RANDOM SURFACES AND QUANTUM GRAVITY: LOOKING FOR THE EMERGENCE OF CONTINUUM THEORIES FROM TRIANGULATED MANIFOLDS

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ABSTRACT

We investigate the possible existence of continuum theories originating from lattice theories defined on triangulated manifolds. We discuss the $2d$ case, considering theories with extrinsic curvature coupled to embedding bosonic matter fields, and the $4d$ case, where we find that the effect of a non-trivial measure is likely to be relevant.

1. Introduction

In this note we will discuss the output of two related fields of investigation [1, 2]. Both of them are based on the hope that a critical point may allow a continuum theory to emerge out of a discrete lattice theory defined on a (diffeormosphism invariant) triangulated lattice. In the $2d$ case one hopes that the addition to the *free* surface Hamiltonian of an extrinsic curvature dependent term could favour the emergence of a non-trivial continuum theory. In $d = 4$ the hope is that a non-trivial critical point could emerge already from the discretized form of the simple Einstein action. In this case maybe self-consistency of the theory (for example the simple demand of the emergence of a $4d$ manifold at large distances) could help in solving the key point of the determination of the correct measure factor. In both cases the tools used are from Statistical Mechanics combined with a very detailed analysis of the features of the cross-over region.

2. $2d$ Random Surfaces with Extrinsic Curvature

String theory, in a number of guises, has been conjectured to describe the underlying fundamental physics of a wide variety of physical phenomena and models. These include the strong interaction at long distances, the three-dimensional Ising model and unified models incorporating gravity. In its simplest form, the bosonic string, it is a theory of free fluctuating surfaces. The functional integral for the Euclideanized bosonic string is just the partition function for an ensemble of random

[1] Talk presented by E. Marinari

fluctuating fluid surfaces. Such surfaces are also ubiquitous in nature, being found for example in macro-emulsions and the lipid bilayers that form an important part of cell membranes [3]. These systems are fluid because their component 'molecules' are loosely bound. Their constituents are arranged so that the net surface tension (nearly) vanishes; thus these membranes are subject to large thermal fluctuations. In one important respect, however, these chemical/biological membranes differ fundamentally from the surfaces we discuss and simulate; they are self-avoiding. The worldsheets of the bosonic string, in contrast, generically self-intersect.

The bosonic string for surfaces embedded in 26 dimensional space has been studied extensively. Much progress in numerically simulating strings has been made through the use of Dynamically Triangulated Random Surfaces (DTRS)[6, 7]. For theories of surfaces embedded in $D \leq 1$, analysis of the continuum Liouville theory [4, 5] and of matrix models has led to consistent predictions for critical exponents and correlation functions. In the 'double scaling limit', in fact, exact nonperturbative solutions [8] have been found from these matrix models; the functional integral over surfaces has been essentially summed over all genera.

These analytical techniques have failed, for the most part, in probing the theory of random surfaces in the more interesting embedding regime $D > 1$. Indeed the formulas for critical exponents computed in Liouville theory give complex results when continued to the range $1 < D < 25$.[2] The matrix models describing $D > 1$ strings have so far been too complicated to solve.

Monte Carlo simulations for $D > 1$ [10, 11, 12, 13] indicate that these theories do not appear to describe the fluctuations of two dimensional *smooth* surfaces in the continuum limit. Extremely spiky, branched-polymer-like configurations with high (perhaps infinite) Hausdorff dimension dominate the functional integral[3].

Evidence for this pathology was obtained, for instance, when it was shown that the bare string tension, essentially the amount of work per unit area needed to perturb a boundary loop on these configurations, cannot vanish at the critical point [14]. (This implies that the renormalized string tension diverges in the continuum limit.) It has been speculated in ref. [15] that the proliferation of vortex configurations of the internal geometry (the 'Liouville mode') induces the degeneration of these surfaces in the embedding space. A related explanation that is often proferred is that a negative mass-squared particle[4], which comes on shell in the string theory for $D > 1$, creates instabilities which are made manifest by these singular configurations.

The tachyon, and apparently these related instabilities, can be eliminated in particular cases by introducing fermionic coordinates and supersymmetry on the worldsheet, and implementing an appropriate projection of states. Presumably, the fermions effectively smooth out the surfaces. This would be consistent with what has been observed for one-dimensional geometries; the random walk of a spin one-half particle has Hausdorff dimension one and thus appears to be smooth [16]. Many authors have proposed an alternative modification of the string action [17, 18, 19, 20] via the addition of a term that directly suppresses extrinsic curvature[5].

To write down our action we introduce an explicit parametrization of a generic surface $\mathcal{M}$ in R^3 with coordinates (σ_1, σ_2) and the embedding $X^\mu(\sigma_i)$. μ runs from 1 to 3 (since we only study the case of a $3d$ embedding space). The induced metric (the pullback of the Euclidean R^3 metric via the embedding) is given by

$$h_{ij} = \partial_{\sigma_i} X^\mu \partial_{\sigma_j} X_\mu \ . \tag{1}$$

We will use Greek letters for the embedding space indices; they can be raised and lowered at will since our background space is flat. Associated with each point in $\mathcal{M}$ are tangent vectors ($t_i^\mu \in T\mathcal{M}$)

[2] More precisely, one encounters these instabilities in Liouville theory when the quantity $c - 24\Delta > 1$, where c denotes the central charge of the matter theory which describes the embedding of the surfaces and Δ is the conformal weight of the lowest weight state in this theory [9]. Since here we are considering flat space, $c = D$ and $\Delta = 0$.

[3] In the same way singular configurations dominate the Gaussian theory, which is essentially a theory of free random walks.

[4] In some contexts, this is referred to as the 'tachyon'.

[5] In fact, integrating the fermions out of the Green-Schwarz superstring yields an action similar to the one we consider, but with the addition of a complex Wess-Zumino type term [21].

and a normal vector $n^\mu \in T\mathcal{M}^\perp$. The extrinsic curvature matrix K_{ij} (the second fundamental form) can be defined by

$$\partial_i n^\mu = -K_{ij} t^{\mu j} \ . \qquad (2)$$

The eigenvalues of this matrix are the inverses of the radii of curvature of $\mathcal{M}$. One usually describes the geometry of these surfaces in terms of the mean curvature [22, 23]

$$H = \frac{1}{2} h^{ij} K_{ij} \ , \qquad (3)$$

and the Gaussian curvature

$$K = \epsilon^{ik} \epsilon^{jl} K_{ij} K_{kl} \ . \qquad (4)$$

One can show that the Gaussian curvature can be computed solely from the metric h_{ij}, while the mean curvature depends explicitly on the embedding X^μ.

