a^{-1}$) in the fluctuating metric. The propagation of such massive particles is very different from that of massless ones, and new physical phenomena, such as "trapping" by regions with high curvature, occur, which have to be taken into account properly. Interestingly, these problems are related to problems encountered in the statistical mechanics of disordered systems (spin glasses, propagation in random potentials).

### *3.8. Liouville theory at* $c > 25$

There is a very important difference between the models of $2d$ gravity in the weak coupling phase $c \leq 1$ that we have discussed insofar, and Einstein theory of gravity in dimensions $d > 2$. For $c < 1$ the effective theory is the Liouville theory, whose action is given by eqs. (3.1.3,14), with a *positive* coupling constant. This action is bounded from below, and for positive cosmological constant $\Lambda_R$ (i.e. positive vacuum energy density), the stable ground state corresponds to anti-de-Sitter space ($R < 0$).

Following Polchinski (1989), one can try to study the Liouville theory with a *negative* coupling constant, which corresponds formally to the case $c > 25$, as a toy model for the $d > 2$ Einstein theory with positive Newton's constant. The Liouville action, which takes a local form for conformally flat metric, can then be rewritten for general metrics in a *bilocal form*

$$\begin{aligned} S_L[g] = & -\frac{1}{2}\int \mathrm{d}^2x\,\sqrt{g(x)}\int \mathrm{d}^2y\,\sqrt{g(y)}\,R(x)\left(\frac{1}{-\Delta}\right)'R(y) \\ & +\Lambda_R\int \mathrm{d}^2x\,\sqrt{g(x)} \qquad . \end{aligned} \tag{3.8.1}$$

Because of the minus sign in the r.h.s. of eq. (3.8.1), this model admits de Sitter space $R = \Lambda_R$ as classical solution for $\Lambda_R > 0$, but suffers from the conformal instability problem.

One can study the problem of the stability of this action by the dynamical triangulation regularization. Since the configuration space is discrete (for fixed area), the action is in fact bounded from below. One might ask what are the real configurations which minimize this bilocal action, i.e. which *maximize* the Liouville action. Simple, but not rigorous arguments suggest that in that case also, branched polymer configurations minimize the action. Indeed, if one considers a genus 0 metric with fixed area $A$, the Liouville action of the sphere scales as $\ln(A)$, while the action of a long cylinder with radius $a$ (the UV cut-off) and length $L \propto A/a$ scales as $-L/a$. Moreover, one can check that the action for tree-like configurations made out of cylinders with width $a$ is still proportional to $-L/a$, where $L$ is the total length of the tree. Thus one sees that the action is still extensive, i.e. proportional to the area. Moreover, one can argue that such branched polymer configurations do extremize the Liouville action (David, *unpublished*).

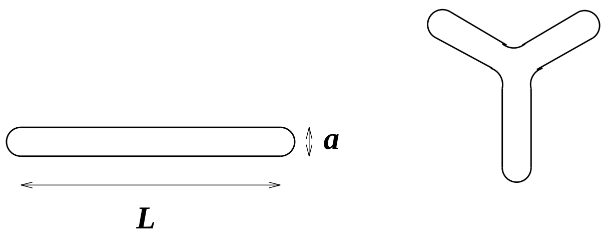



Fig. 18. Configurations which maximize the Liouville action

## 4. Euclidean quantum gravity at $D > 2$

In this lecture I shall give a short review of recent results on lattice models for quantum gravity in dimensions $D > 2$. Since I have not worked myself on these models, my review will be probably incomplete and will reflect some of my prejudices. I shall deal neither with the formulations of $3D$ gravity as Chern-Simon theories (Witten (1989)), nor with the $3D$ topological theories à la Ponzanno-Regge (Ponzanno and Regge (1968), Turaev and Viro (1992)), which allow to construct new topological invariants for $3d$ manifolds. Let me first discuss what one could expect from the lattice approaches of the problem of the quantization of the Einstein theory.

### *4.1. What are we looking for?*

For $D > 2$, the Einstein action given by eq. (1.1.2) is non-renormalizable, since the Newton's constant is dimensionful, with $G_N \sim [length]^{D-2}$. If one still wants to describe gravity by some local field theory, one possibility is to add higher derivative terms to the action. In $D = 4$, the curvature squared theory obtained by adding to the action the two independent dimension 4 operators

$$\int d^D x \sqrt{g}\, (\alpha\, R^2 \;+\; \beta\, \boldsymbol{W}^2) \quad , \tag{4.1.1}$$

where $\boldsymbol{W}$ is the Weyl curvature tensor, is known to be renormalizable and asymptotically free (Fradkin and Tseytlin (1982)). Thus it might define a Euclidean theory sensible at short distance, with dynamical generation of a mass scale (as in QCD), with General Relativity as a low energy effective theory below this scale. Moreover, the action is now bounded from below. However, there are still serious problems with this approach. In particular, such higher derivative theories are known to be non-unitary. Since the conformal instability problem is still present at large scales, it is not clear whether the effective low energy theory is really the Einstein theory at scales $\ell \gg L_{Planck}$.

Another approach has been advocated some time ago by S. Weinberg (Weinberg (1979)). From dimensional analysis, the effective coupling constant of Einstein theory is known to grow with the momentum scale $p$ as $G(p) \propto p^{D-2}$. It is possible to study in perturbation theory the renormalization group flow of $G(p)$ for space-time dimension $D > 2$ within a $D = 2 + \epsilon$ expansion. Such an expansion makes sense in perturbation theory (see Weinberg (1979) and Kawai and Ninomiya (1990)). The interesting result is that the beta function, calculated at one loop, is of the form

$$p\frac{\partial G}{\partial p} \;=\; \beta(G) \;=\; \epsilon\, G \;-\; \boldsymbol{c}\, G^2 \;+\; \ldots \tag{4.1.2}$$

with $\boldsymbol{c}$ some positive constant (at least when no matter fields are coupled to gravity). The standard interpretation of this result for small positive $\epsilon$ is as follows. The non trivial zero of the beta function, $G^\star \sim \epsilon/\boldsymbol{c}$, corresponds to an UV fixed point of the RG flow. The domain $0 < G < G^\star$ is a weak coupling phase for gravity. At large distance $G(p) \to 0$ and General Relativity is recovered. At small distances $G(p) \to G^\star$, and the fixed point governs the non-trivial short distance behavior of gravity at scales $\ell \sim L_{Planck}$. The domain $G > G^\star$ should correspond to a new strong coupling phase for gravity. Let me stress that this picture is valid for zero renormalized cosmological constant $\Lambda_R$. This should require the usual fine tuning of the bare cosmological constant in the regularized theory.

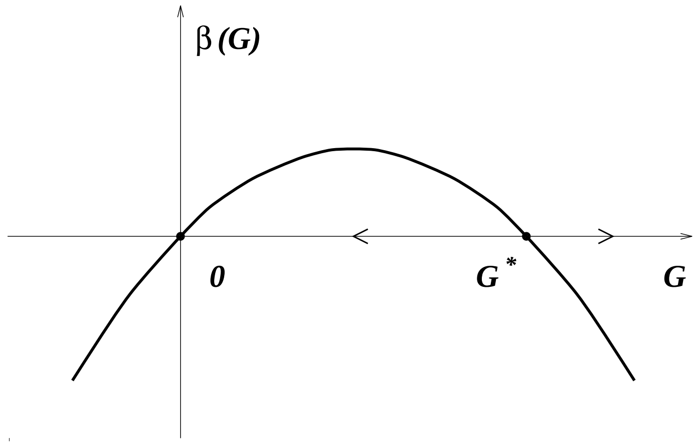



Fig. 19. The perturbative beta function for Einstein theory in $D = 2 + \epsilon$ dimensions

If this picture is valid for the four dimensional theory ($\epsilon = 2$), this would mean that gravity is "asymptotically safe" at short distance, thanks to the existence of a non-perturbative UV fixed point, which corresponds to a continuous phase transition between the weak coupling phase of gravity, with freely propagating gravitons, and a strong coupling phase, possibly with confined gravitons.

However this interpretation of the perturbative calculation should be considered with some caution. The existence of a non-trivial UV fixed point which allows to construct a quantum field theory out of a perturbatively non-renormalizable theory is confirmed for the non-linear sigma models in $2 + \epsilon$ dimensions, while for gauge theories in $4 + \epsilon$ dimensions, a similar phenomenon has never been observed, and in numerical simulations the deconfinement transition (which separates the weak coupling deconfined phase from the strong coupling confined phase) is found to be first order in dimensions $D \geq 5$. The problem of the unboundness of the action, which do not raise inconsistencies in the perturbative renormalization calculations, still precludes a rigorous non-perturbative treatment of these questions.

### *4.2. Simplicial 3D gravity*

One can use dynamical triangulations to discretize the functional integration over Euclidean 3-metrics (see sec. 2). Recently this approach has attracted much interest, in particular for numerical simulations. The basic idea is to start from some given initial $3d$ simplicial manifold, for instance with the topology of the 3-sphere $S_3$, and to explore the space of combinatorially equivalent SM by performing the Alexander moves described in sec. 2, or rather the $(1,4)$ and $(2,3)$ moves (see Fig. 8), which are more efficient in numerical simulations.

