THE DYNAMICAL CRITICAL EXPONENTS OF THE SWENDSEN-WANG ALGORITHM FOR THE 2D ISING MODEL

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Abstract

Cluster update algorithms can greatly reduce critical slowing down in computer simulations of spin models. In the case of the Ising model in two dimensions, the dynamical critical exponent z is very small, and there is some evidence that it may in fact be zero. We are performing high statistics measurements of z using the cluster algorithm of Swendsen and Wang on a number of parallel computers, to try to distinguish between the two possibilities.

1. Introduction

Monte Carlo simulations of spin models have traditionally used local algorithms such as that of Metropolis *et al.* [1] . These algorithms have the major drawback that near a phase transition the number of sweeps needed to generate a statistically independent configuration increases approximately as the square of the linear size of the lattice. New algorithms have recently been developed which dramatically reduce this 'critical slowing down' by updating clusters of spins at a time. The first such algorithm was proposed by Swendsen and Wang (SW) [2] , and was based on an equivalence between the Potts spin model [3] and a percolation model [4] , for which cluster properties play a fundamental role. The q-state Potts model consists of a lattice of spins σ_i which can take q different values, and whose Hamiltonian is

$$H = K \sum_{<i,j>} \delta_{\sigma_i \sigma_j}. \tag{1}$$

By introducing bonds between neighboring sites, with probability

$$p(\sigma_i, \sigma_j) = \delta_{\sigma_i \sigma_j}(1 - e^{-K}), \tag{2}$$

clusters of bonded spins are created. The SW algorithm consists of generating all such clusters, then choosing a random spin value for each cluster and assigning it to all the sites in that cluster.

2. Autocorrelation measurements

We have made a high statistics study of the SW algorithm for the Ising model (i.e. the $q = 2$ Potts model) in two dimensions. We have measured the autocorrelations in the energy (a local operator) and magnetization (a global one). The normalized autocorrelation function for an operator A is given by

$$\rho(t) = \frac{< A(0)A(t) > - < A(0) >^2}{< A(0)A(0) > - < A(0) >^2}. \tag{3}$$

At large t this should go as $e^{-t/\tau_{exp}}$, where τ_{exp} is the exponential autocorrelation time. The quantity which is actually relevant to the error in A is the integrated autocorrelation time [5]

$$\tau_{int} = \frac{1}{2} + \sum_{t=1}^{\infty} \rho(t). \tag{4}$$

Both the integrated and exponential autocorrelation times have an associated exponent z governing critical slowing down given by $\tau(L) \sim L^z$, where LxL is the size of the lattice and $\tau(L)$ is measured at the critical point.

The autocorrelations were measured at the infinite volume critical point (which is known exactly for the $2D$ Ising model) over runs of typically 10^6 sweeps, and 10 or 20 such runs were done for each of the different lattice sizes, from 16^2 up to 512^2. The results from the multiple runs were averaged to give the overall autocorrelation function $\rho(t)$. The autocorrelation time τ_{exp} was extracted by a fit to $log\ \rho(t)$ in a region where it was approximately linear, typically somewhere between τ_{exp} and $3\tau_{exp}$. τ_{int} was calculated by summing $\rho(t)$ for t less than some cutoff value, and using the exponential fit to $\rho(t)$ for the remaining part of the sum, which is then just a geometric series and so can be summed analytically. The cutoff was taken to be the largest point used in the exponential fit, which was usually between $2\tau_{exp}$ and $3\tau_{exp}$. This method is similar to that used by Wolff [6] . The above analysis was also done for each of the 10 or 20 multiple runs, which provided an estimate of the error in the autocorrelation times.

3. Computations

The simulations were performed on a number of different parallel computers. Standard update algorithms (such as Metropolis) can be easily and efficiently parallelized or vectorized, since they are regular and local. However in the case of the new cluster algorithms, the highly irregular and non-local nature of the clusters means that they are much more difficult to parallelize efficiently, and do not appear to vectorize at all. On this problem a CRAY X-MP is only about ten times faster than a SUN4 workstation (CRAY timings are taken from ref. [6]).

For lattice sizes less than about 256^2, it is possible to run independent simulations on each processor of a parallel machine, enabling us to obtain 100% efficiency by running the 10 or 20 runs for each lattice size in parallel. These calculations were done using a 32 processor Meiko Computing Surface, a 20 processor Sequent Symmetry, a 20 processor Encore Multimax, and a 96 processor BBN GP1000 Butterfly, as well as a network of SUN workstations.