We shall be concerned primarily with the Polyakov form of the string action [24], in which an additional *intrinsic metric* g_{ij} is introduced to describe the surface geometry. We discretize our model by triangulating surfaces. In this construction, each triangle is equilateral with area 1 in the intrinsic metric; the coordination number at each vertex determines the intrinsic curvature of the surface. The coordinates i label the vertices of the triangulation. Then the discrete analogue of the intrinsic metric is the adjacency matrix C_{ij} whose elements equal 1 if i and j label neighbouring nodes of the triangulation, and vanish otherwise. Two-dimensional diffeomorphism invariance reduces to the permutation symmetry of the adjacency matrix at this discrete level. One of the keys, in fact, to the power of this construction is the preservation of this symmetry. Each vertex of the triangulation is embedded in R^3 via the mapping X_i^μ. Given the embedding X, we can also associate a unit normal vector $(n^\mu)_{\dot{k}}$ with each triangle on the surface (dotted Roman indices label the triangles). Note that all of the surface curvature of our triangulations is concentrated along the links and vertices. The surface is still flat in the direction tangent (but not transverse) to each link, so that the mean curvature has support on the links, while the Gaussian curvature is non-zero only at the vertices. The intrinsic curvature R_i at vertex i is given by the deficit angle determined solely by the triangulation

$$R_i = \pi \frac{(6 - q_i)}{q_i}, \qquad (5)$$

where q_i denotes the connectivity of the lattice at vertex i. The Gaussian curvature K on the other hand is expressed in terms of the deficit angle in the embedding space.

We shall study the theory defined by the action

$$S = S_{Gauss} + \lambda S_E = \sum_{i,j,\mu} C_{ij} (X_i^\mu - X_j^\mu)^2 + \lambda \sum_{\dot{k},\dot{l},\mu} C^{\dot{k}\dot{l}} (1 - n_{\dot{k}}^\mu \cdot n_{\dot{l}}^\mu) \ . \qquad (6)$$

Thus, for $\lambda > 0$, we have introduced a ferromagnetic interaction in the surface normals. The model defined by this action has been studied in [25, 26, 27, 28, 29, 30] and references therein.

From (2) and the definition of the induced metric, it follows that this is a discretization of the continuum action

$$S = \int \sqrt{|\det g|} (g^{ij} \partial_i X^\mu \partial_j X^\mu + \frac{\lambda}{2} g^{ij} h^{kl} K_{ik} K_{jl}) \ . \qquad (7)$$

Note that the second term in this action is manifestly positive, Weyl and reparametrization invariant, and that λ is a dimensionless coupling. So, naively, it is not clear whether it is relevant or not. If it were relevant, one would then anticipate that (since it obeys all of the appropriate symmetries) it should be effectively generated in any string action, and that it should engender ordering of the normals. It should then lead to another RG fixed point at a finite value of λ, which would characterize a phase transition between the crumpled phase (observed when $\lambda = 0$) and a 'smooth(er)' phase.[6]

A renormalization group analysis [18, 32, 33, 19] indicates, however, that there should be no phase transition at finite coupling when such extrinsic curvature dependent operators are added to the action. The computations of refs. [18, 32, 33] use the action

$$S = \int d^2\sigma\ (\mu_0\sqrt{\det h} + \frac{1}{\alpha}\sqrt{\det h}(h^{ij}K_{ij})^2)\ , \tag{8}$$

in the regime in which the string tension μ_0 is small (unlike the usual particle physics limit of string theory, which is characterized by large μ_0). After integrating out fluctuations of the embedding X^μ between momentum scales Λ and $\tilde{\Lambda}$, it is found that the renormalization of the extrinsic curvature coupling is given to one-loop order by

$$\beta(\alpha) \equiv \Lambda\frac{d\alpha}{d\Lambda} = -\frac{3}{4\pi}\alpha^2\ , \tag{9}$$

so that α is driven to infinity in the infra-red. This theory thus exhibits asymptotic freedom. Surfaces are smooth (the normals are correlated) below a persistence length [34]

$$\xi_p \sim \exp(\frac{4\pi}{3\alpha_{bare}})\ , \tag{10}$$

and are disordered above this scale. Some intuition into this result can be gained by observing that this theory is similar to the $O(3)$ sigma model, which is asymptotically free [16]. The normals to M are the analogues of $O(3)$ vectors, though in this case they are constrained to be normal to a surface governed by the action (7).

Without the extrinsic curvature term, (8) is the Nambu-Goto action, while (7), which we use in our simulations, is based on the action quantized by Polyakov. Classically (when the equations of motion for the Polyakov action are solved and substituted back into the action), the two actions are equivalent. It has also been demonstrated [35] that the two quantizations are equivalent in the critical dimension $D = 26$. In lower dimensions (note that the Nambu-Goto action clearly does not make sense for $D < 2$), it is not so clear that quantizations 'based' on the two actions are indeed the same. The work of Polchinski and Strominger [36] suggests that there are alternative quantizations. Distler ([37]) has also questioned the equivalence of these quantizations in $D = 3$. Indeed, even if the two quantizations are equivalent, it does not automatically follow that the two theories are still the same once an extrinsic curvature dependent term has been added.

In fact, Polyakov in [19] uses a hybrid form of the action (8) and still obtains the same result for the beta function. He introduces an intrinsic metric g_{ij}, chooses the conformal gauge $g_{ij} = \rho\delta_{ij}$ and considers

$$S = \frac{1}{2\alpha}\int d^2\sigma(\mu_o\rho + \rho^{-1}(\partial^2X^\mu)(\partial^2X^\mu) + \lambda^{ij}(\partial_iX^\mu\partial_jX^\mu - \rho\delta_{ij}))\ . \tag{11}$$

Classically, the Lagrange multiplier λ^{ij} constrains the intrinsic metric to equal the induced metric (this equality is not enforced by the classical equations of motion for the original Polyakov action). This constraint should be relaxed quantum mechanically if, as Polyakov [16] argues, the condensate of this Lagrange multiplier assumes a value of the order of the momentum cutoff. If this dynamical assumption is correct, then one can essentially derive the equivalence of this Nambu-Goto-like and the original Polyakov quantizations. In the large D (embedding dimension) limit,

[6] The extrinsic curvature term is also higher-derivative, indicating that the field theory described by this action is non-unitary. This fact alone does not imply that the associated string-scattering amplitudes do not satisfy unitarity. Polchinski and Yang [31] do, however, contend that in this case the string theory will not be unitary. Even if this were so, this model could still be an appropriate description of the statistical mechanics of fluctuating surfaces, although not one corresponding to a physical fundamental string theory.

saddle point calculations [38] show that λ indeed does acquire a large expectation value, and that for small values of the string tension μ_o, the coupling α is asymptotically free, as the RG calculations suggest.

There are, however, a couple of caveats and suggestions in the analytic literature that do allow for the existence of a crumpling transition for fluid surfaces. Polyakov remarks that if, in the infrared region, fluctuations of the internal geometry (ρ) are suppressed relative to fluctuations of the extrinsic metric, then the beta function is proportional to α and hence the continuum limit of the theory exhibits non-trivial scaling behaviour; this presumably cannot be the case in the large D limit. Another RG calculation, performed by Yang [39] using the Polchinski-Strominger action [36] with an extrinsic curvature dependent term, indicates that the two-loop correction (which is proportional to α^3) might be large enough to yield a zero of the beta function, and thus a non-trivial IR fixed point. The Polchinski-Strominger action is based on the assumption that the Liouville mode ρ effectively decouples (its mass is much greater than the momentum scale set by the string tension); it is not clear why this assumption should hold for the model that we simulate. Finally, note that these computations are perturbative (in $1/D$ or α). It is possible that non-perturbative effects could drive a crumpling transition.