The possible discretized actions for the triangulation depend on the numbers of $i$-simplices $N_i$, $i = 1, 4$, but using the geometric relation $4N_3 - 2N_2$ and the fact that the Euler characteristic for odd dimensional manifolds is zero

$$\chi \;=\; N_0 - N_1 + N_2 - N_3 \;=\; 0 \quad , \tag{4.2.1}$$

one sees that there are only two quantities, for instance $N_0$ (number of vertices) and $N_3$ (number of tetraedra), which can vary independently. At variance with the two dimensional case, it is not possible to work in the canonical ensemble where the number if simplices is fixed. In fact, to go from a given small triangulation to another small one, one must pass in general through configurations with a large volume.

Let me also mention that one can think of generalizations of matrix models to generate $3d$ triangulations. The simplest idea is to consider tensorial models (see Gross (1992), Ambjørn et al. (1991)), where the field $T_{ijk}$ depends now of 3 indices, with a quartic interaction term (which is diagrammatically dual to a tetraedron) in the action

$$S(T) \;=\; \frac{1}{2}\, T_{ijk}\, T_{kji} \;-\; g\, T_{ijk}\, T_{jnl}\, T_{lmk}\, T_{imn} \quad . \tag{4.2.2}$$

However, the Feynman diagram for such models are in general not dual to $3d$ simplicial manifolds, but only to $3d$ pseudo-manifolds with non zero Euler characteristic. In addition, such models are not exactly solvable in the large $N$ limit like the matrix models.

In the DT approach the partition function is a sum over $3d$ triangulations $\mathcal{T}$ with a fixed topology, of the form

$$Z(\alpha,\beta) \;=\; \sum_{\mathcal{T}} \mathrm{e}^{\alpha N_0 - \beta N_3} \quad . \tag{4.2.3}$$

This is the discretized form of the functional integral over $3d$ Euclidean metrics given by eq. (1.1.1–2). Indeed $N_3$ corresponds to the volume of the manifold and $N_0$ is linearly related to the total scalar curvature (an exact formula can be written using the deficit angle formula (2.4.3)). Large positive $\alpha$ favors *positive* total curvature, while large negative $\alpha$ favors *negative* total curvature.

It is easy to show that $0 < N_0 \leq cst.\, N_3$. However, to show that this sum makes sense, the number of triangulations with a given topology and a fixed volume $N_3$ must not grow faster than $cst.^{N_3}$. In other words, the entropy of triangulations must be proportional to the volume. This ensures that the sum given by eq. (4.2.3) converges for $\beta$ (the cosmological constant) large enough, and that in MC simulations one reaches an equilibrium configuration for simulation time $t \to \infty$.

Contrarily to the two dimensional case, there is no mathematical proof that such a bound holds! Some plausibility arguments have been given by Ambjørn, Durhuus and Jónsson (Ambjørn et al. (1991)). Numerical estimates (Agishtein and Migdal (1991b), Ambjørn and Varsted (1991), Boulatov and Krzywicki (1991), Ambjørn et al. (1992a)) for triangulations with the topology of the 3-sphere and up to $N_3 \sim 10^4$ indicate that such a bound is valid and that $Z(\alpha,\beta)$ defines a series (in $\mathrm{e}^{-\beta}$) with a finite radius of convergence.

This result, which implies that the partition function $Z(\alpha,\beta)$ exists for $\beta > \beta_c(\alpha)$, and becomes singular at some critical value $\beta_c(\alpha)$ of the bare cosmological constant, is a first non-trivial outcome of numerical simulations. As for the $2d$ case, one expects that at $\beta_c$ the average volume $\langle N_3 \rangle$ diverges and that the renormalized cosmological constant vanishes. The next important issue is to understand this critical point, and whether it has some relationship with continuum $3d$ gravity.

For this purpose, one must study how the observables of the model depend on $\alpha$ along the critical curve $\beta = \beta_c(\alpha)$. The simplest observable is the total curvature $\langle N_0 \rangle$. The numerical simulations show the existence of two different phases. For $\alpha > \alpha_c$, where $\alpha_c \simeq 4.0$, $\langle N_0 \rangle$ is large and is found to be proportional to $\langle N_3 \rangle$ (cold phase). For $\alpha < \alpha_c$, $\langle N_0 \rangle$ is much smaller, and seems not to grow linearly with $\langle N_3 \rangle$, but rather as $\langle N_0 \rangle \propto \langle N_3 \rangle^\delta$ with $\delta \sim .6$ (hot phase). It is crucial to determine the nature of the transition between these two phases, in particular to see whether it is discontinuous (first order), or continuous, since in the latter case there should correspond a continuum field theory, with two relevant couplings ($\alpha$ and $\beta$), in analogy with the results of the perturbative calculations ($2+\epsilon$ expansion) discussed above. Unfortunately, from the most recent MC simulations, an hysteresis cycle is observed around $\alpha_c$ (see Fig. 20). This indicates that the transition is probably first order.

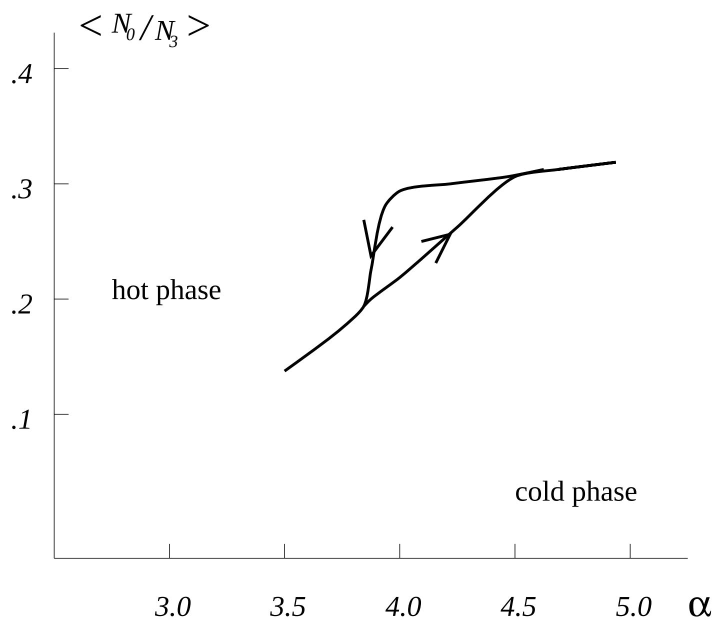



Fig. 20. Hysteresis cycle for the curvature at the hot/cold phase transition for 3d DT (redrawn from Ambjørn et al. (1992a), no error bars represented)

Let me propose a tentative interpretation of these results. In the cold phase the curvature is large and positive. It is tempting, by analogy with the $2d$ case, to assume that in this case branched polymer configurations are dominant. Such BP configurations for $3d$ $S_3$-like metric can be obtained very easily by starting from the smallest triangulation of $S_3$ (the boundary of the $4d$ simplex), and by iterating the $(1,3)$ move, i.e. by expanding one tetraedron into three tetraedra, in randomly chosen tetraedra. We have no proof that such configurations maximize $N_0$, but it is easy to see that asymptotically $N_0 \simeq \frac{1}{3} N_3$, which agrees with the numerical results for $\alpha > 4$. (see fig. 20).

