Newton–Cartan Space-time

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Newton–Cartan Space-time:

its structure and recent applications to dynamical physical theories

by

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Ισως το φως θαναι μια νεα τυραννια.
Ποιος ξερει τι καινουργια πραγματα θα δειξει.
Καβαφης, Τα Παραθυρα (Greek Poet of the 20th century)

To Ioanna

Abstract

Newton–Cartan theory presents a four-dimensional covariant geometrical formulation of Newton's gravity. This report, reviews the history and structure of Newton–Cartan space-time and its applications to dynamical theories like Newton's Gravity, Lagrangian mechanics or even Quantum Mechanics. Ideas of classical physics, combined with Einstein's equivalence principle, mesh with quantum mechanics, as we present a way to describe a quantum test particle moving in a gravitational Newton–Cartan space-time.

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I Prologue

"But the years of Anxious searching in the dark, with their intense longing, their alternations of confidence and exhaustion, and the final emergence into the light – only those who have experienced it can understand that"

A Einstein, as quoted by M Klein (1971)

"In questions of science the authority of a thousand is not worth the humble reasoning of a single individual."

Galileo Galilei (1632)

1. Overview

One of the most important advances in the understanding of physics that this century has offered is the unification of Space and Time, to entities that stayed separate for three centuries in the course of Classical Physics. Historically, Minkowski was the first one to introduce the notion of a mathematical structure named Space-time. In his own words

> *"Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."*

The new entity, space-time, is explored in this essay. More specifically we will explore the structure, properties and applications of the Newton–Cartan space-time, a theory first developed by Cartan in the mid-twenties. Under the new view

of the background space, dynamical theories like Newton's mechanics and gravity, Lagrangian and Hamiltonian mechanics, even quantum mechanics take a totally different form.

The chapters are so written as to permit a stand-alone approach to their contents; nevertheless the whole essay follows a step-by-step approach to the subject. In chapter II, we will explore the historical appearance of the notions of space and time and their meshing to space-time. A first inking of Cartan's ideas will be given here.

In chapter III, we will explore the structure of space-time imparted by classical mechanics and Newton's gravitational theory. This task, is tackled by a bottom-to-top approach i.e. we accept the existence of a general mathematical space and try to reveal its structure by imposing the condition that Newton's laws must hold. Because this is an extensive approach, at the end of the chapter, we present an axiomatic step-by-step approach in the spirit of Trautman(1963). We conclude the existence of a four-dimensional manifold endowed with three inter-related structures. No metric for this space can be supplied, so the theory produced is not a metric theory. The laws of Newtonian mechanics and gravity are restated in the spirit of the new four-dimensional space-time in a manifestly covariant form.

The next chapter, chapter IV, is devoted to Galilei and Bargmann structures. This approach, initiated by Künzle (1972) offers a group-theoretical approach to space-time and its structure and is ultimately connected with the theory of principal and vector bundles. The properties of the Galilei group, i.e. the group whose elements connect physics in two different Galilean co-ordinate systems, are explored. We also state the equations of the Quasi-Galilean gauge group that will be used to simplify equations by the correct choice of gauge. The Bargmann group is merely introduced in this report as the non-trivial extension of the Galilei group acting on a five-dimensional space.

In chapter V, we start exploring the application of Lagrangian mechanics in Newton-Cartan space-time. We accept the existence of an action principle, and locate the Lagrangian for which the action principle yields –through the Euler-Lagrange equations– the geodesic equation of motion for a test particle as produced in chapter IV. We then change to Hamiltonian mechanics, producing what we will call the super-Hamiltonian of a particle. This will lead the way to a direct quantisation with Dirac's constraint quantisation method. We also classify different preferred observers in order to render our task easier in what it is to follow.

In chapter VI, we face the Quantum Challenge. Based in Kuchař's approach, we construct step by step, a Hilbert space of states and a Schrödinger equation to describe the motion of a test quantum particle. This is done by quantising the Hamiltonian produced in the previous chapter using Dirac's constraint quantisation method i.e. changing coordinates and momenta to operators and imposing a limiting constraint condition on the state function.We conclude this chapter by stating the Schrödinger equation for different observers.

In the Epilogue we review the work done and we are faced with open questions.

The appendices were designed as foldouts. This will enable the reader to use them while reading the text, thus making the task of comparing equations easier; it may also remind the reader of the notation and conventions used while typing this report.

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2. Notation and Conventions

In this report, we mainly will use the "Landau–Lifshitz Spacelike Convention" (LLSC) MTW conventions. Greek indices $(\alpha, \beta, \ldots \mu, \nu, \ldots)$ run from 0 to 3 while latin indices $(a, b, \ldots, i, j, \ldots)$ run from 1 to 3. The Lorentzian space-time metric $g_{\mu\nu}$ has signature $(-1, +1, +1, +1)$.

Symmetrization and antisymmetrization of tensors will be given by the use of square brackets and parentheses respectively i.e.

$$\omega_{[\alpha\beta]} \equiv \omega_{\alpha\beta} - \omega_{\beta\alpha}$$
$$\omega_{(\alpha\beta)} \equiv \omega_{\alpha\beta} + \omega_{\beta\alpha}$$

Partial derivatives will be noted by "," or "∂" and Covariant derivatives by semicolon ";" or "∇". Depending on the physical problem, we will also use different types of covariant differentiation denoted by "$_{|\gamma}$".

As far as the Riemmannian geometry is concerned we build the Ricci tensor as

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^{\lambda}{}_{\mu\lambda\nu}.$$

and the Riemmann tensor as

$$\mathcal{R}^{\kappa}{}_{\mu\lambda\nu} = \Gamma^{\kappa}{}_{\mu\nu,\lambda} - \Gamma^{\kappa}{}_{\mu\lambda,\nu} + \Gamma^{\kappa}{}_{\rho\lambda}\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\kappa}{}_{\rho\nu}\Gamma^{\rho}{}_{\mu\lambda}$$

For vectors, we will normally use upper indices and for covectors lower. Operators will be noted by an underbar e.g. $\underline{\nabla}$ and tensors by an undertilde e.g. $\underset{\sim}{\gamma}$. When the geometrical picture needs to be more apparent, three-vectors will be noted by bold-face characters $\mathbf{a}, \mathbf{b}$ etc. Space-time events will be noted by calligraphy letters $\mathcal{P}, \mathcal{Q}, \mathcal{T}, \mathcal{S}$.

II Space, Time and SpaceTime

Nothing puzzles me more than time and space;
and yet, nothing troubles me less, as I
never think about them.
Charles Lamb (1775-1834)

Ye gods! annihilate but space and time, I
And make two lovers happy.[1]
Alexander Pope(1688-174)

Space and Time, have always been though of as different and independent entities, absolute and eternal in their construction. Einstein's mind helped change the view of physicists not only about space and time but also about geometry and its role in physics. New geometrical physical theories appeared, incorporating even Newtons child: Newtonian Mechanics. So, space-times are just mathematical constructions (models) trying to establish a one-to-one correspondence with physical reality.

But let's look in Space and Time in little more detail.

1. Space and Time

"Absolute Space, in its own nature,
without relation to anything external, remains
always similar and immovable."
"Absolute, True, and Mathematical Time,
of itself, and from its own nature, flows
equably without relation to anything external."
Issac Newton, *Scholium in the Principia*

Although the notion of *space* is intuitive to humans, scientists have tried to postulate its existence. This is usually done from the dynamical behavior of matter

[1] *The Art of Sinking in Poetry.*

and energy in small at very low densities and in large at dense concentrations taking in mind the conservation laws. In order to describe space and its properties, a proper language had to be found: mathematics. Thus, the mathematical properties of space-time will represent physical properties such us relative separations of material an non-material objects. Classical space is full of *classical particles*;their idealized properties allow them to evade quantum treatment and yet to be considered small compared with all other characteristic lengths of the system. This heralds the role of the background space: to provide a means of describing the relative locations of our classical particles. The properties of space are thus easy to imagine, properties like orientability (our ability to distinguish left from right and so on), or dimensionality (possible different directions up,left, forward). But mathematicians, have well defined notions of these properties in their own jargon. In this language, for our physical properties we seem bound to use a three-dimensional orientable mathematical space, with the additional property of spatial ordering in each direction.

The notion of *time* seems to emanate from the notion of relative *motion* of objects. We can think of time as an additional capability of ordering (earlier and later) and orienting (future and past) the states of a physical object. Under these assumptions, time must hold a one-dimensional mathematical structure.

In the language of mathematics, both spaces, the one-dimensional time and the three-dimensional space, are to be real *manifolds* mathematical structures endowed with the property of smoothly labeling its points. These real numbers will play the role of *spacetime coordinates.*

The views of Newton about space and time are vividly described in the passage given above. Space and time are, under that view, distinguishable concepts. But modern physics has proved that they are in may ways similar and interrelated so that theoretical physics collectively refers to them as *space-time* even if this has to describe Newtonian Physics. In that framework, in the language of modern mathematics, space and time together form an independent mathematical object; a trivial fibre bundle[2]. Since time is absolute –everybody agrees about simultaneous

[2] See for example Schutz B F 1980 *Geometrical Methods of Mathematical Physics.*

events– we can construct a bundle with base space $\mathbb{E}^1$ (time) and with typical fiber $\mathbb{E}^3$ (space). Note that there is *no* natural relation between points in different fibers (that is points in space at different time); this is in agreement with the relativeness of Newtonian Physics. To different observers, moving with respect to each other, will *not* agree on what constitutes a fixed point in space. This mathematical picture of space-time as the product $\mathbb{E}^1 \times \mathbb{E}^3$ was to be destroyed by Einstein's Relativity and replaced by an affine metric space.