Monte Carlo simulations of the action (7) on dynamically triangulated random surfaces (DTRS) were first performed by Catterall [25], and shortly thereafter by Baillie, Johnston, and Williams [26, 27] and Catterall, Kogut and Renken [30]. They simulated triangulations with the topology of the sphere, and measured the specific heat

$$C(\lambda) \equiv \frac{\lambda^2}{N}(< S_E^2 > - < S_E >^2) \ , \tag{12}$$

on surfaces with up to $N = 144$ nodes (and $N = 288$ nodes in the last reference). They found a peak in the specific heat; the peak size appeared to grow with N. A similar model that can be vectorized rather straightforwardly was also considered; the set of planar ϕ^3 graphs was simulated [30, 40]. Each vertex of these ϕ^3 graph was embedded in R^3 and the action (7) was used; graphs of up to 1000 nodes were simulated (these would be dual to 500 node triangulations). It was found that the specific heat peak grew with N, albeit slowly, as

$$C_{max} = AN^{\omega} + B \ , \tag{13}$$

with $\omega = 0.185(50)$. Further work in [28, 29], using dynamical triangulations with the topology of the torus and lattices with up to $N = 576$ nodes, indicated that the rate of increase of the peak height severely diminishes with increasing N. The data strongly suggests that in fact the specific heat peak height does **not** diverge as $N \to \infty$. These authors also measured the bare string tension and mass gap, by embedding the torus in a background toroidal space spanned by a loop, and measuring the dependence of the free energy on the loop size. They found that these measurements (when taken for λ values near the peak position) are consistent with the appropriate scaling relations (with vanishing bare string tension and mass gap) that should characterize a phase transition to smooth surfaces. This measurement, which may turn out to be quite important (since it constitutes the best evidence we have, up to now, that there could really be a phase transition for $\lambda = \lambda_c$), is still quite an indirect way of measuring correlation functions. As we will discuss, these scaling relations could contradict other observed phenomena (such as the absence of diverging correlation times and increasing finite size effects at the pseudo-critical point).

Thus it appears that numerical evidence could allow for the existence of a crumpling transition (most probably of higher order), while analytical calculations generally indicate that no such transition should occur.

In [41] the peak was measured in a DTRS simulation that incorporated self-avoidance and the extrinsic curvature term S_E, with a solid-wall potential substituted for the Gaussian term in the action. The results for the specific heat turned out to be very similar to those found in the simulations we have just discussed, for example, in [29]. The specific heat peak is, in this context, considered to be a lattice artifact, because the peak height levels off with large N (of order 500). These simulations included a crude block-spin measurement that suggests that the renormalization group flow of λ is consistent with the analytical result of asymptotic freedom.

Simulations using other discretizations for the extrinsic curvature dependent term have yielded somewhat different results [25, 26]. The specific heat peak, measured in simulations employing what is referred to as the 'area discretization', is rather feeble, and levels off for small values of N (by $N = 72$) (the authors interpret this as being indicative of perhaps a 'third' order transition). Actions based on these various discretizations have been simulated for fixed, triangular meshes. These systems model tethered or crystalline membranes, in which the constituent molecules are tightly bound together. In the tethered case, the specific heat peak obtained from simulations of the edge action (7) grows vigorously as a function of N for very large (128×128) lattices [42]. This is strong evidence for the existence of a second order transition which, in this case, is in accord with the analytic results – these calculations are reviewed by Nelson [43] and David [3, 44] and involve mean field and large D computations which suggest that the β function is linear at leading order, with a zero for finite α, i.e. a UV fixed point. When the alternate area discretization is used in the tethered case, the specific heat peak again stops growing. Recent work has demonstrated that this other discretization is pathological in the tethered case; the class of 'corrugated' surfaces, which are singular in one direction and smooth in the other, then dominates the path integral [42].

Thus, given the muddle of somewhat contradictory evidence, it is unclear whether or not a crumpling transition exists for fluid surfaces. We have pursued this question by taking high statistics measurements of the specific heat peak, and by measuring other observables describing the geometry of these surfaces, since observables with different quantum numbers can give quite different information. For example, in the Ising model the magnetization behaves quite unlike the internal energy (which is invariant under the standard Z_2 transformation).

To analyze and interpret this data, we have applied insights gained from work on better understood systems, primarily spin models and lattice gauge theories. Issues of the equivalence of the Nambu-Goto and Polyakov quantizations have also motivated us to compare the intrinsic and induced geometry of the surfaces that we simulate.

3. Numerical Simulations in $d = 2$

To minimize finite size effects, we have considered triangulations with the topology of the torus. The action (7) was used, with the BRST invariant measure utilized also by Baillie, Johnston, and Williams [26], so that

$$Z = \sum_{G \in T(1)} \int \prod_{\mu,i} dX_i^\mu \prod_i q_i^{\frac{d}{2}} \exp(-S_{Gauss} - \lambda S_E) \ , \tag{14}$$

where $d = 3$, q_i is the connectivity of the ith vertex, and $T(1)$ refers to the set of triangulations of genus 1. The authors of [28, 29] do not include this connectivity dependent term in their measure. The long-distance physics of the simulations is presumably insensitive to the presence of this term. Because we have chosen a different measure, though, our quantitative results cannot be precisely compared with theirs.

We measured a variety of quantities that characterize the extrinsic and intrinsic geometry of these surfaces. These observables include:

1. The edge curvature S_E and the associated specific heat $C(\lambda)$, which is a sensitive indicator of the presence of a phase transition.

2. The squared radius of gyration R_G;

$$R_G \equiv \frac{1}{N} \sum_{i,\mu} (X_i^\mu - X_{\mathrm{com}}^\mu)^2 \ , \tag{15}$$

where the com subscript refers to the center of mass of the surface. By measuring the N dependence of the gyration radius, we can extract a value for the extrinsic Hausdorff dimension, which is given by

$$R_G \sim N^\nu \sim N^{\frac{2}{d_{\mathrm{extr}}}} \ . \tag{16}$$

3. The magnitude of the extrinsic Gaussian curvature. We measure a discretization of $\int \mid K \mid \sqrt{\mid h \mid}$, with

$$\mid \mathcal{K} \mid = \frac{1}{N} \sum_i \mid 2\pi - \sum_j \phi_i^j \mid \ . \tag{17}$$

Here ϕ_i^j denotes the angle subtended by the jth triangle at the ith vertex. This quantity, therefore, measures the magnitude of the deficit angle in the embedding space averaged over all vertices. We also record the mean square fluctuation of $\mid \mathcal{K} \mid$, denoted by $F[\mid \mathcal{K} \mid]$.

4. The corresponding intrinsic quantity, $\mid \mathcal{R} \mid$, given by

$$\mid \mathcal{R} \mid = \frac{\pi}{3N} \sum_i \mid 6 - q_i \mid \ , \tag{18}$$

and its fluctuations. When the intrinsic and extrinsic metrics are equal, the intrinsic and extrinsic deficit angles are identical, and $K = R/2$.