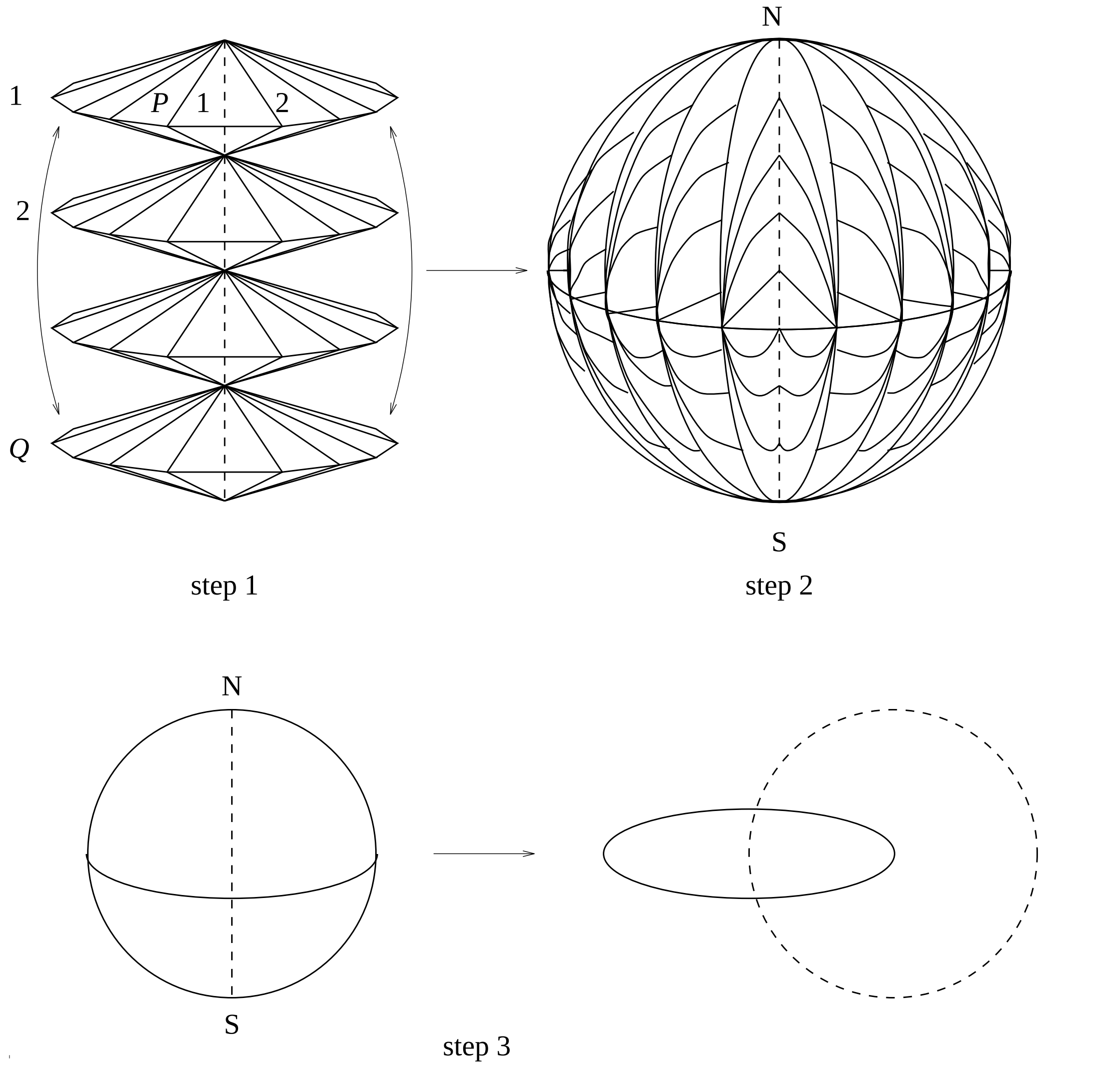



Fig. 21. A singular 3-sphere

In the hot phase the curvature is large and negative. We expect that in this case collapsed configurations such that a few links carry very large negative curvature dominate. Let me give a simple example of such a configuration, such that $N_0 \sim N_3^{1/2}$ (see fig. 21). Glue $P$ tetraedra along a common edge to form a "pancake" (step 1). Then stack $Q$ such pancakes to get a "ball" with an "equator" made of $P$ links and with $Q$ interior links between the "north" and "south" poles (step 2). Finally identify the north and south poles and the corresponding hemispheres to obtain a 3-sphere with $N_3 = PQ$ and $N_0 = P + Q$ (step 3). Such a configuration is in fact quite similar to the singular $2d$ triangulations that we discussed in subsec. 3.6. The two points with very large negative curvature are replaced here by two intertwinned loops.

If this still speculative interpretation is correct, this would mean that the phase diagram for the $3d$ DT model is somehow similar to the phase diagram of $2d$ gravity in the strong coupling regime $c \gg 1$. For positive Newton's constant the conformal instability would lead to branched polymer configurations with large positive curvature $R \propto a^{-2}$ (where $a$ is some UV cut-off), while for negative Newton's constant collapsed configuration with large negative curvature localized along defects (points in $2d$, loops in $3d$) are dominant. It is not clear to me if there is space for a weak coupling phase where the curvature is small, and such that the Einstein theory is recovered in the classical limit. Of course we expect that numerical simulations with larger systems will lead to a better understanding of the hot and cold phases and of the nature of the transition.

Finally let me mention that numerical simulations of Ising models on $3d$ dynamical triangulations have been performed recently (see Ambjørn et al. (1992b)). The first results lead to qualitatively similar behavior when $3d$ gravity is coupled to that kind of matter fields.

### *4.3. Simplicial 4D gravity*

The same techniques can be developed for $4d$ simplicial gravity. $4d$ simplicial manifolds are constructed by gluing 4-simplices along their $3d$ faces (tetraedra). Ergodic moves in the space of equivalent triangulations can be achieved by the three moves $(1,5)$, $(2,4)$ and $(3,3)$. The condition that 4-simplices are glued along common 3-simplices implies $5N_4 = 2N_3$. Using the fact that the boundary of the 4-ball formed by the 4-simplices which share a common vertex must be a 3-sphere (since the triangulation is a simplicial manifold), with Euler characteristic $\chi = 0$, one obtains, by summation over all the vertices, the additional constraint $2N_1 - 3N_2 + 4N_3 - 5N_4 = 0$. These two constraints, together with the Euler characteristic identity $\chi = N_0 - N_1 + N_2 - N_3 + N_4$, imply that in a given topological class of triangulations only two quantities (for instance the number of vertices $N_0$ and the volume $N_4$) can vary independently.

Numerical simulations have been performed by several groups for $4d$ DT with the topology of the 4-sphere ($\chi = 2$), with an action similar to that of eq. (4.2.3)

$$S = -\alpha N_0 + \beta N_4 \tag{4.3.1}$$

for triangulations with up to $N_4 \simeq 10^4$. The results are qualitatively similar to the $3d$ case. I refer to Agishtein and Migdal (1992), and to Ambjørn and Jurkiewicz (1992). One obtains a critical point at a finite value of $\beta$, indicating that the entropy of $4d$ triangulations still grows linearly with the volume. As the curvature coupling $\alpha$ vary one goes from a hot phase, where $\langle N_0 \rangle \ll \langle N_4 \rangle$, to a cold phase where $\langle N_0 \rangle$ grows linearly with $\langle N_4 \rangle$. The simulations on large systems do not indicate hysteresis at the transition between the cold and hot phases. Therefore it is possible that the transition is continuous, at variance with the $3d$ case. One also observes that in the cold phase, for large $\alpha$, $\langle N_0 \rangle \simeq .25 \langle N_4 \rangle$, which is consistent with a branched polymer interpretation where $N_0 = N_4/4$. However, finite size effects are expected to larger in $4d$ (a lattice with $10^4$ simplices has a diameter $< 10$ !) and it is clear that simulations on larger systems have to be performed, and will probably reveal unexpected phenomena.

### *4.4. 3D and 4D quantum Regge calculus*

I have no time to discuss here the results of numerical simulations using fixed triangulations with varying edges lengths. I refer to the recent reviews by Hamber (Hamber (1991a,b)). In these simulations also one observes a phase transition between a hot phase with large negative curvature and a cold phase with large positive curvature. The connection between these systems and the results of the DT is not yet very clear to me, and I think that the problems that I discussed for the two dimensional case still exist in higher dimensions.

## 5. Non-perturbative problems in 2D quantum gravity

In this last lecture I shall come back to two dimensional gravity and to matrix models, and discuss the issue of topology changes. This is interesting for two reasons. Firstly, topology changes are much simpler in two dimensions than in higher dimensions, since the topology is characterized by the genus, and explicit results can give hints about the higher dimensional models. Secondly, $2d$ gravity theories correspond to solutions of string theory in some simple backgrounds, and the topological expansion corresponds to the perturbative expansion around these backgrounds. Therefore one might hope to get a better understanding of string theories beyond perturbation theory through the matrix models. I shall mostly discuss here the one-matrix model, which allows to construct "pure gravity" ($c = 0$), and gravity coupled to non-unitary matter (described by $(2, q)$ minimal models in the classification of CFT).

### *5.1. The continuum double scaling limit*

To fix notations, let me recall how the so-called double scaling limit (Brézin and Kazakov (1990), Douglas and Shenker (1990), Gross and Migdal (1990)) is obtained for the $m = 2$ critical point, which corresponds to pure gravity (see the lectures by E. Brézin in this volume and Gross et al. (1992) for details). The Hermitian matrix integral (3.3.1)

$$Z \;=\; \int d\Phi \; \mathrm{e}^{-N\mathrm{Tr}\big(V(\Phi)\big)} \tag{5.1.1}$$

can be rewritten as an integral over the $N$ eigenvalues (e.v.)

$$Z \;=\; \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \, \Delta(\lambda_i)^2 \, \mathrm{e}^{-N\sum V(\lambda_i)} \quad ; \quad \Delta(\lambda_i) \;=\; \prod_{i<j} (\lambda_i - \lambda_j) \; . \tag{5.1.2}$$

In the planar limit $N \to \infty$, this integral is dominated by the "master field" configuration corresponding to a continuous distribution of e.v's described by the local density $\rho_0(\lambda)$, which extremizes the effective action (Brézin et al. (1978))

$$S_{\mathrm{eff}}[\rho] \;=\; \int \mathrm{d}\lambda \, \rho(\lambda) \, V(\lambda) \;-\; \int \mathrm{d}\lambda \int \mathrm{d}\mu \, \rho(\lambda) \, \rho(\mu) \, \ln|\lambda - \mu| \tag{5.1.3}$$

subjected to the normalization constraints

$$\int \mathrm{d}\lambda\, \rho(\lambda) \;=\; 1 \qquad ; \qquad \rho(\lambda) \geq 0 \quad . \tag{5.1.4}$$

The first term in eq. (5.1.3) represents the external potential $V$, the second term corresponds to the repulsion between e.v's. The fluctuations of the e.v. density are subdominant (of order $1/N^2$), and the dominant contribution to the free energy is just

$$F \;=\; \ln\big(Z\big) \;\simeq\; N^2\, S_{\mathrm{eff}}[\rho_0] \quad . \tag{5.1.5}$$

In the cases we shall be interested in, the potential $V$ is such that the e.v's are localized along a single segment $[a,b]$ on the real axis, and generically the e.v. density behaves at the end-points $a$ and $b$ as

$$\rho(\lambda) \;\propto\; |\lambda_e - \lambda|^{1/2} \tag{5.1.6}$$

($\rho$ is a deformation of the celebrated Wigner semi-circle distribution obtained for the quadratic potential $V = \lambda^2/2$).

