CS 222: Linear Algebra Review

Vector Space

Definition: A *vector space* (a.k.a. *linear space*) consists of:

- a field F of scalars (for us, $F = \mathcal{R}$)
- a set V of vectors
- an operation, $+$, called vector addition, which for all $x, y, z \in V$ satisfies:
  - closure: $x + y \in V$
  - commutativity: $x + y = y + x$
  - associativity: $x + (y + z) = (x + y) + z$
  - existence of identity: $\exists$ a vector, called 0, such that $x + 0 = x$
  - existence of inverse: $\exists$ an inverse, called $-x$, such that $x + (-x) = 0$
- an operation, $\cdot$, called scalar multiplication, which for all $x, y \in V$ and $\alpha, \beta \in F$ satisfies:
  - closure: $\alpha x \in V$
  - existence of identity: $\exists$ an identity, called 1, such that $1x = x$
  - associativity: $(\alpha\beta)x = \alpha(\beta x)$
  - distributivity: $\alpha(x + y) = \alpha x + \alpha y$ and $(\alpha + \beta)x = \alpha x + \beta x$

Examples

What we normally think of when we think of vectors:
$V = \mathcal{R}^n = \{x | x = (x_1, \ldots, x_n), x_i \in \mathcal{R}\}$
$x + y = (x_1 + y_1, \ldots, x_n + y_n)$
$\alpha x = (\alpha x_1, \ldots, \alpha x_n)$
Matrices are also vector spaces:
$V = \mathcal{R}^{m\times n} = \{m \times n \text{ matrices}\}$
$(A + B)_{ij} = A_{ij} + B_{ij}$
$(\alpha A)_{ij} = \alpha A_{ij}$

Linear Combination

Definition: A *linear combination* of vectors $x_1, \ldots, x_n$ is

$$\sum_{i=1}^{n} \alpha_i x_i = \alpha_1 x_1 + \ldots + \alpha_n x_n$$

where $\alpha_1, \ldots, \alpha_n \in F$.

Linear Independence

Definition: A set of vectors $\{x_1, \ldots, x_n\}$ is *linearly independent* iff

$$\sum_{i=1}^{n} \alpha_i x_i = 0 \qquad \Rightarrow \qquad \forall i, \alpha_i = 0$$

Fact: If the set of vectors is linearly dependent, then at least one of them can be written as a linear combination of the others.

Basis

Definition: A linearly independent set of vectors $\{x_1, \ldots, x_n\}$ is a *basis* for a vector space V iff $\forall x \in V$, $\exists \alpha_1, \ldots, \alpha_n \in F$, such that $x = \sum_{i=1}^{n} \alpha_i x_i$

Fact: All bases of a space have the same number of vectors (the *dimension* of the space).

Fact: Every vector x has a unique representation in a given basis. Thus, we can represent a vector by its coefficients, $x = (\alpha_1, \ldots, \alpha_n)$.

Terminology: We say that a basis *spans* a vector space.

Linear Transformation

Definition: Given vector spaces V and W, a *linear transformation* is a mapping $A : V \to W$, such that $\forall \alpha, \beta \in F$ and $\forall x, y \in V$,

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)$$

Column Space

Definition: The *column space* of a matrix $M \in \mathcal{R}^{m \times n}$ is the vector space formed by using the n columns vectors of M as the basis for the space.

Singular Matrices

Definition: A matrix $M \in \mathcal{R}^{n \times n}$ is *singular* iff the n column vectors of M are linearly dependent.

Transpose

Definition: The *transpose* of a matrix $M \in \mathcal{R}^{m \times n}$ is a matrix $N \in \mathcal{R}^{n \times m}$ such that $M_{ij} = N_{ji}$.

Inverse

Definition: The *inverse* of a matrix $M \in \mathcal{R}^{n \times n}$ is a matrix $M^{-1} \in \mathcal{R}^{n \times n}$ such that $MM^{-1} = M^{-1}M = I$.