For the largest lattice sizes, 256^2 and 512^2, a parallel cluster algorithm is needed, due to the large amount of calculation (and memory) required. We have used an algorithm which gives fairly good efficiencies of about 70% on reasonable numbers of processors (64 processors for the 512^2 lattice, 32 for 256^2) for the machines we have used, an NCUBE-1 and a Symult S2010 [7] . Using all 512 nodes of the NCUBE, by running multiple 64 or 32 processor simulations, gives a performance approximately five times that of the CRAY, while all 192 nodes of the Symult is equivalent to about six CRAYs for this problem.

4. Results

Results for the integrated and exponential autocorrelation times of the SW algorithm are given in Table 1. We do not yet have sufficient statistics on the 512^2

lattice to reliably extract autocorrelation times, so only data for lattice sizes up to 256^2 are shown. The energy and magnetization have very similar values of τ_{exp} (and thus of z_{exp}), however the magnetization decorrelates faster than the energy, giving smaller values of τ_{int}, and smaller values of z_{int} as well. The results for z are shown in Table 2, along with the chi-squared values for the power law fit. Also shown are the chi-squared values for a fit to a logarithm, which corresponds to $z = 0$. These fits are somewhat subjective, since power law (or logarithmic) behavior is only expected *asymptotically*, so there is some uncertainty as to which data points should be included in the fit (the same is true in fitting the autocorrelation function to obtain τ, which will also affect the chi-squared values). In fact the slopes of lines connecting successive points in a log-log plot of L vs. τ (i.e. the approximations to z using only two lattice sizes) tend to smaller values as L increases. The difficult problem is to determine whether this trend eventually disappears, resulting in a non-zero value of z, or continues, signifying logarithmic behavior. The fits we have done include only data for $L = 50$ and above, which means that we are only fitting to five data points, which will also affect the reliability of the chi-squared values.

L	τ_{exp} E	τ_{exp} M	τ_{int} E	τ_{int} M
16	3.314(13)	3.281(16)	3.258(4)	2.956(5)
32	4.117(18)	4.094(18)	4.016(5)	3.468(7)
50	4.740(9)	4.709(25)	4.585(5)	3.811(6)
64	5.081(21)	5.002(28)	4.899(10)	3.986(7)
100	5.778(42)	5.707(49)	5.510(17)	4.304(12)
128	6.160(38)	6.105(53)	5.873(16)	4.509(10)
256	7.434(35)	7.354(49)	6.974(15)	5.023(15)

Table 1. Autocorrelation times for the energy (E) and magnetization (M) for the $2D$ Ising model using the SW algorithm.

	z	χ^2 power	χ^2 log
E exp	0.276(4)	0.4	6.4
E int	0.258(3)	2.2	15.5
M exp	0.275(5)	0.5	4.2
M int	0.171(3)	2.7	0.9

Table 2. Dynamical critical exponents for the exponential and integrated autocorrelations in the energy and magnetization. Also shown are the chi-squared values for the power law fit, as well as for a logarithmic fit.

In their original paper [2], Swendsen and Wang measured the autocorrelations on fairly small lattice sizes, and thus obtained a rather high value of 0.35(1) for z. Wolff [6] has also measured z_{int} for the Ising model using the SW algorithm, and found values of 0.27(2) and 0.20(2) for the energy and susceptibility (the square of the magnetization) respectively. This agrees well with our data. Burkitt and Heermann [8] were the first to suggest that the increase in the autocorrelation time is logarithmic. They present data only for the magnetization, which we find to show

the best evidence for logarithmic behavior, at least for the integrated autocorrelations. However there is no reason to suppose that the values of z are the same for the energy and the magnetization, and our data suggest that z is non-zero for the energy.

5. Conclusions

It is very difficult to distinguish whether the autocorrelation times for the $2D$ Ising model using the SW algorithm increase logarithmically in L, or as a small power. We still do not have enough data to decide the matter conclusively, although we will soon have enough statistics to extract reliable numbers for a lattice of size 512^2, which should allow us to distinguish between the two possibilities. However so far it seems likely that z is zero for the integrated autocorrelations in the magnetization, while in all the other cases considered the exponent has a non-zero value, with z_{exp} being the same (approximately 0.275) for both the energy and the magnetization. In any case, the performance of the cluster algorithms for the $2D$ Ising model is quite extraordinary, giving autocorrelation times of less than 8 on a 256^2 lattice, which is hundreds of times smaller than for the Metropolis algorithm.

References

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