Although the first to understand the importance of a unified space-time was Minkowski by reformulating Einstein's Special Relativity in four dimensions, here we will not follow the flow of time. Instead, we will begin with ideas conceived later starting with Cartan(1924) and presented by several authors in the past 50 years. Their sole purpose: to formulate physics in a four-dimensional way and to explore the relation of the geometry of space-time with physical laws. The theory of General Relativity was directly formulated on this basis and its basic law incorporates the geometrical features of space-time itself. We will at first make a distinction between Newtonian and Einstein Physics; but then we will try to reconcile them by showing their different structure to be based on the action of two different groups: the homogeneous Galilei and the Lorentz Group (Künzle 1972).

2. From Newton to Cartan

No human investigation can be called real science if it cannot be demonstrated mathematically.

Leonardo da Vinci (1452–1519)

In ascending from Newtonian mechanics to the special and general theories of relativity and thereby approaching a more perfect agreement with the experiment, theoretical physics introduced two successive basic versions of its fundamental concepts. However, Newtonian mechanics, still represents a mathematical limit of the theory of relativity. Shouldn't it be possible to formulate it directly in terms of concept much closer to those of relativity than to those of Newton or its 19th century refinements, these concepts having meaning within the framework of Newtonian mechanics itself rather than only as remnants of entirely different structures?

The passageway to the Newton–Cartan theory, is done through geometry. Modern mathematics are used to describe the properties of a space-time which are once and for all related to the gravitational field. The gravitational field under the view of Einstein no longer exists. It is the curvature of the space that shows as gravitational effects. The development of this theory, is in the spirit of Special and General Relativity. The formulation is four-dimensional and the equivalence principle appears to play the most important rôle.

The power of mathematics rests on its evasion of all unnecessary thought and on its wonderful saving of mental operations.

Ernst Mach (1838–1916)

III The Newton–Cartan Space-time

Tout écart décèle une cause inconnue et peut être la source d'une découverte.

Urbain Jean Joseph Le Verrier (1811–1877)

This chapter, is devoted the classical Newton–Cartan space-time, that is mathematical spaces, that can accommodate Newton's Mechanics and Gravity. Since these spaces acquire dynamical properties, we will actually be looking at two classical dynamical theories from a geometrical point of view[1]. This approach, was initiated by Cartan (1923,1924) and continued later by Trautman.

Since our scope is to unravel the geometrical structure of space-time we will proceed with a classical approach. We will assume a general mathematical space as a minimum requirement, and then try to make apparent its hidden properties, by considering the classical laws of mechanics and gravity. We will finally arrive in a space-time of a four-dimensional differentiable manifold in which are defined three interrelated structures. After defining physical quantities, the Laws of Nature, will be re-written in a covariant way; gravity will be connected once and for all with the geometry of the surrounding space. Time will play a special rôle; in the words of Wheeler "*Time is defined so that motion looks simple*"[2]. We will conclude this chapter by a very interesting axiomatic approach in the spirit of Trautman (1963), in which we retrieve all previously acquired results. This will lead the way to what might appear a different mathematical model described in the next chapter: there,

[1] By *dynamical theories* we mean the theories that are concerned with a comparison between the motion of bodies subject to forces and force-free motions.

[2] This view is not new. Note Poincaré: *Le temps doit être défini de telle façon que les équations de la mécanique soient aussi simples que possible.*

symmetry groups will play an essential rôle and the mathematical notion of *fiber bundle* will be used.

1. The general idea

Though this be madness, yet there is method in't[3].

William Shakespeare 1564–1616

How can we actually incorporate Newton's absolute space into Cartan's "*Newtonian Space-time*"? The key to this question is a close examination of the basic principles of Newtonian mechanics under the light of the already successful theory of special and general relativity as described by Einstein. Einstein's main assumption was, that the laws of physics, when formulated in the proper way should be of the same form in *all* special systems. For the special theory that would mean for all *inertial systems*[4] while for the general theory of relativity for *all* coordinate systems. Newton's mechanics is actually compatible with Einstein's requirement. Nevertheless, we must also bear in mind the main difference of Einstein's theory: the constancy of speed of light. Newton permitted his light signals to travel at infinite velocities. As it will be shown, this will lead to the existence of a universal time, kept patiently by the timekeeper of the universe.

It is this detail, the absolute time, that will render our task easier (and at a certain point more difficult). We will assume that *time* is defined once and for all as a scalar function on the mathematical space representing space-time:

$$t = t(\mathcal{P})$$

where $\mathcal{P} \in$ space-time. As for our mathematical space, we will assume a very general one: a 4-dimensional differentiable manifold $\mathbb{M}$. Up to now, we know nothing of the structure of the manifold. In the next sections we will try to unravel the structure of the particular space-time that proves compatible with Newtonian theory. We

[3] Hamlet II.ii.(211).

[4] Here, we will consider as inertial systems, those ones in which Newton's Second Law holds.

will use Newton's Laws and ideas and use a tensorial mathematical language which will help us formulate the theory in a covariant way. We will also consider the way different observers can interchange and compare information i.e. how are different reference frames connected. The importance of the group of transformations on the co-ordinates of reference frames will become more apparent in the next chapter where it will play the salient rôle.

2. The space and its structure group

We will start our study from the basic assumption asserted in the last paragraph: space-time is a 4-dimensional differentiable manifold of which we know nothing, but its definition; yet, this is enough to let us label its points $\mathcal{P}, \mathcal{S}, \ldots$ *locally* by four cartesian co-ordinates: $x^0 \equiv t, x^1, x^2, x^3$. Let us suppose that this co-ordinate system, represents a Galilean Observer (that is an observer who, after experimenting, concludes that Newton's second law is valid). How can we compare the results with another observer? In order to transfer ourselves in a different co-ordinate system we perform a co-ordinate transformation $(x^\mu) \to (x'^\mu)$ of a linear form:

$$x'^\mu = \alpha^\mu{}_\rho x^\rho + \xi^\mu . \tag{3.1}$$

As expected, in that way we have no guarantee that we'll end up in a Galilean co-ordinate system; so we pose restrictions. All transformations that will lead us to a new inertial frame will form a group[5] called the *proper (inhomogeneous) Galilei Group* $\tilde{\mathrm{G}}^0$ and are defined as the affine[6] transformations of $\mathbb{R}^4$ such that

$$(x^0, x^i) \to (x^0 + \xi^0, \alpha^i{}_j x^j + \alpha^i{}_0 x^0 + \xi^i) \tag{3.2}$$

[5] The fact that they form a group, can be seen even from physical arguments.

[6] Reference[5,Cartan 1923]. "En Geometrie ordinaire, il existe des propriétés des figures que l'on appelle *propriétés affines*: ces sont celles qui se conservent lorsqu'on effectue une transformation homographique quelconque conservant le plan de l'infini. Les notions de *vecteur*, d'*equipolance de deux vecteurs*, de *somme geometrique de deux vecteurs*, sont des notions affines; il ne pas de même de la notion de *longeur d'un vecteur*, qui est une notion metrique.".

with $\alpha^i{}_j \in \mathrm{SO}(3, \mathbb{R})$. In other words, we impose the conditions[7]

$$\alpha^0{}_r = 0, \quad \alpha^i{}_j \alpha^j{}_k = \delta^i{}_k, \quad \alpha^0{}_0 = 1 \tag{3.3}$$

The subgroup of these transformations that will leave the origin of $\mathbb{R}^4$ invariant is called the *Proper homogeneous Galilei group* G^0; these satisfy the additional condition

$$\xi^\alpha = 0 \tag{3.4}$$

The physical invariance under these subgroups is even more important than that of the full group as they correspond to uniform relative motion of frames of reference without reflections and time reversal i.e. they express the equivalence if inertial frames.

The independent elements of the matrix $(\alpha)^\mu{}_\rho$ are decreased to six by virtue of the above restrictions and together with the four ξ^μ's make the inhomogeneous Galilei Group $\tilde{\mathrm{G}}^0$ a 10-parameter group of transformations.

3. Fundamental tensors

Up to now, we have restricted our possible transformations between inertial systems, but those relations do not take an invariant(tensorial) form. Nevertheless, it is easy to see[8] that these equations are equivalent to the tensor equations:

$$\begin{aligned} h_{\mu\nu} \alpha^\mu{}_\rho \alpha^\nu{}_\sigma &= h_{\rho\sigma} \\ g^{\mu\nu} \alpha^\rho{}_\mu \alpha^\sigma{}_\nu &= g^{\rho\sigma} \end{aligned} \tag{3.5}$$

for a suitable choice of $h_{\mu\nu}, g^{\mu\nu}$. From the linear transformation equations we have

$$h_{\mu\nu} = (\alpha^{-1})^\rho{}_\mu (\alpha^{-1})^\sigma{}_\nu h_{\rho\sigma} \tag{3.6}$$

and thus $h_{\mu\nu}$ and $g^{\mu\nu}$ have numerically the same components in all co-ordinate systems and we can therefore choose

$$\begin{aligned} h_{00} &= 1 \qquad \text{(time metric)} \\ g^{11} = g^{22} = g^{33} &= -1 \qquad \text{(space metric)} \end{aligned} \tag{3.7}$$

[7] We are only considering orthochronous transformations. For antiorthochronous $\alpha^0{}_0 = -1$.