The $m = 2$ critical point is reached when the potential is such that the e.v. density becomes singular at one of the end-points, $\lambda_e$, and behaves as

$$\rho(\lambda) \;\propto\; |\lambda_e - \lambda|^{3/2} \quad . \tag{5.1.7}$$

For the cubic potential of eq. (3.3.2), $V(\lambda) = \lambda^2/2 - g\lambda^3/3$, this occurs at $g_c = 2{\cdot}3^{-3/4}$. For $g < g_c$, i.e. close to the critical point, $\rho$ still has a square root singularity at the end-point $\lambda_e(g) < \lambda_c = \lambda_e(g_c)$, and the explicit solution shows that $(\lambda_c - \lambda_e) \propto (g_c - g)^{1/2}$.

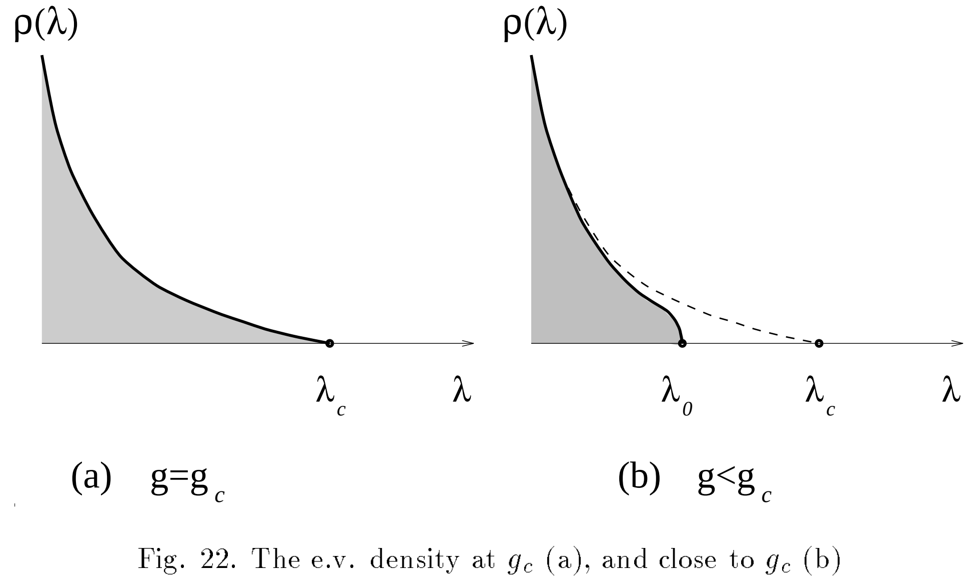



Fig. 22. The e.v. density at $g_c$ (a), and close to $g_c$ (b)

One obtains the double scaling limit by considering a small interval of width $\Delta\lambda$ around the critical end-point $\lambda_c$, and by rescaling the parameters $g \simeq g_c$ and $N \simeq \infty$ so that there is only a finite average number of e.v's $\langle n\rangle \sim \gamma^{-1}$ in this "critical interval". From eq. (5.1.7) this average number of e.v's in the critical interval $[\lambda_c - \Delta\lambda, \lambda_c + \Delta\lambda]$ is easily estimated to be of order

$$\langle n\rangle \;\simeq\; N\int_{\lambda_c-\Delta\lambda}^{\lambda_c} \mathrm{d}\lambda\,\rho(\lambda) \;\simeq\; N\left(\Delta\lambda\right)^{5/2} \quad . \tag{5.1.8}$$

Thus we require that $N$ scales as $\Delta\lambda \sim \left(N\gamma\right)^{-2/5}$. The size of the critical region for the coupling constant $g$ such that the end-point $\lambda_e$ stays in the critical interval $[\lambda_c - \Delta\lambda, \lambda_c + \Delta\lambda]$ is therefore $|g - g_c| \sim \Delta g \sim \left(\Delta\lambda\right)^2 \sim \left(N\gamma\right)^{-4/5}$.

Let us compare this with the continuum limit for the DT model (at fixed topology) discussed in subsec. (3.2). Since $g$ is identified with the bare cosmological constant $\lambda$ of eq. (3.2.7), the short-distance cut-off $a$ corresponds to the size of the critical region $\Delta\lambda$, and we can write the renormalized cosmological constant $\Lambda_R$, which will be denoted by $t_0$ in the following, as

$$t_0 \;\equiv\; \Lambda_R \;=\; a^{-2}\left(g_c - g\right) \quad , \tag{5.1.9}$$

while we rescale the e.v. position $\lambda$ as

$$p \;=\; a^{-1}\left(\lambda - \lambda_c\right) \quad . \tag{5.1.10}$$

$p$ corresponds to the "loop momentum", conjugate to the renormalized loop length $\ell$. To keep $1/\gamma$ e.v's in the critical interval $|p| < 1$ amounts to rescale $N$ (the total number of e.v's) as

$$\gamma^{-1} \;=\; a^{5/2}\,N \quad . \tag{5.1.11}$$

In the double scaling limit we let $a \to 0$ while $t_0$, $\gamma$ (and $p$) are kept finite. From the KPZ scaling prediction (given by eq. (3.1.10) with $\gamma_s = -1/2$ for the $c = 0$ pure gravity case), the term of order $h$ of the topological expansion of the partition function $F$, which is given by the sum over connected closed surfaces with genus $h$, is expected to behave for $g \simeq g_c$ as $(t_0^{5/2}\gamma^{-2})^{(1-h)}$, and the topological expansion for $F$ should become a series in the single scaling variable $t_0$

$$F(t_0) \;=\; \sum_{h=0}^{\infty} \gamma^{2(h-1)}\, f_h(t_0) \quad ; \quad f_h(t_0) \;=\; \left(t_0\right)^{5/2(1-h)} f_h \;. \tag{5.1.12}$$

Of course, in this simple case, the string coupling constant $\gamma$ can be reabsorbed in $t_0$. We keep it explicitly in order to simplify the discussion of the planar limit $\gamma \to 0$. In this limit the number of e.v's in the critical interval becomes infinite, and their fluctuations can be neglected.

*5.2. The Painlevé I string equation*

It is known since some time that there is a deep connection between the statistics of e.v's of random matrices and some integrable systems (see Jimbo et al. (1980) and the book by Mehta (1991)). One way to understand this connection is to reformulate the integral over the e.v's as a problem of non-interacting fermions in one dimension. Introducing the orthonormal polynomials $\pi_n(\lambda)$ for the measure $\mathrm{d}\lambda\ \mathrm{e}^{-NV(\lambda)}$,

$$\deg(\pi_n) = n \quad , \quad \int \mathrm{d}\lambda\ \mathrm{e}^{-NV(\lambda)}\, \pi_m(\lambda)\pi_n(\lambda) = \delta_{mn} \quad , \tag{5.2.1}$$

the Vandermonde determinant is proportional to the Slater determinant for the first $N$ polynomials

$$\Delta(\lambda_i) \propto \det\left(\pi_i(\lambda_j)\right) \quad , \tag{5.2.2}$$

and one can consider the "wave functions"

$$\Psi_n(\lambda) = \pi_n(\lambda)\ \mathrm{e}^{-\frac{N}{2}V(\lambda)} = \langle\lambda|n\rangle \tag{5.2.3}$$

as one-particle states for a system of free fermions. The vacuum state $|\mathbf{o}\rangle$ is obtained by filling the first $N$ levels, since there are $N$ e.v's. The correlations functions are encoded into the "position operator" $Q$, which acts by multiplication by $\lambda$, $\pi_n(\lambda) \mapsto \lambda\pi_n(\lambda)$, whose one-particle matrix elements are defined by

$$Q_{nm} = \langle n|Q|m\rangle = \int \mathrm{d}\lambda\ \mathrm{e}^{-NV(\lambda)}\, \lambda\pi_n(\lambda)\, \pi_m(\lambda) \quad . \tag{5.2.4}$$