5. To study the correlation between intrinsic and extrinsic geometry, we also measure the quantity which we refer to as $\mathcal{K} * \mathcal{R}$:

$$\mathcal{K} * \mathcal{R} \equiv \frac{\int KR}{\sqrt{\int K^2 \int R^2}} = \frac{\sum_i (2\pi - \sum_j \phi_i^j)(6 - q_i)}{\sqrt{\sum_i (2\pi - \sum_j \phi_i^j)^2 \sum_i (6 - q_i)^2}} \ . \tag{19}$$

This quantity is 1 when the metrics are equal, 0 if they are un-correlated, and negative when these curvatures are anti-correlated.

6. We measure, finally, the average maximum coordination number of the surface vertices, $\max_i q_i$.∎

In our simulations we have used the standard Metropolis algorithm to update the embedding fields X_i^μ. To sweep through the space of triangulations we performed flips (see reference [12]) on randomly chosen links. Flips were automatically rejected if they yielded a degenerate triangulation; i.e. one in which a particular vertex has fewer than three neighbors, or in which a vertex is labeled as its own neighbor, or where more than one link connects two vertices. (It has been proven in ref. [12, 13] that the entire space of graphs of a given topology can be spanned by only performing these flips.) After a set of $3N$ flips was performed, $3N$ randomly selected embedding coordinates were updated via random shifts from a flat distribution,

$$X^\mu \rightarrow X^\mu + \delta X^\mu \ . \tag{20}$$

The mean magnitude of these shifts

$$< \delta X^\mu \delta X^\mu > \tag{21}$$

was chosen so that the acceptance rate for updates of the X^μ was roughly 50 percent. Most of the Monte Carlo simulations were performed on HP-9000 (720 and 750 series) workstations; we also

N=144	.8	1.25	1.35	1.40	1.45	1.50	2.0	λ
	3	3	3	3	3	3	3	$\times 10^6$ sweeps
N=288	.8		1.375	1.40	1.425	1.475	2.0	λ
	14.4		21.0	15.0	16.2	13.5	14.4	$\times 10^6$ sweeps
N=576	.8	1.325	1.375	1.40	1.425	1.475	2.0	λ
	12.0	27.0	27.0	27.0	27.0	27.0	9.6	$\times 10^6$ sweeps

Table 1: A record of the number of sweeps performed at each different λ value for 3 different lattice sizes.

collected some data by simulating lattices on each of the 32 nodes of a CM-5. Our code was in Fortran, with a Fibonacci random number generator.

In Table 1 we summarize our runs. Note that we have performed quite long runs on the larger lattice sizes. We will discuss later why we believe runs of this length are just sufficient to yield accurate values of the observables for the largest lattice size ($N = 576$).

In all of our figures the different dots will be printed with their associated statistical error (sometimes too small to be visible). The statistical error is computed by means of a standard binning procedure. We will explicitly discuss the cases in which our estimator for the statistical error is not asymptotic.

The lines in these figures are from a histogram reconstruction (see for example [45, 46]). We patch different histograms [47, 48, 49] by weighting them with the associated statistical indeterminacy (which we estimate by a jack-knife binned procedure); this procedure seems to be very effective and reliable. All of the reconstruction curve sets (3: dotted, dashed and continuous for 3 surface sizes on each figure) consist of 3 curves (which sometimes appear as a single one). The middle curve is the histogram reconstruction, and the upper and the lower ones bound the data within the errors obtained by the procedure we have just described.

For $N = 144$ we have patched the four histograms originating from $\lambda = 1.35$, 1.40, 1.45, 1.50. For $N = 288$ we have used $\lambda = 1.375$, 1.40, 1.425 and 1.475. For $N = 576$, we chose $\lambda = 1.375$, 1.40 and 1.425.

We have only drawn the reconstructed, patched curves (with their reliable errors) in the regions where we trust them. For example, close to the pseudo-critical region we can trust a peak pattern only when we can reconstruct the peak by starting from both sides of the transition (without multi-histogram patching). So we have always used single histogram reconstructions to check these criteria, before constructing the final, multi-histogram data.

We have measured a large number of local observables. We will see that a mixed picture emerges from these measurements. For example the observables related to the dynamical triangulations exhibit a characteristic pattern, to be discussed in detail below.

We start by showing, in Fig. 1, the edge curvature S_E as a function of λ. The crossover region is around $\lambda \simeq 1.4$. For small values of λ, the surface is crumpled (see the latter part of this section). In this region, finite size effects are already negligible for our lattice sizes, and our 3 data points are on top of each other. We can see weak finite size effects by comparing the continuous lines in the transient region. The $N = 144$ dotted line is far from the ones of the two larger lattices, which lie, on the contrary, on top of each other. Finite size effects are larger in the large λ phase. One would expect, close to a phase transition with a diverging correlation length, an increase of the finite size effects which we do not observe here. The lattice should feel the presence of the zero mass excitation, and the finite size corrections should be larger than everywhere else (in the case of periodic boundary conditions they would obey a power-law, rather than decaying exponentially with size). This is surely not firm evidence against the presence of a phase transition, but it does show that the putative critical behaviour is atypical.

The errors in the 'flat phase' ($\lambda = 2.0$) are not under control. Our estimators do not plateau under repeated iterations of the binning procedure. In this regime, correlation times are large, as

Figure 1: The edge curvature S_E as a function of λ. As in all other pictures, filled circles and a dotted line correspond to $N = 144$, crosses and a dashed line indicate $N = 288$, and empty squares and a solid line represent $N = 576$.

N	$C(\lambda)^{\max}$	λ_c
144	5.37(14)	1.395(30)
288	5.55(7)	1.410(25)
576	5.81(17)	1.425(30)

Table 2: The maximum of the specific heat and its position, with their errors, for 3 different lattice sizes.

we will discuss in next section. This *caveat* holds for this figure and for all the quantities we have measured.

In Fig. 2 we show the related specific heat $C(\lambda)$, in the same λ region. In Fig. 3 we enlarge the pseudo-critical λ region, in order to show the reconstructed peak of the specific heat. As already noted our reconstruction procedure is quite reliable here.

In Table 2 we give the maximum of the edge curvature specific heat and its location for the 3 different lattice sizes. We can extract from these data a specific heat exponent $\omega = .06 \pm .05$, with ω defined as in equation (13), and the constant B set to zero. If we estimate an effective exponent from the two smaller lattice sizes we get $.05 \pm .06$, and from the two larger ones we get $.07 \pm .06$; this demonstrates that we do not see, within our statistical precision, any sign of a non-pure-power, non-asymptotic behavior. Note that if the constraint that B vanishes is relaxed, our data is not accurate enough to yield a meaningful fit to equation (13). A very small (asymptotically finite) correlation length is sufficient to produce such a small effect on our quite small lattice sizes. These results appear to be consistent with those of the Copenhagen group [29], and they are not so far from the ones of the Urbana group [30, 40].

The critical value of λ shifts (very slowly, i.e. in a way compatible with no increase given our statistical precision) to higher values for increasing N. Also the shape of the specific heat (the width, for example), is basically unchanged as we go to larger lattices. From Figs. 2 and 3 we do not infer any evidence of criticality.