[8] See for example the review paper by Havas (1964).

These –obviously singular– tensors satisfy the orthogonality relation

$$h_{\mu\nu} g^{\rho\nu} = 0 \tag{3.8}$$

A particularly helpful vector[9] can be introduced, viz.

$$\omega^{\mu} = (1; 0, 0, 0) \tag{3.9}.$$

The components of this vector are numerically the same in any coordinate system. It is clear that

$$\begin{aligned} g^{\mu\nu}\omega_{\nu} &= 0 \\ h_{\mu\nu} &= \omega_{\mu}\omega_{\nu} \end{aligned} \tag{3.10}$$

Our space has already acquired quite a bit of structure: *space slices*–that is hypersurfaces of constant t– are flat and endowed with a euclidean geometrical structure presented by the existence of the space metric tensor g^{ij}. But, although our three-dimensional space is flat as well as our time space, we have yet said nothing of the manifold as a whole. In fact, we will *not* limit the possibility of a curved space time under certain conditions as for example under the presence of matter. Curvature is described by the Riemmann tensor $\mathcal{R}^{\kappa}{}_{\mu\lambda\nu}(\Gamma)$ where Γ represents the connection coefficients. Since we can decide to take all Γ's to be equal to zero for a preferred co-ordinate system they must also vanish (by virtue of the linear transformations and the transformation properties of the Γ's everywhere. In that case we would end up with a flat space-time.

4. A way to measure

Up to now there is no mention of a metric tensor for the whole manifold. We could try to introduce one, but then we would come up with a difficult question. Which one? All possible tensors we've come up to, are singular. A four dimensional "distance" *can* be defined by the time metric $h_{\mu\nu}$ as

$$ds^2 = h_{\mu\nu} dx^{\mu} dx^{\nu} \tag{3.11}$$

but $h_{\mu\nu}$ being singular, we would be lead to a non-Riemmannian space. Although other non-singular tensors can be introduced, there is no guarantee as to if their

[9] the fact that it is a vector is based on its transformation law under $\tilde{G}^0$.

numerical values will be the same in all co-ordinate systems. The defined "length" is then *not* form-invariant i.e. physical length would need a different measurement procedure altogether, in different co-ordinate systems.

Actually, the way ds^2 was defined in equation (11), it is a pure time interval, and so $h_{\mu\nu}$ allows us to distinguish between spacelike and timelike vectors following the rule:

$$h_{\mu\nu}V^{\mu}V^{\nu} \begin{cases} > 0 & \Leftrightarrow \quad V^{\mu} \quad \text{timelike} \\ = 0 & \Leftrightarrow \quad V^{\mu} \quad \text{spacelike} \end{cases} \tag{3.12}$$

A non-trivial characterization of events can be then given to all events by the newtonian way, by using the space metric for standard space measurements:

$$d\ell^2 = \sum_m dx^m dx^m, \quad dx^0 = 0 \qquad \text{(length} \quad \text{measurement)} \tag{3.13}$$

It can be shown[10] that going the other way round is still possible i.e. to recover the Galilei Group $\tilde{G}^0$ from the form invariance of $h_{\mu\nu}$ and $g^{\mu\nu}$ in much the same way that this is done for the –more studied– Lorentz Group.

In order to parametrise events $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \ldots$ –and bearing in mind the treatment of special relativity– we introduce the parameter τ by the relation:

$$d\tau^2 = h_{\mu\nu} dz^{\mu} dz^{\nu} \tag{3.14}$$

where $\mathcal{P} = \{z^{\mu}\}$. This is identical[11] with $z^0 \equiv t$ but helps formulate things in a way more parallel to the special relativistic treatment and more clearly leading to a covariant formulation of our theory. Thus, τ will rather play the role of a parameter along a world line than that of a co-ordinate $z^0 \equiv t$.

5. Defining Physical Objects

In the previous section, we have come up with the conclusion that the mathematical space under consideration is non-metric. It is this absence of a non-singular metric that does not allow us to create an easy one-to-one correspondence between vectors and covectors. In order to bypass this difficulty, we will introduce some

[10] See Havas 1964.

[11] Except from a possible sign difference.

auxiliary vectors of physical importance and hope that –if properly chosen– their properties will help us to establish the correspondence needed.

Let us start by considering an event $(\mathcal{Z}) \equiv \{z^\mu\}$. Then, according to equation (3.11), we define the parameter τ as

$$d\tau^2 = h_{\mu\nu} dz^\mu dz^\nu. \tag{3.11}$$

It is clear that τ differs from $z^0 \equiv \pm t$ by at most an additive constant (in general $\tau = \lambda t + \beta$). We define a *newtonian four-velocity* and *four acceleration* by

$$u^\rho = \frac{dz^\rho}{d\tau}, \qquad \alpha^\rho = \frac{d^2 z^\rho}{d\tau^2} \tag{3.15}$$

We note that $u^i = \frac{dz^i}{dt}$ and $\alpha^i = \frac{d^2 z^i}{dt^2}$ i.e. the spatial parts of the four-vectors, correspond to the usual three-velocity $\mathbf{u}$ and acceleration $\mathbf{a}$.

We come back now, to use the fiducial vector ω_μ defined in equation (3.9), by restating its definition:

$$\omega_\mu = h_{\mu\nu} u^\nu \tag{3.16}$$

and verify that it satisfies the relations

$$u^\rho \omega_\rho = 1, \qquad \omega_\rho \alpha^\rho = 0. \tag{3.17}$$

These relations are the Newtonian counterpart of the special relativistic analogue[12].

Given a three-vector u^i, it is fairly simple to convert this to a four-vector, by adjoining a zero component:

$$u^\alpha = (0; u^j). \tag{3.18}$$

An invariant version of this equation would be

$$u^\alpha \omega_\alpha = 0 \tag{3.19}$$

The following lemma, connecting a vector to a covector can be proved[13]: given a vector V^ρ subject to the above condition, we can define a new co-vector V_ρ under the Galilei Group by

$$V_\rho = (V^i u^i; -V^j) \tag{3.20}$$

[12] It is easy to take the limit of the special relativistic equations when $c \to \infty$.

[13] For a simple proof see Havas 1964.

where *Euclidean* summation convention is implied: repeated Euclidean latin indices $(i, j, \ldots)$ are summed over even if they appear both as subscripts or superscripts.

The co-vector V_α, satisfies the condition

$$V_\alpha u^\alpha = 0. \tag{3.21}$$

Furthermore, V_α is connected to V^α by the relations

$$V^\alpha = g^{\alpha\beta} V_\beta, \qquad V_\alpha = k_{\alpha\beta} V^\beta \tag{3.22}$$

where $g^{\alpha\beta}$ is the space metric and $k_{\alpha\beta}$ is defined as

$$k_{00} = - \mid u^i u^i \mid^2, \quad k_{0r} = k_{r0} = u^r, \quad k_{11} = k_{22} = k_{33} = -1. \tag{3.23}$$

The tensor $k_{\alpha\beta}$ will prove of physical importance in what is to follow and satisfies the relation

$$k_{\alpha\beta} u^\beta = 0 \tag{3.24}$$

parallel to this for $g^{\mu\nu}$

$$g^{\mu\nu} \omega_\nu = 0. \tag{3.25}$$

We have managed in this way to establish a one-to-one correspondence between vectors and covectors (transforming in the proper way under the action of $\tilde{\mathrm{G}}^0$), thing which in general, is only possible for metric spaces. Thus, we arrive to the practical rule:

Greek indices are raised by the space metric$g^{\mu\nu}$

If two events happen to be simultaneous, applying the procedure explain above, we can associate with each co-ordinate differential a co-vector $dx_\alpha = k_{\alpha\beta} dx^\beta$ (for these events). In that case, the three-dimensional length $d\ell^2 = dx^m dx^m, dx^0 = 0$ can be re-expressed as

$$d\ell^2 = -g^{\mu\nu} k_{\mu\lambda} k_{\nu\sigma} dx^\lambda dx^\sigma \tag{3.26}$$

However, from the definition of $g^{\mu\nu}$ and $k_{\mu\nu}$ we compute $g^{\mu\nu} k_{\mu\lambda} k_{\nu\sigma} = k_{\lambda\sigma}$ and we are lead to

$$d\ell^2 = -k_{\mu\lambda} dx^\mu dx^\lambda \tag{3.27}$$

subject to the condition required in our procedure

$$dx^\alpha \omega_\alpha = 0. \tag{3.28}$$

Although it is possible to define some four-vectors directly from the Newtonian three-vector counterparts viz. the acceleration **a** and the force **F**, there is *no* four-vector to be constructed from the newtonian *position vector* **z** and the *velocity vector* **u**. From the acceleration **a** and the force **F** we can construct the four-vectors

$$\alpha_\nu = (a^i u^i; -a^j), \quad F_\nu = (F^i u^i; -a^i) \tag{3.29}$$

where again, Euclidean summation convention is implied. We must note that α_ν is *no* derivative of a four-vector in our formulation.