$Q$ is Hermitian and real and elementary properties of orthogonal polynomials show that only its first subdiagonals are non-zero, i.e. $Q_{nm} = 0$ if $|n-m| > 1$. For instance the v.e.v. for the (Laplace transformed) one-loop operator $W(\lambda) = \mathrm{Tr}\big((\lambda-\Phi)^{-1}\big)$ is given by

$$\langle W(\lambda)\rangle = \langle\mathbf{o}|\frac{1}{\lambda - Q}|\mathbf{o}\rangle = \sum_{n=0}^{N-1}\langle n|\frac{1}{\lambda-Q}|n\rangle \quad . \tag{5.2.5}$$

The matrix elements of $Q$ satisfy the so-called "recursion relations" (Itzykson and Zuber (1980), Chada et al. (1981)), which follows from the Heisenberg algebra satisfied by $Q$ and the derivation operator $P$, $\Psi_n \mapsto \frac{\partial}{\partial\lambda}\Psi_n$. $P$ is antisymmetric ($P = P^t$), and from eq. (5.2.3), $P + \frac{N}{2}V'(Q)$ is upper triangular. Therefore it can be written as

$$P = \frac{N}{2}\left(V'(Q)_+ - V'(Q)_-\right) \tag{5.2.6}$$

where $A_{+(-)}$ is the upper (lower) triangular part of the operator $A$ ($\langle m|A_+|n\rangle = \langle m|A|n\rangle$ if $m \leq n$, 0 otherwise). The commutation relation

$$[P, Q] = 1 \quad , \tag{5.2.7}$$

together with eq. (5.2.6), gives non-trivial equations for the matrix elements of $Q$, which allow to compute them recursively. This commutation relation can be considered as an isomonodromic deformation condition on the linear system obtained from eq. (5.2.6), which is of the form

$$\frac{\partial}{\partial\lambda}\Psi - P\,\Psi = O \quad , \tag{5.2.8}$$

where $\Psi(\lambda, n)$ is a 2-components field constructed out of the $\Psi_n$'s (see Moore (1991)).

In the double scaling limit, we may wonder which "states" $|n\rangle$ contribute to the e.v's density $N\rho(\lambda) = \pi^{-1}\mathrm{Im}\big(\langle W(\lambda + i\epsilon)\rangle\big)$ for $\lambda$ in the critical interval $|\lambda - \lambda_c| < a$, i.e. for $|p| < 1$. It appears that these are the states which are close to the "Fermi surface", and more precisely those states $|m\rangle$ with level number $m$ such that $N - m < \big(\gamma\sqrt{\Delta\lambda}\big)^{-1}$. Therefore the number of such states, although small w.r.t. the total number $N$ of states, is large w.r.t. the average number of fermions in the critical interval. In the double scaling limit the label $m$ can be rescaled in a continuous parameter $x$

$$x = a^{-2}\left(1 - \frac{m}{N}\right) \quad . \tag{5.2.9}$$

For the $c = 0$ critical point one can start from the cubic potential ($\deg(V) = 3$). In the double scaling limit $Q$ will become a Hermitian differential operator of degree 2 (remember that only the first subdiagonals of $Q$ are non-zero), and from eq. (5.2.6) $P$ becomes an anti-Hermitian differential operator of at most degree 3. The explicit calculation shows that $Q$ takes the form

$$Q = \gamma^2\,\frac{\partial^2}{\partial x^2} - 2\,u(x, t_0) \tag{5.2.10}$$

(Douglas (1990)) and the recursion relations for $Q$ become the so-called "string equation" for $u$, which is the Painlevé I equation

$$-\frac{\gamma^2}{6}\frac{\partial^2 u}{\partial x^2} + u^2 = x + t_0 \quad . \tag{5.2.11}$$

The vacuum energy $F$ is explicitly given by the integral (defined by a finite part prescription at $\infty$)

$$F(t_0) = -\gamma^{-2}\int_0^\infty \mathrm{d}x\, x\, u(x, t_0) \quad . \tag{5.2.12}$$

Hence $u$ depends only on $x + t_0$ and $-u$ is the second derivative of the partition function $u(t_0) = -F''(t_0)$. The one-loop operator is (Banks et al. (1990))

$$\langle w(p)\rangle = \int_0^\infty \mathrm{d}x\,\langle x|\frac{1}{p - Q}|x\rangle \quad . \tag{5.2.13}$$

In the "classical limit" $\gamma \to 0$, one should recover the results of the large $N$ calculations. One has explicitly at leading order in $\gamma$

$$
\begin{aligned}
u(x,t_0) &= \sqrt{x+t_0} \quad , \qquad F(t_0) \;=\; -\gamma^{-2}\,\frac{4}{15}\,(t_0)^{5/2} \quad , \\
\langle w(p)\rangle &= -\gamma^{-1}\,\frac{1}{3}\left(t_0^{1/2}-p\right)\left(p+2t_0^{1/2}\right)^{1/2} \quad .
\end{aligned}
\tag{5.2.14}
$$

*5.3. Non-perturbative properties of the string equation*

The Painlevé I equation, that we rewrite by shifting $x \leftarrow x + t_0$ as

$$
-\frac{\gamma^2}{6}\,u'' + u^2 \;=\; x \quad , \tag{5.3.1}
$$

is the simplest second-order differential equation whose solutions have the so-called Painlevé property. Any solution of eq. (5.3.1) has an essential singularity at $x=\infty$, and is a meromorphic function in the whole complex plane, with only double poles as "movable singularities" (i.e. singularities depending on the initial conditions). These double poles have the same residue, and in fact at any pole $x_0$

$$
u(x) \;=\; \frac{\gamma^2}{(x-x_0)^2} + \mathcal{O}\big((x-x_0)^2\big) \quad . \tag{5.3.2}
$$

The initial condition at $+\infty$, $u(x) \simeq \sqrt{x}$, and the Painlevé equation are sufficient to specify uniquely all the terms of the asymptotic expansion of $u$ at large $x$

$$
\begin{aligned}
u(x) &= \sum_{k=0}^{\infty} u_k\,\gamma^{-2k}\,x^{\frac{1-5k}{2}} \\
u_0 &= 1,\ u_1 \;=\; -\frac{1}{48},\ u_2 \;=\; -\frac{49}{4608},\ \ldots
\end{aligned}
\tag{5.3.3}
$$

However, this is not sufficient to fix the whole solution. Indeed, let us assume that there exists two solutions $u$ and $\tilde{u}$ with the same asymptotics at $+\infty$. Linearizing eq. (5.3.2) for the difference $\delta u = \tilde{u} - u$ we obtain at leading order

$$
-\frac{\gamma^2}{6}\,\delta u'' + 2\sqrt{x}\,\delta u \;\simeq\; 0 \quad , \tag{5.3.4}
$$

whose solutions with growth slower than $\sqrt{x}$ are exponentially small

$$
\delta u(x) \;\propto\; x^{-1/8}\ \mathrm{e}^{-\gamma^{-1}\frac{4}{5}2\sqrt{3}x^{5/4}} \quad . \tag{5.3.5}
$$

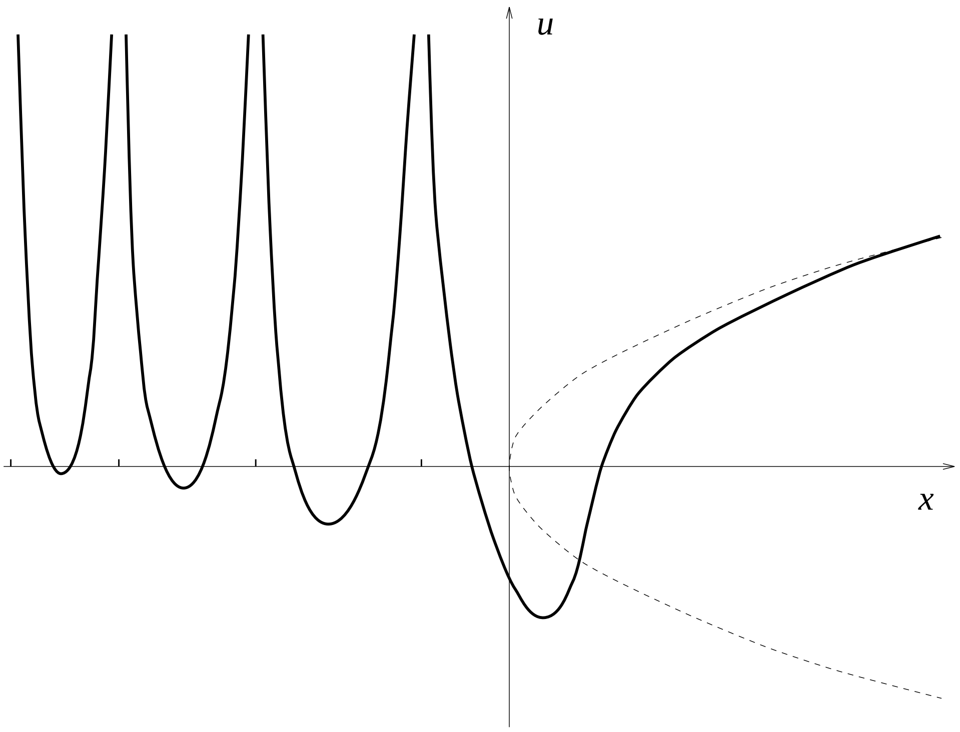



Fig. 23. Schematic view of a real solution of the $P_{\mathrm{I}}$ string equation