Figure 2: The edge curvature specific heat, $C(\lambda)$. The edge curvature S_E as a function of λ. As in all other pictures, filled circles and a dotted line correspond to $N = 144$, crosses and a dashed line indicate $N = 288$, and empty squares and a solid line represent $N = 576$.

Figure 3: As in Fig. 1, but with the multi-histogram reconstruction in the transient region. The edge curvature S_E as a function of λ. As in all other pictures, filled circles and a dotted line correspond to $N = 144$, crosses and a dashed line indicate $N = 288$, and empty squares and a solid line represent $N = 576$.

Figure 4: The gyration radius R_G defined in (15), plotted as in Fig. 1. The edge curvature S_E as a function of λ. As in all other pictures, filled circles and a dotted line correspond to $N = 144$, crosses and a dashed line indicate $N = 288$, and empty squares and a solid line represent $N = 576$.

In Fig. 4 we show the radius of gyration of the surface, R_G, as defined in (15). Here obviously the volume scaling is non-trivial: larger surfaces have larger radius. The histogram reconstruction ceases to work already for quite low values of λ for the larger lattice. This effect could be related to the interesting finite size scaling behavior of this quantity, which we illustrate in better detail in Fig. 5 and 8.

Here we plot

$$\nu(N) \equiv \frac{\log \frac{R(N)}{R(\frac{N}{2})}}{\log(2)} \ . \tag{22}$$

This is an effective inverse Hausdorff dimension, which is a function of λ. In the large λ limit $\nu \rightarrow 1$ and $d_{extr} \rightarrow 2$, as expected for flat surfaces. In the low λ limit d_{extr} becomes very large. In the pseudo-critical region ν is a linear function of λ. Curiously enough, the latter curve yields a Hausdorff dimension of 4, a value characteristic of branched polymers, near the location of the specific heat peak. Due to finite-size effects, this value is not particularly reliable, though (and a value changing rapidly from 2 to a large number must to pass through 4 somewhere in the crossover region). In ref. [29] a value compatible with ours is quoted for the critical theory ($D_H(\lambda_c) > 3.4$), Which is compatible with our findings. We stress however (also here, in complete agreement with remarks of [29]) the fact that the Hausdorff dimension in the pseudo-critical region depends heavily and in a quite unusual manner on N.

In both high and low λ regions finite size effects are quite small (compatible with zero in one standard deviation). On the contrary, in the pseudo-critical region, finite size effects are strong. This effect cannot be explained by the shift in λ which one gets from the shift of the peak of the specific heat, which is far too small. This behaviour is very different from the one which we have discussed for S_E, and it seems to indicate the possibility of some sort of critical behavior close to $\lambda = 1.4$.

In Fig. 6 we plot the expectation values of the magnitude of the extrinsic Gaussian curvature $\mid \mathcal{K} \mid$. If the induced metric is equal to the intrinsic metric, then $\mid \mathcal{K} \mid = \frac{|\mathcal{R}|}{\epsilon}$. This plot is not

Figure 5: The effective inverse Hausdorff dimension ν as a function of λ, as defined in (16). The filled dots and the dashed curve are from a fit to the $N = 288$ and $N = 144$ data, while the empty dots and solid curve represent the fit to $N = 576$ and $N = 288$.

Figure 6: The extrinsic Gaussian curvature $| \mathcal{K} |$ defined in (17), plotted as in Fig. 1.

Figure 7: The fluctuations of $\mid \mathcal{K} \mid$.

substantially different from that of S_E. We note that finite size effects are somewhat larger in this case than for the edge action, but they follow the same pattern (exhibiting a big increase in the flat phase).

The plot of the fluctuations of the extrinsic Gaussian curvature, $F[\mathcal{K}]$ which we present in Fig. 7, shows something very new. A very sharp crossover, with perhaps a peak developing for large N, dominates the pseudo-critical behavior. Fluctuations do not seem to depend on λ in the crumpled phase, while they drop dramatically, in a very small λ interval, in the flat region. Here, again, finite size effects are sizeable in the pseudo-critical region. The position of the crossover does not depend sensitively on N, while the detailed shape at λ_c seems to change slightly with N. It is difficult to give a precise interpretation of a plot like this, but, as we said, the change of regime is very clear here.

In Fig. 8 we give the intrinsic curvature $\mathcal{R}$, and in Fig. 9 its fluctuations. Both plots are very similar to the related, extrinsic curvature $\mathcal{K}$ plots. $\mid \mathcal{R} \mid$ drops off rapidly, just as $\mid \mathcal{K} \mid$ also does. Through the peak region, though, $\mid \mathcal{K} \mid$ decreases by about a factor of 5 while $\mid \mathcal{R} \mid$ diminishes to only about .6 of its value on the left-hand side of the peak. Since the action explicitly suppresses mean curvature, and the mean and extrinsic Gaussian curvature are closely related (for instance, $H^2 > \frac{K}{2}$), we would expect that for large λ extrinsic fluctuations would be suppressed much more than fluctuations of intrinsic geometry.

In Fig. 10 we plot the intrinsic extrinsic curvature correlation. The plot of $\mathcal{K}$*$\mathcal{R}$ indicates that intrinsic and extrinsic geometry are strongly correlated for small λ, but as one passes through the peak region, they become decorrelated. This is not particularly surprising, given that the action directly suppresses only extrinsic fluctuations.

In Fig. 11 we plot the expectation value of the maximum correlation number, which has non-trivial scaling behavior. In Fig. 12 we give its scaling exponent, defined analogously to the exponent we have exhibited for the gyration radius. In the pseudo-critical region q_{max} scales (for our 3 lattice sizes) as a power, with an exponent close to 0.1; we do not know if this scaling is meaningful.

We have also computed correlation times for different observables. As we already noticed, correlation times become very large in the large λ region. In agreement with ref. [29] we do not see any increase of the correlation times close to the pseudo-critical point.

Figure 8: The intrinsic curvature $| \mathcal{R} |$ defined in (18), plotted as in Fig. 1.

Figure 9: The fluctuations of $| \mathcal{R} |$.

Figure 10: The intrinsic extrinsic curvature correlation, as defined in (19), plotted as in Fig. 1.

Figure 11: The average maximum connectivity number of the surface vertices, $\max_i q_i$, plotted as in Fig. 1.

Figure 12: The scaling exponent of $\max_i\ q_i$, plotted as in Fig. 1.

We will not present precise estimates of correlation times (exponential or integrated): they are too large to get precise estimates. We will limit ourselves to a discussion of a few figures, which give quite a clear idea of what is happening. The comparison with ref. [29] cannot be very direct, since our action is different, and because their dynamics may be more effective than ours. Still, the comparison is quite puzzling, since we estimate and exhibit correlation times which are much (orders of magnitude) larger than the ones of [29]. Applying usual methods to estimate τ_{int} can lead to an underestimate of short correlation times if more than one time scale is present (that does surely happen to our data if we integrate our data on a window of a reasonable size).