6. Covariant Re-formulation

Our discussion on space-time, has lead us to the fact that it is flat i.e. for a choice of possible affine connections Γ's the Riemmann–Christoffel tensor vanished

$$\mathcal{R}^\alpha{}_{\beta\gamma\delta}(\Gamma) = 0. \tag{3.30}$$

Since all our formulas are expressed in tensor notation, the transcription to a generally covariant form would be as easy as replacing ordinary derivatives with covariant ones. We must also take into account the fact that covariant derivatives commute due to the absence of curvature. In that case, we understand that the relations are valid not only under the transformations previously discussed (invariance under the Galilei Group $\tilde{\mathrm{G}}^0$) but under *all* analytical transformations.

This new treatment, certainly has its disadvantages; the quantities $g^{\mu\nu}$ and $h_{\mu\nu}$ become tensors and we cannot anymore predict their exact form (co-ordinate numerical values) under any transformation. So, something must change. Our previous requirements for $g^{\mu\nu}$ and $h_{\mu\nu}$ will be changed to: $g^{\mu\nu}$ and $h_{\mu\nu}$ are both symmetric tensors of rank 4 with signatures -3 and 1 respectively. The compatibility condition between the metrics is maintained under the form

$$h_{\mu\nu;\rho} = g^{\mu\nu}{}_{;\rho} = 0. \tag{3.31}$$

The vanishing curvature, together with this condition, will guarantee the existence of a co-ordinate system in which both tensors will take the numerical values that we assumed they had for Galilean systems.

In the covariant formulation, the fiducial vector ω_μ is still defined by equation (3.16)

$$\omega_\mu = h_{\mu\nu} u^\nu \tag{3.16}$$

which satisfies

$$\omega_{\mu;\rho} = 0 \tag{3.32}$$

by its own definition. Using the same arguments as before, we conclude that there exists a system in which ω_μ takes the special form

$$\omega_\mu = (1; 0, 0, 0) \tag{3.9}$$

and still

$$u^\alpha \omega_\alpha = 1. \tag{3.17}$$

Although no mathematical restrictions have to be posed to the analytical transformations under which our equations remain valid, our requirement of a sensible description of physical phenomena will probably restrict the transformations. Certainly, we wouldn't like to find ourselves in a situation where signals will be allowed to travel forward *and* backwards in time at the same time[14]. Since our signals are allowed to travel even at infinite velocities we must demand that for two adjacent events in the world line of a signal

$$ds^2 = h_{\mu\nu} dx^\mu dx^\nu \geq 0. \tag{3.34}$$

In the case of an infinite velocity the equality sign will hold leading to the simultaneity of events $dx^0 = 0$. Following this part and using physical arguments it can

[14] "*It is bad enough to know the past; it would be even more intolerable to know the future*" W. Maugham 1874–1965.

be proved[15] that the form of the fundamental tensors $g^{\mu\nu}$ and $h_{\mu\nu}$ is restricted in *any* co-ordinate system to the form

$$\begin{aligned} h_{\mu\nu} = 0 \qquad &\text{for} \quad \text{all} \quad \mu, \nu \neq 0 \\ g^{\mu 0} = g^{0\mu} = 0 \qquad &\text{for} \quad \text{all} \quad \mu \end{aligned} \tag{3.35}$$

which will also lead to

$$\omega_\mu = 0 \qquad \text{unless} \quad \mu = 0. \tag{3.36}$$

In much the same way, k_{mn} must transform independently of $k_{\mu\nu}$ leading to the condition

$$k_{mr} g^{rn} = \delta_m{}^n \tag{3.37}$$

in *all* co-ordinate systems. This is an important relation; the physical meaning of the "artificially" introduced tensor $k_{\mu\nu}$ is starting to appear. The spatial part of $k_{\mu\nu}$, k_{mn} is actually the inverse of the restriction of the space metric $g^{\mu\nu}$ in three-dimensions g^{mn}.

Since Γ's vanish in any inertial system, we must be able to calculate them n arbitrary co-ordinates by the use of their transformation laws. The physical arguments used before lead us again to

$$\Gamma^0_{\mu i} = \Gamma^0_{i\mu} = 0. \tag{3.38}$$

It can be proved, that the rest of the Γ's components in an arbitrary co-ordinate system are given by

$$\begin{aligned} \Gamma^0_{00} &= \partial_0 \log \sqrt{h_{00}} \\ \Gamma^r_{mn} &= \frac{1}{2} g^{rs}(k_{ns,m} + k_{sm,n} - k_{mn,s}) \end{aligned} \tag{3.39}$$

which is nothing but the definition of the three-dimensional Christoffell symbol formed in a space with metric k_{mn}

$$\Gamma^m_{ij} = \left\{ \begin{matrix} & m & \\ i & & j \end{matrix} \right\} \tag{3.39b}$$

Based on the definition of u^μ, we will define four-acceleration as

$$\alpha_\rho \equiv \mathbf{D}_\tau u^\rho \equiv \frac{du^\rho}{d\tau} + \Gamma^\rho_{\mu\nu} u^\mu u^\nu \tag{3.40}$$

[15] For the proof see Havas 1964.

where $\mathbf{D}_\tau \equiv \nabla_\tau$ denotes covariant differentiation with respect to τ. By defining $P^\alpha = mu^\alpha$ we can restate Newton's second law as

$$m\alpha^\lambda = \mathbf{D}_\tau P^\lambda = F^\lambda \tag{3.41}$$

with

$$F^\lambda u_\lambda = 0, \qquad \alpha^\lambda u_\lambda = 0. \tag{3.42}$$

We can introduce a scalar potential ϕ producing the force F^λ in the way that

$$\mathbf{D}_\tau P^\lambda = g g^{\lambda\sigma} \phi_{;\sigma} \tag{3.43}$$

where g is the "interaction" constant.

In the case of a non-interacting particle, i.e. in the total absence of forces we end up with the equation

$$\frac{du^\rho}{d\tau} + \Gamma^\rho_{\mu\nu} u^\mu u^\nu = 0. \tag{3.44}$$

In the more general case of the presence of a field the equation becomes

$$\boxed{\frac{du^\rho}{d\tau} + (\Gamma^\rho_{\mu\nu} - h_{\mu\nu} g^{\rho\sigma} \phi_{;\sigma}) u^\mu u^\nu = 0. \qquad (3.45)}$$

Law of Motion in a Field

7. The world is curved!

Some hypothesis are dangerous– first and foremost those which are tacit and unconscious. . . . Here again, there is a service that mathematical Physics may render us[16].

Henri Poincaré 1854-1912

Our treatment has made it easy to proceed further on. We will follow Einstein's steps, and assume that our covariant Law of Motion is nothing but the geodesic line

[16] H. Poincaré, *Science and Hypothesis*.

of a curved space time. In that case we simply read off the term $\Gamma^\rho_{\mu\nu} - gg^{\rho\sigma}\phi_{;\sigma}$ as being the affine connection of our space. Changing our notation, we will name the new connection $\Gamma^\alpha_{\beta\gamma}$ and rename the old one to $\mathring{\Gamma}^\alpha_{\beta\gamma}$

$$\mathring{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + \Omega^\alpha_{\beta\gamma} \tag{3.46}$$

with $\Omega^\alpha_{\beta\gamma} \equiv h_{\beta\gamma}g^{\alpha\sigma}\phi_{;\sigma}$ being a tensor of rank $\binom{1}{2}$. Our space-time is not flat any more but the Riemmann tensor still satisfies

$$\mathcal{R}^\kappa{}_{\mu\lambda\nu}(\mathring{\Gamma}) = 0 \tag{3.47}$$

where all components of $\mathring{\Gamma}$ vanish in a preferred (Cartesian) co-ordinate system of reference. In the new notation, the Law of motion becomes

$$\frac{du^\rho}{d\tau} + \Gamma^\rho_{\mu\nu}u^\mu u^\nu = 0. \tag{3.48}$$

Although the Riemmann–Christoffel curvature tensor formed by the old affine connections $\mathring{\Gamma}$ vanish, this is not generally true for the one formed with the new Γ's. We conclude that the presence of a potential ϕ has produced a curvature in our mathematical construction.