In fact a closer analysis of the Painlevé I equation shows that it has an infinite family of solutions with the same asymptotic expansion at $+\infty$ (see Boutroux (1913) and the book by Hille (1976)). These solutions differ, for instance, by the position of the largest double pole on the real axis. The difference between two solutions is exponentially small, from eq. (5.3.5), and is invisible in perturbation theory. This kind of phenomenon is typical of non-Borel summable series (see below). The analysis of the asymptotic behavior of $u$ for negative $x$ shows that the real solutions must necessarily have an infinite series of double poles along the negative real axis. Indeed, it is natural to rescale

$$X \;=\; \frac{4}{5}\, x^{5/4} \qquad , \qquad U(X) \;=\; x^{-1/2}\, u(x) \quad , \tag{5.3.6}$$

so that the Painlevé I equation becomes an equation for $U(X)$, which is asymptotic for $|X| \to \infty$ to an elliptic equation (I have set $\gamma = 1$)

$$U'' \,-\, 6\,U^2 \,+\, 6 \;=\; -\,\frac{U'}{X} \,+\, \frac{4}{25}\,\frac{U}{X^2} \;=\; \mathcal{O}(X^{-1}) \quad . \tag{5.3.7}$$

The generic solution when the r.h.s vanishes is a Weierstrass function $\wp(X)$ with a doubly periodic lattice of double poles in the complex plane. A careful analysis shows that if $u$ is asymptotic to $\pm\sqrt{x}$ as $x \to +\infty$, generically this analytic asymptotics is valid in two-fifth of the complex plane, namely for $-2\pi/5 < \mathrm{Arg}(x) < 2\pi/5$. In the remaining three sectors, which include the negative real axis, $u$ has necessarily an infinite number of poles. There is a *unique* solution, called the "triply truncated solution", which is asymptotic to $\pm\sqrt{x}$ in the larger "sector of analyticity" $-4\pi/5 < \mathrm{Arg}(x) < 4\pi/5$. This solution is real, but still has poles on the negative real axis, and has the wrong sign asymptotics $u \simeq -\sqrt{x}$ for our problem.

Going back to the "physical solutions" of the string equation, which must satisfy $u(x) \simeq \sqrt{x}$, we would like to understand the origin of these non-perturbative terms, and to know which physical constraint, if any, allows to choose between the different real solutions of eq. (5.3.1). The unavoidable existence of an infinite number of double poles of $u$ for negative $x$ is a serious problem for defining the resolvent $\langle x|(p-Q)^{-1}|y\rangle$, and the continuum limit for the loop operators (David (1990)).

### *5.4. Divergent series and Borel summability*

Asymptotic but divergent series expansions are very common in quantum mechanics and in quantum field theories. A mathematical recipe to make sense out of these series is the Borel summation method. I refer to Zinn-Justin (1983) and to Le Guillou and Zinn-Justin (1990) for details. It relies on the following Theorem:

*If the function $f(x)$ is analytic in the sector $\mathcal{S} = \{x : |\mathrm{Arg}(x)| < \alpha\pi/2,\ |x| < cst.\}$, and has a Taylor expansion $f(x) = \sum_{k=0}^{\infty} f_k x^k$, such that the following bound on the rest of the expansion holds uniformly for any $x$ in $\mathcal{S}$ and at any order $N$, for some $\beta < \alpha$*

$$\left| f(x) \ - \ \sum_{k=0}^{N} f_k\, x^k \right| \ \leq \ B\, A^{-N}\, (\beta N)!\, x^{N+1} \ , \tag{5.4.1}$$

*then $f$ is unique. It can be defined from its Borel transform (of order $\beta$)*

$$B(b) \ = \ \sum_{k=0}^{\infty} \frac{f_k}{(\beta k)!}\, b^k \quad , \tag{5.4.2}$$

*which is analytic in the domain $\{b : |b| < A \text{ or } |\mathrm{Arg}(b)| \leq \alpha/2\}$ by the Borel summation formula*

$$f(x) \ = \ \int_0^\infty \mathrm{d}t\ \mathrm{e}^{-t}\ B(xt^\beta) \quad . \tag{5.4.3}$$

*In fact, at least for $\beta = 1$, it is sufficient to have a sector with $\alpha = \beta$ (Nevanlinna-Sokal theorem).*

In general, in quantum mechanics and in quantum field theories, one has $\beta = 1$ (but $\beta$ may be smaller for theories with fermions, such as QED). The singularity of the Borel transform closest to the origin, $b_0$, is usually real and negative, and from eq. (5.4.3) the terms of the series $f_k$ grow factorially, but with alternate signs, as $f_k \propto k!(-1)^k|b_0|^{-k}$. These singularities are in fact related to the existence of possibly complex Euclidean solutions of the equation of motion with *finite action*, called *instantons*. Additional singularities of the Borel transform, usually called *renormalons*, arise from short distance divergences (for strictly renormalizable theories), and from large distance divergences (for perturbatively massless theories). I shall not discuss these renormalon singularities here (see Le Guillou and Zinn-Justin (1990)). As a crude argument, let us consider the Euclidean functional integral over field configurations $\phi$ for a theory with action $S(\phi)$, and coupling constant $g$ (which plays the role of $\hbar$). The partition function can be rewritten as

$$Z(g) \;=\; \int \mathrm{d}[\phi]\,\mathrm{e}^{-S(\phi)/g} \;=\; \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}b}{2\mathrm{i}\pi}\,\mathrm{e}^{-b/g}\;\mathcal{B}(b) \quad , \tag{5.4.4}$$

with

$$\mathcal{B}(b) \;=\; \int \mathrm{d}[\phi]\,\frac{1}{S(\phi)-b} \quad . \tag{5.4.5}$$

The function $\mathcal{B}$ is expected be analytic for negative $b$, but as $\mathrm{Re}(b)$ increases, it will become singular for $b = b_0 = S(\phi_0)$, where $\phi_0$ is the extremal configuration with lowest action. By appropriate contour deformations, one can analytically extend $\mathcal{B}$ into a function of $b$ with cuts starting at some $b_i = S(\phi_i)$ corresponding to saddle points configurations $\phi_i$ of $S$, with finite action. These classical solutions are the instantons. In the semi-classical limit $g \to 0$, the perturbative expansion for $Z(g)$ is obtained by moving to the right the contour over $b$ in eq. (5.4.5). The dominant contribution is given by the first cut, associated to the lowest action configuration—the classical vacuum— with action $S_0$. The Borel transform of the perturbative expansion around the vacuum is simply the discontinuity of $\mathcal{B}$ around the first cut

$$B_0(b) \;=\; \frac{1}{\pi}\,\mathrm{Im}\big(\mathcal{B}(b)\big) \quad , \quad Z(g) = \int_{S_0}^{\infty} \mathrm{d}b\;\mathrm{e}^{-b/g}\;B_0(b) \quad . \tag{5.4.6}$$

However, if the $\phi$ integration for $\mathcal{B}$ "catches" other instantons, the contributions of the other cuts associated to these instantons have to be added in eq. (5.4.6). Through the Stokes phenomenon, they will contribute to cuts in the $b$ plane for the perturbative Borel transform $B_0(b)$.

For simple theories, such as the massive scalar $\lambda\phi^4$ theory (with $\lambda > 0$ and in dimensions $D < 4$), there are no real instantons with positive action, the perturbative Borel transform $B_0$ is well-defined for $b > 0$, and the theory is Borel summable. The closest singularity of $B_0(b)$ is negative. The corresponding instanton is an imaginary solution of the equations of motion, or equivalently a real solution for the upside-down potential ($\lambda < 0$). However in theories with inequivalent topological sectors, such as non-abelian gauge theories, real instantons which describe tunneling between these vacua do contribute to the functional integral, and have to be taken into account. They will give non-perturbative contributions of order $\mathrm{e}^{-S_{\mathrm{inst}}/g}$.

Another situation where instantons are crucial is of course the case of theories with an action unbounded from below. Instantons describing the decay (through quantum tunneling) of the unstable perturbative vacuum give singularities of $B_0(b)$ on the positive real axis, but in this case there is in general no way to sum consistently perturbation theory. A simple 0-dimensional example is given by the integral

$$Z(g) \;=\; \int \mathrm{d}\lambda\ \mathrm{e}^{-\lambda^2/2+\sqrt{g}\lambda^3/3} \;=\; \sum_{k=0}^{\infty} g^k\ \frac{(6k)!}{3^{2k}(2k)!} \quad . \tag{5.4.7}$$

It describes a single particle in a cubic potential. The perturbative expansion is obtained by expanding around the metastable vacuum $\lambda_0 = 0$, but there is a second saddle point at $\lambda_1 = 1/\sqrt{g}$ corresponding to the particle at the top of the wall. This 0-dimensional instanton has an action $S_1 = 1/(6g)$, it gives a singularity of the Borel transform at $b = S_1$, and the coefficient of order $g^k$ behaves as $k!\,6^k$ (but with no alternate signs), so that the series of eq. (5.4.7) is not Borel summable. This is of course a consequence of the divergence of the integral as $\lambda \to +\infty$.