In Fig. 13 we plot S_E for $N = 144$, $\lambda = 1.4$, and in Fig. 14 the gyration radius for these values (with a different time scale). Clearly, the correlation time is at least of order $40,000$ sweeps in the first case and $100,000$ sweeps in the second one. In Figs. 15 and 16, we plot the same quantities for $\lambda = 1.5$. Here correlation times are larger, of order $50,000$ steps for S_E and larger than $150,000$ steps for R_G. In Figs. 17 and 18 we draw the same plot on the largest lattice we study ($N = 576$) for $\lambda = 1.4$. Here we can see dramatic correlations, with times of at least $100,000$ steps for S_E and of at least $1,000,000$ steps for R_G.

In Fig. 19 we plot, for the same time history and on the same scale, both S_E and R_G. This figure shows a clear anticorrelation: larger surfaces are flatter, and have smaller curvature (this is apparent in the region close to the 2000th step).

4. A Discussion of the $2d$ Results

We have shown that the transition from crumpled to flat surfaces with increasing λ is quite complex. Our interpretation of what is going on is, at present, mainly based on these data and on the results of ref. [29].

On lattices with up to 576 vertices we can clearly see a sharp crossover, but the absence of a diverging specific heat, of diverging correlation times and of strong finite size effects suggests that we are not observing a usual second order phase transition. On the other hand, quantities like the intrinsic curvature, or the mass measurements of [29] show that something non-trivial is happening.

Figure 13: S_E as a function of Monte Carlo time $(80,000$ steps) for $N = 144$, $\lambda = 1.4$.

Figure 14: R as a function of Monte Carlo time $(300,000$ steps) for $N = 144$, $\lambda = 1.4$.

Figure 15: S_E as a function of Monte Carlo time $(80,000$ steps) for $N = 144$, $\lambda = 1.5$.

Figure 16: Fig. 16 R as a function of Monte Carlo time $(300,000$ steps) for $N = 144$, $\lambda = 1.5$.

Figure 17: S_E as a function of Monte Carlo time $(300,000$ steps) for $N = 576$, $\lambda = 1.4$.

Figure 18: R as a function of Monte Carlo time $(3,000,000$ steps) for $N = 576$, $\lambda = 1.4$.

Figure 19: S_E and R from the same Monte Carlo run, $N = 576$, $\lambda = 1.325$, $20,000$ steps.

Let us review the crux of our observations again. This model of crumpled surfaces appears to exhibit sharp crossover behavior in the region around $\lambda = 1.4$. The sharp growth in the gyration radius and the suppression of curvature fluctuations indicate that the normals acquire long-range correlations, up to the size of the systems we examine. Presumably the zero string tension measurement of [29] also shows that the disordered regime differs from the regime in which the surfaces are ordered (up to scale of the lattices that are simulated) by only a small shift in λ. This evidence might indicate the presence of a phase transition at this point. If so, it is very likely to be of order higher than 2 (or, rather implausibly, it could be second order with an extremely low negative specific heat exponent; our lattices are much too small for us to confidently extrapolate the value of the specific heat exponent as $N \to \infty$).

If the transition were higher order, the peak should exhibit a cusp, but we would need far more accurate data to detect this. The existence of this phase transition would then suggest the existence of a new continuum string theory, though many other issues would have to be resolved (e.g. unitarity) to determine if such a theory is physically desirable.

There are other possible interpretations of our data. We need to consider the influence of finite-size effects, since the surfaces which we simulate are quite small, even smaller than one might naively assume because they are not intrinsically smooth. For instance, random surfaces characteristic of $D = 0$ gravity have a Hausdorff dimension of roughly $d_{intr} = 2.8$ [50, 51]; it has been predicted that surfaces embedded in 1 dimension have Hausdorff dimension $2 + \sqrt{2}$ [50]. Thus, for instance, if the surfaces in our simulations had an intrinsic dimension of 3, they would have a linear size of fewer than 9 lattice spacings[7].

Perhaps the simplest alternative explanation for the presence of this peak is suggested by the arguments of Kroll and Gompper [41]. They argue that the peak occurs when the persistence length of the system approaches the size of the lattice ($\xi_p \sim N^{\frac{1}{d}}$)[8]. For larger λ, fluctuations on a larger scale become more important, but when this scale is greater than the lattice size, these fluctuations are suppressed. Thus one might surmise that the specific heat will drop for large λ. (It clearly goes

[7] Of course, our lattices are too small, by one or two orders of magnitude, to really exhibit a convincing fractal structure.

[8] For couplings above this point, our simulations would simply be measuring finite size effects.

to zero for small λ because of the presence of the prefactor λ^2; the lattice implements a ultraviolet cutoff that freezes out very short-range fluctuations.)

The one-loop renormalization group calculation (10) predicts that the persistence length grows as $\xi_p \sim \exp(C\lambda)$; C is inversely proportional to the leading coefficient of the beta function. We would expect that the peak position should shift to the right with increasing N in this scenario as

$$\Delta = \frac{\delta \ln N}{d_{intr} C}. \tag{23}$$

Quite a large value of C is needed to explain the rapid crossover; roughly values of $C \sim 10, d_{intr} \sim 3$ are more or less consistent with the magnitude of the peak shift and crossover width. The RG calculations using different forms of the action yield $C = \frac{4\pi}{3}$ (see equation 10), but this may not apply to the action we simulate.

This reasoning also indicates that the peak should widen as the lattice size increases; we do not observe this at all. It seems plausible though that these arguments, based only on the leading term of the high lambda expansion, are too naive.

An alternative scenario, which builds on the ideas in the above paragraph, is suggested by the tantalizing similarities between the results of our fluid surface simulations and what has been observed for the $d = 4$ $SU(2)$ Lattice Gauge Theory [45] and for the $d = 2$ $O(3)$ model. Let us discuss the case of the $O(3)$ model.

The $O(3)$ model, which is asymptotically free, exhibits a specific heat peak near $\beta = 1.4$. This peak was first measured via Monte Carlo simulations by Colot [53]. It can also be obtained by differentiating the energy data measured by Shenker and Tobochnik [54, 55]. The origin of this peak is understood [55, 56]; it is due to the fluctuations of the sigma particle, a low-mass bound state of the massless $O(3)$ pions. The sigma induces short-range order, and contributes to the specific heat as a degree of freedom only at high temperatures (when the correlation length in the system becomes smaller than its inverse mass). The peak thus occurs at the beginning of the crossover regime, when the correlation length is several lattice spacings.

According to the low temperature expansion, the correlation length grows as $\xi \sim \exp(2\pi\beta)/\beta$. Thus one would expect a fairly rapid crossover in the $O(3)$ model; the correlation length should increase by roughly a factor of 9 when β is shifted by about .35[9]. Such a crossover is indeed observed, though it is somewhat less dramatic than the behavior observed for fluid surfaces; the specific heat peak we measure is roughly half the width of the peak for the $O(3)$ model.

Recent simulations of the $O(3)$ model [52] indicate that the specific heat peak grows significantly when the lattice size L is increased from 5 to 15, and that virtually no growth in peak height is evident as L is increased further up to 100. Also, the peak position shifts to the right as L grows, and then appears to stabilize for large L. This is more or less what we observe in our simulations of fluid surfaces, on lattices of small size. We point out these similarities largely to emphasize that there does exist an asymptotically free theory (with low mass excitations) which exhibits crossover behavior qualitatively similar to that observed in our simulations.