We must be happy with our results, even though our space is not metric. Our fundamental tensors being singular, it is impossible to directly calculate the values of Γ's as a function of our fundamental tensors $g^{\mu\nu}$, $h_{\mu\nu}$, and the scalar potential ϕ. At least we know one thing for sure: both covariant derivatives of $g^{\mu\nu}$ and $h_{\mu\nu}$ vanish –thing easy to prove in a preferred system and then generalize. So,

$$\begin{aligned} h_{\mu\nu;\rho} &= h_{\mu\nu,\rho} - \Gamma^\sigma_{\mu\rho}h_{\sigma\nu} - \Gamma^\sigma_{\nu\rho}h_{\mu\sigma} = 0 \\ g^{\mu\nu}{}_{;\rho} &= g^{\mu\nu}{}_{,\rho} + \Gamma^\mu_{\rho\sigma}g^{\sigma\nu} + \Gamma^\nu_{\rho\sigma}g^{\mu\sigma} = 0 \end{aligned} \tag{3.49}$$

It is obvious that these equations cannot be solved for Γ's since both $g^{\mu\nu}$ and $h_{\mu\nu}$ are singular. It will prove helpful, to introduce a new covariant derivative with respect to the old affine connections $\mathring{\Gamma}$. This derivative, to avoid confusion, will be denoted by "$_{|\gamma}$". Under this derivative, analogous equations to (3.49) must hold

$$\begin{aligned} h_{\mu\nu|\rho} &= h_{\mu\nu,\rho} - \mathring{\Gamma}^\sigma_{\mu\rho}h_{\sigma\nu} - \mathring{\Gamma}^\sigma_{\nu\rho}h_{\mu\sigma} = 0 \\ g^{\mu\nu}{}_{|\rho} &= g^{\mu\nu}{}_{,\rho} + \mathring{\Gamma}^\mu_{\rho\sigma}g^{\sigma\nu} + \mathring{\Gamma}^\nu_{\rho\sigma}g^{\mu\sigma} = 0 \end{aligned} \tag{3.50}$$

In order to be able to calculate the equations satisfied by the affine connections $\Gamma^\alpha{}_{\beta\gamma}$, we will produce a set of new equations from the Ricci tensor. In the preferred system of reference

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^\lambda{}_{\mu\lambda\nu} = \Gamma^\lambda{}_{\mu\nu,\lambda} - \Gamma^\lambda{}_{\mu\lambda,\nu} + \Gamma^\lambda{}_{\rho\lambda}\Gamma^\rho{}_{\mu\nu} - \Gamma^\lambda{}_{\rho\nu}\Gamma^\rho{}_{\mu\lambda} = -h_{\mu\nu}g^{\rho\sigma}\phi_{,\rho\sigma}. \tag{3.51}$$

Since both $\mathring{\Gamma}^\alpha{}_{\beta\gamma}$ and $g^{\alpha\beta}\Gamma^\varepsilon{}_{\alpha\beta}$ vanish in the preferred co-ordinate system, we can rewrite the equation as either

$$\mathcal{R}_{\mu\nu} = -h_{\mu\nu}g^{\rho\sigma}(\phi_{,\rho\sigma} - \mathring{\Gamma}^\kappa{}_{\rho\sigma}\phi_{,\kappa}) = -h_{\mu\nu}g^{\rho\sigma}\phi_{|\rho\sigma} \tag{3.52a}$$

or

$$\mathcal{R}_{\mu\nu} = -h_{\mu\nu}g^{\rho\sigma}(\phi_{,\rho\sigma} - \Gamma^\kappa{}_{\rho\sigma}\phi_{,\kappa}) = -h_{\mu\nu}g^{\rho\sigma}\phi_{;\rho\sigma} \tag{3.52b}$$

Since $h_{\mu\nu}$ is singular, we can not obtain a bianchi identity, but it can be proved that the field equation will be in exact equivalence with Einstein's theory of General Relativity

$$\begin{gathered} \mathcal{R}_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\mathcal{R} = \kappa h_{\mu\rho}h_{\nu\sigma}T^{rho\sigma} \\ \mathcal{R} = g^{\rho\sigma}\mathcal{R}_{\rho\sigma} \quad , \quad \kappa = 4\pi G. \end{gathered} \tag{3.53}$$

The curvature tensor itself, is proved to be in direct connection with the potential ϕ and the universal time T by the equation

$$\mathcal{R}^\alpha{}_{\beta\gamma\delta} = T_{,\beta}\phi^\alpha{}_{;[\gamma}T_{\delta]} \tag{3.54}$$

.

8. An axiomatic approach

We have shown that it is possible to reduce Newton's Mechanics and Gravity to a geometrical theory of space-time in complete analogy with Einstein's General Theory of Relativity. Let us now put everything together and reverse our process; since we already have a deeper understanding of the intrinsic structures of the Newton–Cartan theory, we will restate the whole theory in an axiomatic way. This could have been done from the beginning and then recover Newton's Laws from the new formulation; but then, all axioms as well as any quantities introduced, would look arbitrary and without justification. In this section, we will give such an axiomatic approach to Newton–Cartan theory.

The space-time of Newton–Cartan theory is a four-dimensional manifold $\mathbb{M}$ endowed with an affine connection. There exists a function

$$T : \mathbb{M} \longrightarrow \mathbb{R}$$

creating a family of hypersurfaces of T which we require that they carry a three-dimensional Euclidean structure i.e. when being on a hypersurface $T = const$ we can perform distance measurements under the normal Euclidean laws. The family of hypersurfaces T therefore allows a stratification of our manifold $\mathbb{M}$. Local co-ordinates can be introduced for any event $\mathcal{X}$ as $\{x^\alpha\}$ and all events lying on the hypersurface of T obey the equation[17]

$$\tau(\mathcal{X}) = \text{constant}. \tag{3.55}$$

The basic geometrical structures defined on the manifold $\mathbb{M}$ are the affine connexion Γ, the curvature tensor $\mathcal{R}$, the covariant derivative ∇_α and two symmetric tensors $g^{\mu\nu}$ and $h_{\mu\nu}$. The last two tensors, will play the role of space and time metric respectively. These structures are interrelated and follow special conditions. Any event –that is any point of our space time– can be given co-ordinates on $\mathbb{M}$ which doesn't intersect with each of the hypersurfaces in more than one point. The equation for the world line would therefore be

$$x^\alpha = z^\alpha(\tau) \tag{3.56}$$

I. Both space and time metrics are degenerate with signatures (0;1,1,1) and (1;0,0,0) respectively while they are mutually orthogonal

$$g^{\alpha\beta} h_{\beta\gamma} = 0 \tag{3.57}$$

From these metrics we introduce the proper time τ by the equation

$$d\tau^2 = h_{\mu\nu} dx^\mu dx^\nu. \tag{3.58}$$

[17] The formal definition of τ is given later on.

Using these metrics, we can associate vectors on the tangent space of the manifold to co-vectors on the dual space using the convention that greek indices are *raised* by the space metric and for lowering we will use the auxiliary tensor $k_{\mu\nu}$ introduced in equation (3.23)

$$V^\alpha = g^{\alpha\beta} V_\beta, \qquad V_\alpha = k_{\alpha\beta} V^\beta. \tag{3.22}$$

The space metric, also helps us distinguish between space-like and time-like vectors,

$$h_{\mu\nu} V^\mu V^\nu \begin{cases} > 0 & \Leftrightarrow \quad V^\mu \quad \text{timelike} \\ = 0 & \Leftrightarrow \quad V^\mu \quad \text{spacelike} \end{cases} \tag{3.12}$$

II. The affine connection Γ is without tortion i.e. $\Gamma^\alpha{}_{[\beta\gamma]} = 0$.

III. The covariant derivative ∇_α is compatible with both metrics i.e.

$$\nabla_\gamma g^{\alpha\beta} = 0 = \nabla_\gamma h_{\alpha\beta} \tag{3.59}$$

IV. The curvature tensor of the affine connection has the properties

$$\begin{aligned} h_{\alpha[\beta} \mathcal{R}^\gamma{}_{\delta]\mu\nu} &= 0 \\ \mathcal{R}^{[\alpha}{}_\beta{}^{\gamma]}{}_\delta &= 0 \end{aligned} \tag{3.60}$$

Note that the curvature tensor is introduced under the convention[18]

$$V^\alpha{}_{;[\mu\nu]} = V^\varepsilon \mathcal{R}^\alpha{}_{\varepsilon\mu\nu} \tag{3.61}$$

where V is any four-vector.

In order to establish a contact with physical reality, we introduce the correlation axiom[19]

V. Timelike intervals can be measured by standard clocks, while spacelike intervals by standard rods. The motion of the free particles is such that the world lines are geodesics with respect to the affine connexion Γ.

[18] See the Review paper of Kuchař 1980.

[19] The adjective *free* will be used as corresponding to a neutral particle which is only affected by gravitational phenomena.

VI. The equation of motion in free space is described by

$$\mathcal{R}_{\alpha\beta} = 0. \tag{3.62}$$

Given a vector field ϕ^α, we can define new connections[20] $\mathring{\Gamma}^\alpha{}_{\beta\gamma}$ by the equation

$$\Gamma^\alpha{}_{\beta\gamma} = \mathring{\Gamma}^\alpha{}_{\beta\gamma} + \phi^\alpha \partial_\beta \tau \partial_\gamma \tau. \tag{3.63}$$

from which is easy to calculate that

$$\mathring{\mathcal{R}}^\alpha{}_{\beta\gamma\delta} = 0 \tag{3.64}$$

From the postulates and the previous equations we conclude that the law of motion may be written as

$$\frac{d^2 x^\alpha}{d\tau^2} + \mathring{\Gamma}^\alpha{}_{\beta\gamma} \frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = -\phi^\alpha. \tag{3.65}$$

Thus, the vector ϕ^α represents the gravitational force and $\mathring{\Gamma}^\alpha{}_{\beta\gamma}$ is the connexion that defines the inertial systems of reference. We have to accept a global character for both the vector field ϕ^α and the connexion $\mathring{\Gamma}^\alpha{}_{\beta\gamma}$ and therefore conclude that the latter is a flat connexion.In the contrary, the connexion $\Gamma^\alpha{}_{\beta\gamma}$ is only locally defined and thus might in general correspond to a curved space-time.