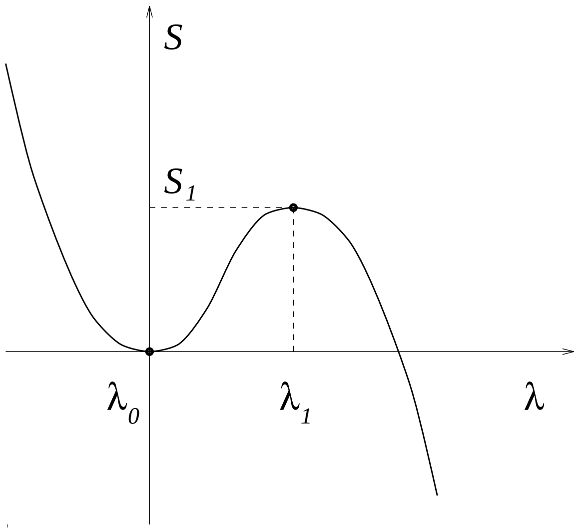



Fig. 24. A zero-dimensional instanton

The above integral defines nothing but the number of Feynman diagrams for a bosonic theory with cubic interaction. The fact that the total number of Feynman diagram with $k$ internal loops grows like $k!\,(constant)^k$ is a general feature of local quantum field theories. Moreover (see Rivasseau (1991b)), it is possible to show that the amplitude of a "typical" diagram at order $k$ grows also as $(constant)^k$ (not taking into account the UV and IR singularities that occur in some specific subclasses of diagrams, and which give rise to the renormalon singularities mentioned above). Thus it is the instanton with smallest action which governs the large order behavior of each term of the perturbative expansion.

### *5.5. Non-perturbative effects in 2D gravity and string theories*

For the $c = 0$ pure gravity theory, the form of the non-perturbative terms, given by eq. (5.3.5), indicates that the term of order $k$ of the topological expansion will not grow like $k!$, but like $(2k)!$ Indeed, the expansion parameter is $\gamma^2$, but the non-perturbative terms behave as $\exp(-\gamma^{-1}\,cst.)$, not as $\exp(-\gamma^{-2}\,cst.)$. This suggests to use a $\beta = 2$ Borel transform w.r.t. the string coupling constant $\gamma^2$, that is an ordinary Borel transform ($\beta = 1$) w.r.t. $\gamma$. The free energy $F(t)$ is for instance

$$F(t) \;\simeq\; \int_0^\infty \mathrm{d}b\; \mathrm{e}^{-b\gamma^{-1}t^{5/4}}\; B(b) \quad . \tag{5.5.1}$$

The existence of a one-parameter family of solutions of the string equation, with the same perturbative expansion, but which differ (from eq. (5.3.5)) by terms of order $\delta F \propto t^{-5/8}\exp\big(-\gamma^{-1}b_1 t^{5/4}\big)$, with $b_1 = 8\sqrt{3}/5$, implies that $B(b)$ has a singularity at $b_1$, and behaves as $(b_1 - b)^{-1/2}$, so that the perturbative expansion (5.1.12) for $F$ is not Borel summable, and that each term $f_k$ has the same sign and grows like

$$f_k \;\propto\; \Gamma(2k - \frac{5}{2})\; b_1^{-2k} \quad . \tag{5.5.2}$$

It is possible to understand this $(2k)!$ large order behavior, and the corresponding singularity of the Borel transform at $b_1$, in terms of an instanton effect, but in this case only one among the $N$ eigenvalues is tunneling (David (1991)). Indeed, in order to reach the $m = 2$ critical point, one must start from a potential $V$ unbounded from below in eq. (5.1.2), and for $N$ large but finite the integral over the $N$ real eigenvalues is divergent (but it can be used as a formal generating functional of the topological expansion). Looking at the variation of the effective action given by eq. (5.1.3) when one eigenvalue is moved from its equilibrium position to some $\lambda$ outside the support of e.v's, (this amounts to change the distribution density by $\delta\rho(\mu) = N^{-1}(\delta(\mu-\lambda) - \delta(\mu-\lambda_0)$ with $\lambda_0 \in \mathrm{Supp}(\rho)$), one obtains the effective potential $\Gamma(\lambda)$ for one eigenvalue in the background created by the $N-1$ remaining e.v's, which are supposed to stay at equilibrium. $\Gamma$ is given by

$$\Gamma(\lambda) \;=\; N\left[V(\lambda) \,-\, 2\int \mathrm{d}\mu\,\rho(\mu)\,\ln|\lambda-\mu|\right] \quad . \tag{5.5.3}$$

The saddle point equation for $\rho$ is equivalent to the equilibrium condition that the force

$$f(\lambda) \;=\; -\,\Gamma'(\lambda) \;=\; N\left[-\,V'(\lambda) \,+\, 2\,⨍\, \mathrm{d}\mu\,\frac{\rho(\mu)}{\lambda-\mu}\right] \tag{5.5.4}$$

must vanish wherever $\rho(\lambda) \neq 0$. This implies that $\Gamma = \mathit{constant}$ on the support of $\rho$. This condition, together with the normalization and positivity constraints of eq. (5.1.4), fixes the eigenvalue distribution $\rho$.

In the double scaling limit, only the critical neighborhood of the e.v. distribution end-point is important. If we perform the rescalings given by eqs. (5.1.9–11) and take $\gamma \to 0$, i.e. the number of e.v's in the critical region to be large, we obtain the force, which is given by $f(p) = 2\,\mathrm{Re}\langle w(p)\rangle$, with the loop operator $\langle w(p)\rangle$ given by eq. (2.2.13). Integrating w.r.t. $p$ we obtain the expression for the effective potential

$$\Gamma(p) \;=\; \mathrm{Re}\left(\gamma^{-1}\,\frac{4}{15}\left(3t_0^{1/2}-p\right)\left(p+2t_0^{1/2}\right)^{3/2}\right) \quad . \tag{5.5.5}$$

The shape of $\Gamma$, together with that of the density $\rho(p) = \mathrm{Im}\langle w(p-\mathrm{i}\epsilon)\rangle/2\pi$, is depicted for $t_0 > 0$ on Fig. 25.

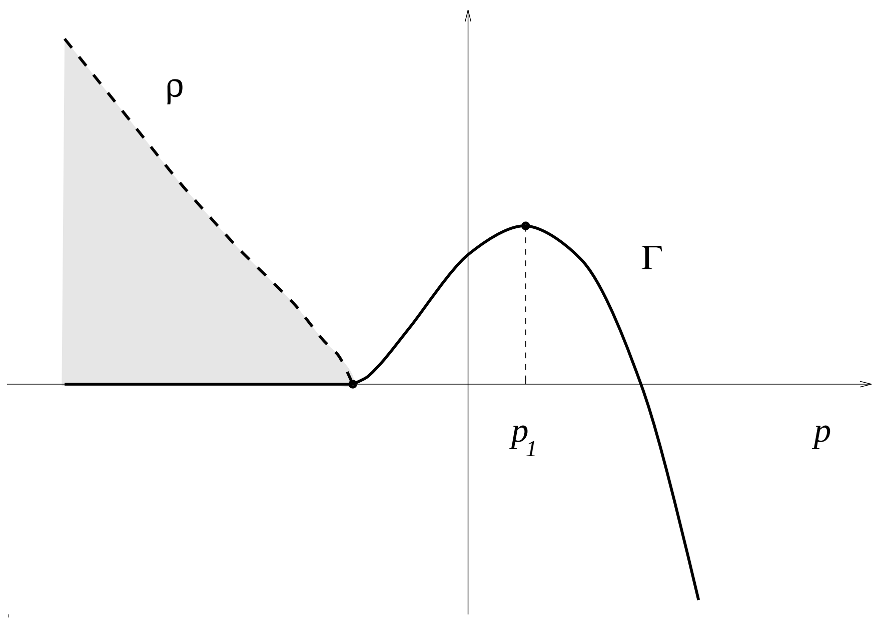



Fig. 25. One e.v. effective potential $\Gamma$ (black) and the density $\rho$ (dashed).

Perturbation theory is constructed by taking into account the fluctuations of the e.v's around the "classical vacuum" where all the e.v's are on the constant potential half-line $p < p_0 = -2t_0^{1/2}$ One may consider the instanton configuration where *one* eigenvalue sits at the top of the potential well $p_1 = t_0^{1/2}$, while the others stay in the classical vacuum. At leading order in $\gamma$ one may neglect the backreaction of this e.v. on the background configuration, and the action for this configuration is

$$\Gamma_{\mathrm{inst}} \;=\; \Gamma(p_1) \;=\; \gamma^{-1}\frac{8\sqrt{3}}{5}\,t_0^{5/4} \quad , \tag{5.5.6}$$

which corresponds exactly to the strength of the non-perturbative effects given by eq. (5.3.5) in the string equation.