The analogy is perhaps deeper, though, since the fluid surface action (with extrinsic curvature) in certain guises looks like a sigma model action. So, perhaps it would not be so surprising from this point of view to find a sigma particle in these theories perhaps associated with $(\hat{n}^2 - 1)$, in which $\hat{n}$ denotes the unit normal to our surfaces.

Another additional possibility is that fluctuations of the intrinsic geometry (the Liouville mode) are responsible for short-range order and contribute to the specific heat peak.

There remains much to be done to clarify whether or not a crumpling transition occurs for a finite value of the extrinsic curvature coupling λ. It would be interesting (and probably a fair amount of

[9] In fact, the presence of the sigma significantly modifies this low-temperature expansion result [56] in this intermediate regime, but does not qualitatively destroy the rapidity of the crossover. Indeed, despite heroic efforts, it has been impossible to extend computationally beyond this regime and precisely verify the asymptotic low-temperature relation for the correlation length [57, 58].

work) to apply Wilson renormalization group techniques to the actual action (7) which we simulate, to determine the leading coefficient of the beta function. Additionally, perhaps a calculation of $1/D$ corrections to the large D computations already performed could unearth evidence of a sigma-type excitation in these theories (the effects of the sigma appear as $1/N$ corrections in the $O(N)$ model).

We also are histogramming our data to examine the behavior of complex zeroes (in complex λ space) of the partition function of our simulations [59]. It has been shown (in the case of $SU(2)$ lattice gauge theory) that such zeroes, when they are near but do not approach the real axis in the infinite volume limit, occur in theories which exhibit specific heat peaks with no associated phase transition [45]. High temperature expansions also indicate that the $O(3)$ model susceptibility has a complex singularity near the real axis [60]– presumably this corresponds to a zero of the partition function and is a manifestation of the sigma.)

Of course, simulations on large lattices, with better statistics, should also help us evaluate whether a crumpling transition exists. We are testing algorithms, such as simulated tempering [61], in order to evade the long auto-correlation times that have characterized our simulations so far.

Even if no such transition exists for finite λ, one could still attempt to study a continuum theory in the strong coupling limit, as is done for QCD, for instance. To do so, we would like to examine global quantities, such as masses extracted from normal-normal correlation functions, rather than just the local quantities (energy, e.g.) that we have measured. Measuring these correlations requires a definition of distance on these triangulated lattices; the most successful definition of the metric is based on the propagation of massive particles (via inversion of the Laplacian) on these lattices [62].

We have observed dramatic crossover behavior for particular observables in our Monte Carlo simulations, but on the other hand, the correlation times and certain finite-size effects do not behave as one would expect in the presence of a phase transition. The behavior of other lattice models also indicates that it is possible that we are observing the effects of finite-mass excitations on small lattices, rather than a phase transition. We hope that future work will clarify this murky state of affairs, to determine if there indeed exists a crumpling transition for fluid surfaces.

5. $4d$ Triangulated Manifolds and Quantum Gravity

Dynamically triangulated random surfaces (DTRS) [6] play an important role in the efforts to develop a coeherent description of quantum gravity. The (Euclidian) space-time is approximated by a d-dimensional simplicial triangulation, where the link length is constant, equal to 1, but the connectivity matrix is a dynamical variable.

The most important advances have been obtained in two-dimensional quantum gravity, where DTRS are simplicial triangulations of a $2d$ manifolds. The analytic success of matrix models, which can be for example exactly solved in the case of pure $2d$ gravity [8], has strongly encouraged this approach. The results obtained in the triangulated approach and in the continuum lead to consistent predictions for correlation functions and critical exponents.

Dynamical triangulations are also potentially relevant in four dimensions. One can hope that a sensible, non-perturbative definition of the quantum gravity theory can be obtained in some *scaling* limit of the theory of $4d$ hyper-tetrahedra. This approach has much in common with Regge calculus, where the connectivity is fixed but the functional integration runs over the link lengths. The underlying principle is clearly very similar, and one could say that DTRS have the status of an improved Regge calculus. We face the usual problem inherent in discretizing a theory, i.e. the discretization scheme can break some of the continuous symmetries, which will have to be recovered in the continuum limit (if there is one). Indeed, Wilson lattice gauge theories have taught us an important lesson. The fact that gauge invariance is exactly conserved in the lattice theory, for all values of the lattice spacing a, is in that case crucial: it would have been very difficult to establish firm numerical results if one would have had to care about the presence of non gauge-invariant correction, which would disappear only in the $a \to 0$ limit. In the case of quantum gravity, diffeomorphism invariance plays such a crucial role, and DTRS are diffeomorphism invariant by construction, at least on the space of piecewise flat manifolds. Hence part of the difficulties Regge calculus has in forgetting about the lattice structure are eliminated a priori in the DTRS lattice approach.

There are two more important points to stress. The first one is that in the DTRS approach in $3d$ and $4d$, as opposed to $2d$, we can try to make sense out of the pure Einstein action, without, for example, curvature squared terms. Even though the partition function formally diverges, at fixed volume the local curvature is bounded both from below and from above. Therefore we can study the theory at fixed (or better quasi-fixed, see later) volume, and look for the existence of a stable fixed point in the large volume limit. A second order phase transition with diverging correlation lengths, in the statistical mechanics language, would allow us to define a continuum limit which is universal and is not influenced by the details of the underlying discrete lattice structure. Precisely this scenario constitutes one of the best hopes we have to find a consistent quantum theory of gravity. If Euclidean quantum gravity based on the Einstein action does have non-perturbative meaning, then we can exhibit it in this way.

The second problem is to determine the measure one should use to define the quantum theory. This problem, far from being solved in the continuum, is completely open in the lattice approach (for a good review see for example [63]). In quantum Regge calculus the influence of the measure has been examined in [64] but there is no direct relation to DTRS. The problem of the measure is the main point we address in this note, and we want to suggest that the DTRS approach may be powerful enough to solve it.

Recently two groups have pionereed Monte Carlo simulations of DTRS [65, 66, 67]. Numerical simulations (even if on quite a small scale) turned out to be feasible, and lead to very non-trivial results. One clearly observes a phase transition structure. Although quantitative statements are not easy to make, given the limited statistics and the small lattice size, it is clear that the situation is different from the $3d$ case, where the phase transition is manifestly of first order, and there is no continuum theory. In $4d$ [65] there is an open possibility that the transition is second order (although that cannot be claimed without a much more detailed finite size study). One does not observe hysteresis cycles, and the crossover is less sharp then in $3d$. More involved statements about critical exponents have to be taken at this point, we believe, *cum grano salis*, but there is evidence for the possible presence of a critical point with a non-trivial continuum theory in the phase diagram for the Euclidean Einstein action. This observation certainly warrants further careful investigations.

These first simulations have been run with uniform measure, where all triangulations have the same weight in the sum which defines the path integral of the quantum theory. There are no particularly good reasons for this choice to be the correct one, and in the following we investigate the changes introduced by defining simplicial quantum gravity with a non-trivial measure.