From the definition of the metrics and the new connections we conclude that both metrics are compatible with the new connection

$$\mathring{\nabla}_\gamma g^{\alpha\beta} = 0 \quad \mathring{\nabla}_\gamma h_{\alpha\beta} = 0 \tag{3.66}$$

This shows that a preferred system of reference exists (Galilean system of reference) in which

$$\mathring{\Gamma}^\alpha{}_{\beta\gamma} = 0 \tag{3.67}$$

and at the same time

$$g^{\alpha\beta} = \mathrm{diag}[0\,1\,1\,1]. \tag{3.68}$$

[20] We perform a slight change of notation by not replacing the new connection with the old one.

The Galilean transformations, are exactly the ones which preserve these two last equations. If we are concerned with infinitesimal transformations of the form

$$x'^\alpha = x^\alpha + \xi^\alpha d\lambda \tag{3.69}$$

these are Galilean transformations, if and only if

$$\mathcal{L}_\xi \mathring{\Gamma}^\alpha{}_{\beta\gamma} = 0 \quad and \quad \mathcal{L}_\xi g^{\alpha\beta} = 0 \tag{3.70}$$

where $\mathcal{L}$ is the Lie derivative.

In the Newtonian theory, there exist of course other different systems from the already examined inertial ones; these are the more general Cartesian systems which, using time as one of their co-ordinates, they are limited by the requirement of a constant metric tensor but not of a vanishing connection; they are the non-inertial Cartesian systems which give rise to inertial forces like the centrifugal and the Coriolis force. That is a consequence of the non-vanishing of the expression

$$\mathring{\Gamma}^\alpha{}_{\beta\gamma} \frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} \tag{3.71}$$

In a metric theory of Gravity, where space-time has a well defined metric, the connection is a direct function of this metric and every transformation which leaves the metric invariant will also leave the connexion invariant; the theory of relativity offers such a metric space, either in the way described by Minkowski (1908) incorporating the Special Theory, or as described by Einstein in the theory of General Relativity.

IV Galilei and Bargmann Structures: An Overview

Symmetry is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection.

Herman Weyl (1885–1955)

The formulations of space-time discussed up to now, were described by many authors in there urge to find similarities in Newtonian and Einstein Physics. The four dimensional formulation of the Newtonian theory appeared as early as the General Theory of Relativity itself. This formulation, was later also applied to continuum mechanics (Trusdell and Toupin 1960); from these theories stems the rôle of symmetry groups (Toupin 1958). At about the same time the notion of Galilean manifold and Structure appeared with a detailed study of its differential geometry (Dombrowski and Horneffer 1964). Later on, the structure groups were "once and for all connected with the particular space-time structure" (Künzle 1972) and Newtonian and Einstein Physics were put on a strictly equal footing.

The case of the structure groups was extended and carefully studied so as to be connected with quantum mechanics and the "conformal" symmetry group of the Schrödinger equation by Duval *et al.* 1985 following the earlier work of Bargmann 1954. Duval and Künzle (1984) attempted to present Newtonian Gravity as a gauge theory of the Bargmann group (i.e. the 11–dimensional nontrivial extension of the inhomogeneous Galilei Group).

In this chapter, we will take a closer look to the internal properties of the Galilei and extended Galilei structure groups.

1. Galilei Manifolds and Structures

We have seen in the last chapter, how important the rôle of the Galilei group is: it defines the admissible reference frames of an observer at any space-time event. For that reason, we will define a more general mathematical structure on the manifold $\mathbb{M}$ called a Galilei structure. This will be based on the mathematical concept of a G–bundles. This new formalism (Künzle 1972) appears to be useful for the comparison of any "non-relativistic" and the corresponding "relativistic" theory. Less assumptions are needed and most concepts are introduced smoothly instead of the *ad hoc* approach of the previous Newton–Cartan chapter. The general idea of a Galilei structure on a manifold, is a reduction of the bundle of linear frames to a sub-bundle of frames invariant under the homogeneous Galilei Group.

1.1 Galilei Group

The definition of the proper inhomogeneous Galilei Group $\tilde{\mathrm{G}}^0$, was given in section 3 of the last chapter. Here we will deal with the proper homogeneous Galilei group G^0 which can be defined by the left action on $\mathbb{R}^{n+1}$ by

$$\mathrm{GL}(n+1,\mathbb{R}) \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}. \tag{4.1}$$

With respect to this action, $\tilde{\mathrm{G}}^0$ is the semi-direct product $\mathbb{R}^{n+1} \circledS \mathrm{G}^0$. It can be proved (Künzle 1972) that proper homogeneous Galilei transformations of the $(n+1)$-dimensional vector space V are characterized by leaving invariant a given linear 1-form $\psi \in V^*$ and a given positive semi-definite symmetric tensor $\gamma \in V \otimes V$ of rank n such that ψ is in the kernel of γ, i.e. $\psi_r \gamma^{rs} = 0$. This, offers a more convenient characterization than the one using a covariant degenerate metric, which will prove simplifying in notation. If we extend our discussion to the *full* homogeneous Galilei Group (i.e. including space and time inversions), we have to replace ψ by a symmetric 2-form Ψ of rank one, with $\Psi = \psi \otimes \psi$.

1.2 Definitions of Galilei Manifold and Structure

If $\mathbb{M}$ is an $(n+1)$-dimensional C^∞-manifold and $(\mathcal{G}l(\mathbb{M}), \pi, \mathbb{M})$ the principal bundle of linear frames over $\mathbb{M}$, we will define as a *Galilei Structure* on $\mathbb{M}$, a reduction $\mathcal{G}(\mathbb{M})$ of $\mathcal{G}l^0(\mathbb{M})$ to the proper homogeneous Galilei group. The manifold will then be named a (proper) *Galilei manifold*.

A proper Galilei structure $\mathcal{G}^0$ on $\mathbb{M}$ defines and is characterized by a pair $(\underset{\sim}{\gamma}, \psi)$ of a positive semi-definite contravariant symmetric tensor field $\underset{\sim}{\gamma} = \gamma^{\alpha\beta}(x)\partial_\alpha \times \partial_\beta$ of rank n and a never vanishing 1-form $\psi = \psi_\alpha dx^\alpha$, subject to the condition

$$\gamma^{\alpha\beta}\psi_\beta = 0. \tag{4.2}$$

Such a structure on a manifold $\mathbb{M}$ will exist iff there exists a time orientable and space orientable Lorentz structure. In particular, it is proved that $\mathbb{M}$ has to be orientable and non-compact or have Euler characteristic zero (Steenrod 1951).

Given a Galilei manifold $(\mathbb{M}, \underset{\sim}{\gamma}, \psi)$, we can define a *Galilei frame* $\{e_\alpha\}$ at a point $x \in \mathbb{M}$, as a basis on the tangent space $T_x\mathbb{M}$ such that $e^\alpha_\varepsilon \psi_\alpha = \delta^0{}_\varepsilon$ and $\gamma^{\alpha\beta}\theta^\varepsilon_\alpha\theta^\rho_\beta = \delta^\varepsilon{}_i\delta^{i\rho}$ with $\{\theta^\alpha\}$ the dual basis of $\{e_\alpha\}$. The 1–form ψ defines an n–dimensional subspace S_x of $\mathrm{T}_x\mathbb{M}$ for each $x \in \mathbb{M}$. Each tangent vector $V \in \mathrm{T}_x\mathbb{M}$ with $V \perp \psi = 0$, is called *spacelike*. A *timelike* vector will then be any $V \in \mathrm{T}_x\mathbb{M}$ with $V \perp \psi \neq 0$. Timelike vectors can be future or past directed, if $V \perp \psi(\gtrless)0$. A *timelike unit vector* will be the one for which $V \perp \psi = 1$.

1.3 Scalar product and foliation

A natural inner product on the tangent spaces $\mathrm{T}_x\mathbb{M}$ can be introduced using the contravariant tensor $\gamma_x \in \mathrm{T}_x\mathbb{M} \otimes \mathrm{T}_x\mathbb{M}$. The scalar product is positive definite on the hyperplanes S_x and is defined by

$$\mathbf{V} \cdot \mathbf{V} = V^i\delta_{ij}V^j \tag{4.3}$$

where V is a vector on the hyperplane S_x for a given Galilei frame.

Every proper Galilei manifold will carry a canonical *volume element* defined by

$$\tau = (1/4!)\varepsilon_{\alpha\beta\gamma\delta}dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta \equiv \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \tag{4.4}$$

for any Galilei frame.

The Galilei manifold will be *integrable* if the 1–form ψ is closed (Künzle 1972). In that case, the hyperplanes S_x define a foliation of space slices on $\mathbb{M}$, locally given by $S_\tau = t^{-1}(\tau)$, where t is a locally defined function satisfying the condition $dt = \psi$. The hyperplanes S_t are then Riemannian manifolds. In that case, local

co-ordinate systems can be introduced with x^0 being the time co-ordinate; the 1–form ψ must then obey $\psi_\alpha = \delta^0_\alpha$, whence $\gamma^{\alpha 0} = 0$. In this so called *adapted co-ordinate system*, the "spatial" part of the tensor γ_{ij} and its inverse γ^{ij} play the role of the Riemmannian metric. It can be proved that if the induced Riemmannian metrics are locally flat so is the Galilei Manifold (Künzle 1972).

