Design and Analysis of Algorithms
Computer Science 45

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Problem Set 2

Due Friday, April 18.

1. (Warm-up, don't hand in.) Exercise 27.4-2 (Generic Preflow-Push Max. Flow Algorithm, p. 614).
2. Potential function analysis of the Edmonds-Karp algorithm.

   At any point during the execution of the Edmonds-Karp algorithm:

   - Define the *shortest path in-degree* of a vertex v to be the number of edges coming in to v that are on shortest paths from the source to v in the residual graph.
     Let $deg_{sp}(v)$ denote v's shortest-path in-degree.
   - Let $level(v)$ denote the distance of v from the source in the residual graph.
   - Define potential function Φ to be $\sum_v level(v)|E| - deg_{sp}(v)$.

   (a) Show that this potential function increases with every round of the Edmonds-Karp algorithm. Conclude that the number of rounds is $O(V^2E)$.

   (b) Modify the potential function to improve the bound to $O(VE)$.
3. Show that the generic Preflow-Push Algorithm can in fact perform:

   (a) $\Omega(V^2)$ lift operations and $\Omega(V^2)$ saturating pushes

   (b) (Extra Credit) $\Omega(V^3)$ nonsaturating pushes.
4. For the maximum-flow problem, the input is a directed graph $G = (V, E)$, a source vertex s and a sink vertex t, and, for each edge $e \in E$, a capacity $c(e) > 0$.

   For the *minimum-cost flow problem*, one is also given a *cost* $w(e) \geq 0$ for each edge $e \in E$. The *cost* of a flow f on G is defined to be
   $$\sum_{e:f(e)>0} f(e)w(e).$$
   The goal is to find a maximum flow that is of minimum cost among all maximum flows.

   In the residual graph $G_f = (V, E_f)$ (as defined in the book for the maximum-flow problem), define the *cost* $w_f(u, v)$ of each edge $(u, v) \in E_f$ to be $-w(v, u)$ if $f(v, u) > 0$, or $w(u, v)$ otherwise.

   (a) Prove that if the residual graph G_f for a flow f contains no negative cost cycle (with respect to the cost function w_f) then f is a minimum-cost flow among flows of value $|f|$.

   (b) Among all $s \leadsto t$ paths in the residual graph, let p be a shortest path, with respect to the cost function w_f. Prove that augmenting the flow f by sending flow along p yields a flow f' that is of minimum-cost among all flows of value $|f'|$.

   (c) Why does this imply that in a network with integer capacities, there is always an integer minimum-cost maximum flow?