One of us has described in [68] the structure of the Monte Carlo simulations, and how the efficiency of the numerical procedure can be optimized. The programming of dynamical triangulations is difficult since a dynamical data structure is required, but it is very relevant in many practical applications. For the same reason the implementation of DTRS on parallel computers is a hard problem. Since a set of ergodic moves preserving the volume for canonical simulations is not known, one has to consider a Markov chain which sweeps out the space of different volume simplicial manifolds. We have used the quasi-canonical method introduced in [69], which allows us to control the systematic distortion arising from the *ad hoc* potential that keeps the system close to a given number of 4-simplices [68]. All the details about the system we study and about the numerical procedures are given in [68].

In the continuum the Euclidean Einstein-Hilbert action for a metric $g_{\mu\nu}$ has the form

$$S_E[g] = \int d^4x \sqrt{g}(\lambda - \frac{R(g)}{G}) \ , \tag{24}$$

where $R(g)$ is the Ricci scalar and λ and G are the cosmological and gravitational constants, respectively. We consider a fixed S^4 topology. On a triangulation T we discretize according to

$$V = \int d^4x\sqrt{g} \rightarrow N_4[T] \ , \tag{25}$$

$$R = \int d^4x\sqrt{g}R(g) \rightarrow \frac{2\pi}{\alpha}N_2[T] - 10N_4[T] \ , \tag{26}$$

where $\alpha \simeq 1.318$, and $N_i[T]$ is the number of i-simplices of the triangulation T. The discrete action is then

$$S_E[T] = k_4N_4[T] - k_2N_2[T] \ , \tag{27}$$

where $k_4 = \lambda + 10/G$ and $k_2 = 2\pi/\alpha G$.

In the discrete quantum theory there exists a critical line $k_4 = k_4^c(k_2)$ such that if k_4 is different from k_4^c for a given value of k_2 then the random walk tends to either zero or infinite volume. All measurements are made for $k_4 = k_4^c(k_2)$.

We have selected not one but a family of measures in order to investigate the influence of the measure in a rather general setting. Our choice is guided by diffeomorphism invariance of the measure [70] but ignores more sophisticated arguments like BRST invariance. We have studied, as a function of n, a measure contribution of the form

$$\prod_x g^{n/2} \ , \tag{28}$$

i.e. in the triangulated theory $S_E[T]$ is replaced by $S[T] = S_E[T] + S_M[T]$, where

$$S_M = -n\sum_a \log\frac{o(a)}{5} \ . \tag{29}$$

The sum runs over all 0-simplices (sites) of the manifold, and $o(a)$ is the number of 4-simplices which include the site a. We considered n in the interval from -5 to 5. The case $n = 0$ repeats simulations with the trivial, uniform measure, which can be compared with previous results.

Let us summarize our results. We have confirmed the fact that the phase transition can be of second order, and that it is plausible that we will be able to define a sensible theory. We find that the measure factor plays an important role, and that the critical behavior does depend on n. This is very different from $2d$ quantum gravity (see for example [1]), where modifications of the measure factor of the same kind we use here do not have any non-trivial effect on the critical behavior. Varying n does not only change non-universal quantities, like for example the value of the critical coupling, but changes the actual (pseudo-)critical behavior.

In figure 20 we plot the average curvature R/V for $V \equiv N_4 = 4000$ as a function of the coupling k_2 for different values of n, $n = -5$ for the lowest curve, then $n = -1$, 0, 1 and $n = 5$ for the upper curve. In figure 21 we plot the average distance (in the internal space) of two 4-simplices. We count the minimum number of steps from 4-simplex to 4-simplex across 3-simplex faces that connect a pair of 4-simplices and average over all 4-simplices and random manifolds.

Both figures show that the measure operator has a pronounced effect. Increasing the coupling of the measure term leads to a continuous, monotonous deformation of the curves. Notice that the curves are not just shifted. In the case of R/V, the singularity seems stronger for $n \simeq 0$, where the jump in R/V is quite sharp. The distance d has a sharper jump for $n = 1$, where it seems to jump from one constant value to another constant. Smaller values of n show a slower increase in d.

For large absolute values of n, especially for $n = -5$, the plots show a weaker singularity. The profile of R/V hints less at a sharp jump than the former cases, and the distance increases very smoothly from a critical value of k_2, $k_2^c(n)$ on. When n increases to the value of 5 the system seems to loose criticality on an absolute scale. Its behavior through the crossover is quite smooth.

Figure 20: Expectation value of R/V for $n = -5, -1, 0, 1, 5$. The curves are ordered from the smallest n at the bottom upwards.

Figure 21: Expectation value of d for $n = -5, -1, 0, 1, 5$, $N_4 = 4000$; curves ordered as in figure 1.

Figure 22: Expectation value of R/V for $n = -5, 0, 5$, $N_4 = 4000, 16000$; curves ordered as in figure 20.

Figure 23: Expectation value of d for $n = -5, -1, 0, 1, 5$, $N_4 = 16000$; curves ordered as in figure 20.

A critical value of k_2^c can be defined, for example, as the point where the distance value starts to change. But for the $n = 5$ case the transition point is not very clear. Let us note that such a value of k_2^c changes its sign as a function of n.

Figure 22 and 23 show results for $N_4 = 16000$ to indicate what kind of finite size effects are present. As evident from figure 22, larger volumes amplify the effect of large absolute n for $k_2 < k_2^c$. Figure 23 for the average distance d displays the same qualitative behaviour as figure 21 for $k_2 < k_2^c$, but for large k_2 the average distance does not remain constant, which might already have been guessed in figure 21. The explanation is that the measure term S_M for positive n introduces a bias towards smaller $o(a)$ which increases d, but for large enough k_2 and k_4 the contribution of S_E dominates and d approaches its $n = 0$ value. This may be the case for $k_2 > k_2^c$, while for $k_2 > k_2^c$ the critical value $k_4^c(k_2)$ is such that $S_E \approx 0$, and at least for the range of negative k_2 considered here S_M is relevant and d remains large for $n = 5$. The same holds for R/V when $n = 5$.

Our conclusion has been that the measure term has a strong effect, which seems difficult to reabsorb in a simple renormalization of the critical coupling. Always keeping in mind that a precise finite size study is required before making quantitative statements [71], we believe there are two basic possibilities. The first possibility is that there is only one universality class, and that all the theories we have studied do asymptotically show the same critical behavior. In this case the rate of approach to the continuum limit is strongly influenced by n. We will select the theory with faster convergence to the continuum.

The second possibility (which is the most interesting one) is that the measure factor changes the universality class. Our results, albeit preliminary, seem to hint in this direction. In this case we could have a critical value of n_c, and transitions belonging to different universality classes. This is a very appealing scenario, and here the lattice discrete theory could make its own original contribution. It could be possible to pick out the correct measure, on the lattice, by requiring a particular expectation value and scaling behavior of some physical observable. Such a prescription would be a powerful tool, turning the discrete version of the theory from a source of indetermination into a completely determined scheme.

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