1.4 Connections and curvature of a Galilei Manifold

Introducing the covariant derivative ∇ on $\mathbb{M}$, we define as linear *Galilei connection* any linear connection Γ for which the covariant derivatives of γ and ψ vanish. In local co-ordinates that would mean

$$\nabla_\alpha\gamma^{\beta\varepsilon} = \gamma^{\beta\varepsilon}{}_{,\alpha} + 2\Gamma^{(\beta}{}_{\alpha\rho}\gamma^{\varepsilon)\rho} = 0 \tag{4.5}$$

and

$$\nabla_\alpha\psi_\beta = \psi_{\beta,\gamma} - \Gamma^\rho{}_{\alpha\beta}\psi_\rho = 0. \tag{4.6}$$

The torsion tensor defined by $\mathrm{T}^\alpha_{\beta\gamma} = \Gamma^\alpha{}_{[\alpha\beta]}$ of the connection Γ satisfies

$$\mathrm{T}^\rho{}_{\alpha\beta}\psi_\rho = 2\psi_{[\beta,\alpha]}. \tag{4.7}$$

If we want a torsion free Galilei Manifold, we must restrict ψ to a closed 1–form: $d\psi = 0$. All the Galilei manifolds of current physical interest, admit a symmetric torsion-free connection and therefore are locally flat Galilei manifolds. Unfortunately, even then, the above equations do not determine uniquely a symmetric connection. Nevertheless, it is proved that with the introduction of an arbitrary timelike unit vector field $\underset{\sim}{u} = u^\alpha\partial_\alpha$ a particular solution is

$$\Gamma^\gamma{}_{\alpha\beta} = \gamma^{\gamma\rho}u^\sigma{}_{,\rho}\gamma_{\sigma(\alpha}\psi_{\beta)} - \gamma_{\rho(\alpha}\gamma^{\rho\gamma}{}_{,\beta)} + \frac{1}{2}\gamma^{\gamma\rho}\gamma^{\mu\nu}{}_{,\rho}\gamma_{\mu\alpha}\gamma_{\nu\beta} - \psi_{(\alpha}u^\gamma{}_{,\beta)} \tag{4.8}$$

with $\gamma_{\alpha\beta}$ defined by $\gamma_{\alpha\rho}u^\rho = 0$ and $\gamma_{\alpha\rho}\gamma^{\rho\beta} = \delta^\beta{}_\alpha - \psi_\alpha u^\beta = \gamma^\beta{}_\alpha$ the projection parallel to u of $\mathrm{T}_x\mathbb{M}$ onto S_x. The following interesting result is proved in Künzle 1972: the Galilei connection of flat Galilei manifolds are in one-to-one correspondence with the set of 2–forms in $\mathbb{M}$. Given a symmetric Galilei connection, *any* other connection can be expressed as

$$\mathring{\Gamma}^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} + 2\psi_{(\beta}k_{\gamma)\rho}\gamma^{\rho\alpha} \tag{4.9}$$

for a suitable choice of a 2–form k. From here, one can easily derive in the adopted co-ordinate system that the submanifolds S_t are autoparallel and the connection induced on them by any symmetric Galilei connection on $\mathbb{M}$ coincides with the connection of the induced Riemmannian metric.

We will call a symmetric Galilei connection a *Newtonian connection* if the curvature produced from this connection satisfies

$$\mathcal{R}^{[\alpha}{}_{(\beta}{}^{\gamma]}{}_{\delta)}(\Gamma) = \gamma^{\rho[\alpha}\mathcal{R}^{\gamma]}{}_{(\beta\delta)\rho} = 0 \tag{4.10}$$

The existence of a Newtonian connections on locally flat Galilei manifolds can be easily proved, and the set of all Newtonian connections is in one-to-one correspondence with the set of *closed* 2–forms on $\mathbb{M}$. We will call the set $(\mathbb{M}, \gamma, \psi, \Gamma)$ a Newtonian manifold.

1.5 How does Newtonian Gravity look under this formalism

The formalism given in this chapter, leads to a straightforward re-formulation of Newtonian Gravity based on two axioms:

I. Space-Time is a four-dimensional Newtonian manifold

II. The gravitational field equations are

$$\mathcal{R}_{\alpha\beta} = 4\pi G\rho\psi_\alpha\psi_\beta \tag{4.11}$$

where ρ is the matter density and G is the Newtonian gravitational constant.

These axioms imply a locally flat Galilei structure and can also accommodate Newtonian cosmological solutions for non-flat three-space sections[1]

2. The Quasi-Galilei gauge Group

The formalism introduced above, helps us to build a general mathematical framework. But, when we come to the need of solving a particular problem, the old

[1] This can be done in the expense of introducing an additional vector field.

tensorial co-ordinate formalism remains our basic tool. Generalizations help acquire a better understanding of the whole but prove insufficient for a bottom-to-top approach.

In this section, we will state an extension of the Galilei group of transformations in the spirit of the tensorial formalism. We will transform the Galilei Group to a gauge group, i.e. we will introduce a set of interrelated parameters of physical interest that are allowed to be varied in a prescribed way so as to "gauge" the theory. Under the choice of an arid gauge, equations will look appalling whereas for a particularly chosen *bona fide* gauge, equations will take a simple form to read and interpret.

In the last chapter, we have concluded that the Riemmann tensor of curvature in space-times with gravitation, depends (uniquely) on the elements T, x^α, u^α and ϕ (equation 3.54). Unfortunately, these elements –although interlocked by relations presented in the previous chapter– are not fixed uniquely and different combinations of them can produce the same curvature tensor. Therefore, Galilean observers are not connected by the Galilei group but by a gauge group which is a generalization of the Galilei group for space-times with gravitational fields.

The relations for the co-ordinate transformations of a Galilei observer, shown here as X^α were proved to be

$$\tilde{X}^i = (\alpha)^i{}_j X^j \quad and \quad \tilde{T} \equiv \tilde{X}^0 = T - T_0 \tag{4.12}$$

under the constraints

$$\delta^{mn}(\alpha)^i{}_m(\alpha)^j{}_n = \delta^{ij} \tag{4.13}$$

where $(\alpha)^i{}_i$ and T_0 are constants. We introduce four arbitrary functions of T viz. $R^i(T)$ and $\varphi(T)$. The gauge transformations allowed will then be given by

$$\begin{aligned} \tilde{X}^i &= X^i - R^i(T) \\ \tilde{u}^\alpha &= u^\alpha + U_i(T)X^i{}_{,\beta} \\ \tilde{\phi} &= \phi + \mathcal{Q}_i(T)\tilde{X}^i + \varphi(T) \end{aligned} \tag{4.14}$$

with

$$U^i \equiv \dot{R}^i \equiv dR^i/dT, \qquad \mathcal{Q}^i \equiv \dot{U}^i \equiv \ddot{R}^i \tag{4.15}$$

The gauge transformations for the potential ϕ show an interesting point: the new potential $\tilde{\phi}$ satisfies directly the Laplace equation and therefore is indistinguishable from the "true" gravitational potential ϕ.

The Quasi-Galilei gauge group introduced here, will play an important rôle in the chapters to come. It will enable us to choose specific gauges for each class of preferred observer and results in Newton–Cartan space-time will often take the familiar form of the dynamical theory under consideration.

3. The quest for higher dimensions: Bargmann Structures

The Bargmann Group B was first introduced in Quantum mechanics by Bargmann (1954). The group-theoretic approach hid its geometrical significance for quite a long time. Bargmann structures in Newton–Cartan space-times were first introduced as non-trivial extension of the Galilei Group G by Duval and Künzle (1978). From then on, the Bargmann group gained a classical status and a geometrical interpretation. An extended paper by Duval *et al.* (1985) looks at possible Bargmann structures on Newtonian space-times and establishes the morphisms between Galilei and Bargmann structures. The rôle of the Bargmann group in Quantum Newton–Cartan space-times was investigated by Prugovečki (1987) following the geometrostochastic quantisation techniques. Here, we are only going to review the basic definitions and characteristics of the Bargmann Group and Manifold.

A ***Bargmann Manifold*** $\mathbb{M}^5$ is uniquely specified by a Lorentzian metric g of signature $(+ + + + -)$ and a global Killing field ξ satisfying

$$g(\xi,\xi) = 0 \quad \text{and} \quad \tilde{\nabla}\xi = 0 \tag{4.16}$$

where $\tilde{\nabla}$ is the unique Levi-Civita connection compatible with the non-degenerate metric tensor g. The integral curves of ξ induce a fibration of $\mathbb{M}^5$ so that it can be viewed as a principal $(\mathbb{R}^1, +)$ bundle with projection $\pi : \mathbb{M}^5 \longrightarrow \mathbb{M}^4$ onto a Galilean space-time $\mathbb{M}^4$. Hence, $\mathbb{M}^5$ can be viewed as a trivial bundle $\mathbb{M}^4 \times \mathbb{R}^1$, whose typical fibre $\mathbb{R}^1$ can be given a quantum mechanical interpretation of including phase factors of the state of the system.