In order to estimate the corrections to this behavior, for instance the $t_0^{-5/8}$ power correction, one needs to take into account the backreaction of the e.v. on the bulk. It is also expected that higher orders non-perturbative terms have a similar interpretation in terms of "multi-instantons", i.e. of several e.v's sitting on the top of the effective potential.

This $(2k)!$ large order behavior is expected to be a very general feature of string theories (Shenker (1991)). Indeed, in the same way as the behavior of the single integrals corresponding to 0-dimensional field theories (such as that of eq. (5.4.7)) allow to recover the $k!$ growth of the number of Feynman diagrams with $k$ loops in quantum field theories, simple matrix models allow to estimate the volume of the moduli space $\mathcal{M}_g$ of closed Riemann surfaces with genus $g$. For instance, pure gravity is equivalent to the Liouville+ghosts system and the term of order $k$ of the topological expansion of $F$ can be written as

$$F_k \ = \int_{\mathcal{M}_k} \mathrm{d}\mu[m]\, Z_{\mathrm{Liouville}}[m]\, Z_{\mathrm{ghosts}}[m] \quad , \tag{5.5.7}$$

where $d\mu[m]$ is the natural measure (the Weil-Peterson measure) over the moduli $m$, and $Z_{\dots}[m]$ are the functional determinants of the fields living on the surface. For the $c < 1$ theories this integral is expected not to diverge on the boundary of $\mathcal{M}_k$ (but tachyonic divergences will occur for $c \geq 1$), and to be of the order of $\mathrm{Vol}(\mathcal{M}_k) = \int_{\mathcal{M}_k} d\mu[m]$. This volume can be estimated directly by some random matrix models that produce triangulations of the moduli spaces (Harer and Zagier (1886), Penner (1987)), and by topological field theory (Witten (1990)).

It is expected that for the other $c \leq 1$ non-critical string models, the non-perturbative effects can also be understood in terms of one e.v. instantons. This has been checked explicitly for the $c = 1$ model. This means that the field configurations that govern the non-perturbative effects in string theory cannot be understood in terms of the low energy field-like modes of the string (tachyon, graviton, dilaton, ...), which would give a $k!$ behavior, but are really "stringy" collective excitations.

### *5.6. Various stabilization proposals*

We have seen on the example of the $m = 2$ critical point of the one-matrix model that instantons correspond to the tunneling of e.v's into the infinitely deep potential well. This means that the "classical vacua" is metastable, and that the theory has no real stable ground state. This observation can be extended to the matrix models which describe $2d$ gravity coupled to $c \leq 1$ unitary matter (Ginsparg and Zinn-Justin (1991). Some non-unitary models appear to be stable, and the topological expansion still behaves as $(2k)!$, but is Borel summable, because of sign alternations of the matter partition function $Z_{\mathrm{matter}}$ with the genus $k$. Thus, for the physically relevant models of non-critical strings, non-perturbative effects point out a serious problem of "unboundness of the action", not a simple tunneling between inequivalent—but stable—vacua as for gauge theories.

Several attempts to modify the formulation of these theories, without changing their perturbative expansion, have been made, in order to "stabilize" the functional integral. I shall discuss briefly some of them.

#### *Stochastic quantization*

Marinari and Parisi (1990) suggested to treat the one-matrix model as a one dimensional supersymmetric matrix model, through stochastic quantization. It is known that in general, the average of any quantity $Q(\Phi)$ w.r.t. a probability density of the form $\mathrm{d}\Phi \exp(-V(\Phi))$ can be rewritten as the ground state expectation value

$$\langle Q\rangle \;=\; \int \mathrm{d}\Phi\; \mathrm{e}^{-V(\Phi)} \;=\; \langle 0|Q|0\rangle \tag{5.6.1}$$

of the observable $Q(\Phi)$ for the quantum mechanical system with Hamiltonian

$$H_B \;=\; P^2 + V_B(\Phi) \quad , \quad V_B \;=\; \frac{1}{4}\left(\frac{\partial V}{\partial\Phi}\right)^2 - \frac{1}{2}\,\frac{\partial^2 V}{(\partial\Phi)^2} \quad , \tag{5.6.2}$$

with $P = i\partial/\partial\Phi$. The time variable for this quantum mechanical system is the Langevin time when one uses stochastic quantization to compute $\langle Q\rangle$, and $H_B$ is the associated Fokker-Planck Hamiltonian. It is the restriction to the bosonic sector of a supersymmetric Hamiltonian $\boldsymbol{H} = \boldsymbol{Q}^2$, where $\boldsymbol{Q}$ is the SUSY generator. When the potential is bounded from below, so that the probability density in eq. (5.6.1) exists, SUSY is unbroken, the ground state wave function is $\Psi(\Phi) = \exp(-V(\Phi)/2)$ and has zero energy. If $V$ in unbounded from below, the Hamiltonian $H_B$ still makes sense, and it was proposed by Greensite and Halpern (1984) that it allows to define a quantum theory for systems with classical action unbounded from below. When applied to the one-matrix model, one obtains indeed a stable one-dimensional matrix model with variable $\Phi(t)$, where $t$ is the Langevin time. It has the same topological expansion than the initial unstable zero-dimensional matrix model, but differs by non-perturbative contributions. These corrections come from the fact that, when the potential is unbounded from below, SUSY is spontaneously broken, and the vacuum energy is positive, but exponentially small.

The properties of this model have been studied by several groups. I refer to Karliner and Migdal (1990), Ambjørn et al. (1990), Miramontes et al. (1991) and Dabholkar (1990) for details. The main question raised by this proposal is that the KdV flow structure, the loop equations and the Virasoro constraints of the original model have disappeared. It is not clear what replaces them, and what is the interpretation of this new theory in terms of strings.

*Complex matrix models.*

In Dalley et al. (1992) it is shown that by using complex (non-Hermitian) matrices $M$ and an action of the form $\mathrm{Tr}\big(V(MM^\dagger)\big)$ one can construct well-defined models which have, in the double scaling limit, the same perturbative expansion than that of the ordinary model. The basic reason for this phenomenon is that, when expressed in terms of an integral over the eigenvalues of the matrix $MM^\dagger$, the partition function can be written exactly as that of eq. (5.1.2), with the restriction that the e.v's must be *positive* (see Morris (1991)). This allows, when adjusting the potential $V$, to put a “wall” at a finite distance from the critical end-point of the e.v. distribution, which forbids the e.v's to fall, but which is invisible in perturbation theory. The string susceptibility $u(x)$ is now found to obey a string equation which is related to the KdV hierarchy, but which is now $4^{\mathrm{th}}$ order instead of $2^{\mathrm{nd}}$ order. A problem with this approach is that an additional non-perturbative parameter appears, which corresponds to the distance between the wall and the end-point. The loop equations are also broken by a term related to the disappearance of one of the Virasoro constraints.

*Complex solutions*

Finally, let me mention that one can stay with the one-matrix model and define the e.v. integral given by eq. (5.1.2) by choosing as integration contour not the real axis but a complex contour such that as $|\lambda| \to \infty$ the potential $V(\lambda)$ is real and *positive*. For the cubic potential this amounts to take, for instance, the contour $\mathcal{C}_+$ going from $-\infty$ to $e^{i\pi/3}\,\infty$. See David (1990), Sylvestrov and Yelkovsky (1990), David (1991). It is possible to show that with this choice, in the double scaling limit, one recovers a triply truncated *complex* solution of the Painlevé I string equation, which is asymptotics to $\sqrt{x}$ for $-6\pi/5 < \mathrm{Arg}(x) < 2\pi/5$, and which has an infinite network of double poles in the remaining sector $2\pi/5 < \mathrm{Arg}(x) < 4\pi/5$. This solution, and the solution associated to the complex conjugate contour $\mathcal{C}_-$, can be obtained from the perturbative series by a well-defined Borel summation prescription (eq. (5.5.1)), where the contour of integration over the Borel variable $b$ is taken above (or under) the instanton singularities located on the real axis. Of course this summation prescription violates the physical prescription of reality of the amplitudes.

## 6. Conclusion

I have not discussed many important issues in these lectures. The $c = 1$ matrix models solutions, and their relationship with $2d$ string theories and other conformal field theories, for instance those describing $2d$ black holes, have raised much interest during the last two years: see for instance Gross et al. (1992) and Polyakov (1992). Another important problem is to extend what has been done for $2d$ gravity to $2d$ supergravity theories, with the aim of obtaining exact solutions and non-perturbative information for superstring theories. Some progresses have been made recently (see the lectures by L. Alvarez-Gaumé), but one does not have yet a simple geometrical discretization procedure of superspace similar to random lattices for ordinary space. Another interesting problem would be to construct discrete models and/or exact solutions corresponding to space-time with Lorentzian signature $(-+)$.

For higher dimensions, in my opinion much remains to be done. I hope that in the next years large scale numerical simulations and new theoretical developments will help us to understand the nature of the different phases of these higher dimensional models, the importance of conformal modes, the connection between the different lattice formulations, and with the topological theories in three and higher dimensions.

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