The ***Bargmann Group*** can be viewed as a group of transformations acting on the Bargmann Manifold $\mathbb{M}^5_0 \simeq \mathbb{R}^4 \times \mathbb{R}^1$ corresponding to the case of a null gravitational

field. Any group element can be represented by $(R, \mathbf{b}, \mathbf{c}, c^0, \lambda)$ with $R \in \mathrm{SO}(3)$ $\mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and $\lambda, c^0 \in \mathbb{R}$. The group law

$$\begin{aligned}(R, \mathbf{b}, \mathbf{c}, c^0, \lambda)(\tilde{R}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}, \tilde{c}^0, \tilde{\lambda}) = \\ = (R\tilde{R}, \mathbf{b} + R\tilde{\mathbf{b}}, \mathbf{c} + \mathbf{b}\tilde{c}^0 + R\tilde{\mathbf{c}}, c^0 + \tilde{c}^0, \lambda + \tilde{\lambda} - \frac{1}{2}\mathbf{b}\cdot\mathbf{b}\tilde{c}^0 - \mathbf{b}\cdot R\tilde{\mathbf{c}}).\end{aligned} \tag{4.17}$$

The applications of the Bargmann Group in classical gravitational physics is still an open field of research and the reconciliation of the Bargmann covariance of classical particle and field theory with the Newton–Cartan gravitational theory an important result; this could give important insight on the connection of gravity as presented in the General Theory of Relativity with Quantum Field Theory.

V Lagrangian Mechanics in Newton Cartan Space-time

In any change which occurs in nature, the sum of the products of each body multiplied by the space it traverses and by its speed (referred to as action) is always the least possible.

Pierre Louis Moreau de Maupertuis (1698-1759)

We proceed now, in possible applications of the Newton–Cartan space-time in dynamical physical theories. Lagrangian and Hamiltonian mechanics has proved useful not only because of the rigorous mathematical formalism meshing with physical reality but also because of the close being to Quantum mechanics. Indeed, using Dirac's constraint quantisation we can be lead from a classical to a quantum system. Since this is our final scope, to explore the formulation of quantum mechanics in Newton–Cartan space-time, in this chapter we will deal with the possibility of producing the particle equations of motion from a generalized action principle. The mathematical framework used is based on the axiomatic approach of Newton–Cartan theory as summarized in the foldout of Appendix 2. The classical material on variational principles, comes from Lanczos 1970 and Doughty 1990. This approach was first introduced by Havas(1964) and treated in detail by Kuchař(1980).

First, we will state the Newtonian geodesics (i.e. the paths of neutral particles) and try to find the action which yields the geodesics. We will then deparametrise this action so as to prepare the way to Dirac's quantisation method for the next chapter.

1. Newtonian Geodesics and the Variation Principle

Geodesics can be defined as "straightest paths" by the mathematical requirement that the tangent vector $\omega^\alpha \equiv \dot{x}^\alpha$ be parallely propagated along them,

$$(\nabla_\omega \omega)^\alpha \sim \omega^\alpha \quad \text{or} \quad \ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma \sim \dot{x}^\alpha. \tag{5.1}$$

In Riemmannian spaces with non-degenerate metrics, the knowledge of the connections Γ comes directly from the metric itself since the latter satisfies the compatibility condition $\nabla \mathbf{g} = 0$. Unfortunately, in Newton–Cartan space-time, the metrics are degenerate (as we saw in chapter 3) and are insufficient to fix the affine connection. Our next hope would be to express the connection as a direct function of the metrics and the auxiliary vectors fields introduced, that is the four-velocity and the gravitational potential. But even then we can do nothing. Only by using a particular gauge, viz. the conditions of a Galilean observer, can we bypass the problem.

We first introduce the *covariant* degenerate space metric $g_{\alpha\beta}$ by the relations (Künzle 1972)

$$g_{\alpha\beta}g^{\beta\gamma} = \delta^\gamma{}_\alpha - T_\alpha u^\gamma, \quad g_{\alpha\beta}u^\beta = 0 \tag{5.2}$$

with T being the universal time, $T_\alpha \equiv \partial_\alpha T$ and $g^{\alpha\beta}$ the degenerate space metric. This covariant metric, will be used for lowering greek indices, although it is degenerate and *not* gauge independent. With the help of the space covariant and contravariant metrics we form with the standard prescription the connection ${}^0\Gamma^\alpha{}_{\beta\gamma}$. A gauge invariant affine connection can then be introduced (Kuchař 1980) with the form

$$\Gamma^\alpha{}_{\beta\gamma} = {}^0\Gamma^\alpha{}_{\beta\gamma} + u^\alpha T_{,\beta\gamma} + \phi^\alpha T_\beta T_\gamma. \tag{5.3}$$

We would like to obtain the geodesic equation from a variational principle of an action

$$S[x] = \int d\tau L(x, \dot{x}) \tag{5.4}$$

as paths of extremal length. If we want the action to be independent of any parametrisation, the Lagrangian function $L(x, \dot{x})$ must be a homogeneous function of the first degree in the four-velocity $\dot{x}^\alpha$. In order to write down a Lagrangian we specifically select a Newton–Cartan space-time $\{g^{\alpha\beta}, h_{\alpha\beta}, \Gamma^\alpha{}_{\beta\gamma}\}$ with Galilean observer $\{u^\alpha, \phi\}$ and assume a unitary mass. The Lagrangian

$$L(x, \dot{x}) \equiv \frac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta (T_\gamma \dot{x}^\gamma)^{-1} - \phi T_\gamma \dot{x}^\gamma \tag{5.5}$$

when grinned through the Euler-Lagrange equations implies the equations

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = N^{-1}\dot{N}\dot{x}^\alpha \tag{5.6}$$

with Γ given from equation (5.3) and

$$N \equiv T_\alpha \dot{x}^\alpha = dT/d\tau \tag{5.7}$$

is called the *lapse function*. For a space parametrised by its absolute time $T = \tau$ the right hand side of the equation (5.6) vanishes proving absolute time an affine parameter.

Having acquired a Lagrangian of a certain form we can easily pass in the standard way of Classical Mechanics to a generalized Hamiltonian formalism[1]. We define a four-momentum conjugate to x^α by the relation

$$p_\alpha \equiv L_{,\dot{\alpha}}. \tag{5.8}$$

Using the Lagrangian introduced we compute that

$$p_\alpha = N^{-1} g_{\alpha\beta}\dot{x}^\beta - ET_\alpha \tag{5.9}$$

where E is given by

$$E \equiv \frac{1}{2}N^{-2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + \phi. \tag{5.10}$$

Note that the canonical variables x^α and p^α are *not* independent since they are subject to a constraint (Kuchař 1980)

$$\mathcal{H} \equiv \frac{1}{2}g^{\alpha\beta}p_\alpha p_\beta + u^\alpha p_\alpha + \phi = 0. \tag{5.11}$$

Under the change of variables the action will take the form

$$S[x,p] = \int d\tau p_\alpha \dot{x}^\alpha \tag{5.12}$$

[1] See for example Lanczos 1970 sec. VI.10, Doughty 1990 sec. 8.6.

Under the variational principle $\delta S = 0$, all variables x^α, p_α can be freely varied, always subject to the constraint (5.11). In order to incorporate the $\mathcal{H}$ function in the variational principle, we add a freely varying Lagrangian multiplier N as

$$S[x,p;N] = \int d\tau(p_\alpha\dot{x}^\alpha - N\mathcal{H}). \tag{5.13}$$

Taking the Hamilton equations and comparing to the geodesic equations of (5.6) we conclude that N is nothing but the lapse function. The name ***super-Hamiltonian*** given to the produced function $\mathcal{H}(x,p)$ in the literature will be used further on.

We note that the super-Hamiltonian stays invariant under the quasi-Galilean gauge group[2]; the change induced in the canonical momenta p_α is exactly compensated by the gauge transformation of the potential ϕ and the auxiliary vector u^α.

2. The view of different Observers

2.1 The Super-Hamiltonian of an arbitrary observer

In the previous section, we were lead in an expression of the super-Hamiltonian from the view-point of a Galilean observer. But this is *not* good enough. We would like to be able to include as many classes of observers as possible. Let as join an arbitrary observer and characterize him by the normalized four velocity vector v^α

$$v^\alpha T_\alpha = 1. \tag{5.14}$$

In order to be able to tell how the motion of the arbitrary observer deviates from that of a Galilean observer we introduce the test vector

$$A^\alpha \equiv v^\alpha - u^\alpha. \tag{5.15}$$

We now introduce the characteristic vector v^α of the observer in the super-Hamiltonian produced in the last section through the relation (5.15). We get

$$\mathcal{H} = v^\alpha p_\alpha + \frac{1}{2}g^{\alpha\beta}(p_\alpha - A_\alpha)(p_\beta - A_\beta) + \phi - \frac{1}{2}g^{\alpha\beta}A_\alpha A_\beta. \tag{5.16}$$

[2] See chapter IV section 2, or use appendix I foldout for a quick reference of the relations.

A new interpretation for this Hamiltonian can be given if we group the last two terms in

$$\phi_v = \phi - \frac{1}{2} g^{\alpha\beta} A_\alpha A_\beta \tag{5.17}$$

Then, A^α plays the role of the vector potential and ϕ_v of the scalar potential of the gravitational field of an observer υ^α. The most general form of the action S can be given by modifying p_α and the gravitational potentials by arbitrary gauge transformations

$$\begin{aligned} p_\alpha &\longrightarrow \tilde{p}_\alpha = p_\alpha + \lambda_{,\alpha}(x) \\ A_\alpha &\longrightarrow \tilde{A}_\alpha = A^\alpha + \lambda^{,\alpha} \\ \phi_v &\longrightarrow \tilde{\phi}_v = \phi_v - \upsilon^\alpha \lambda_{,\alpha} \end{aligned} \tag{5.18}$$

Then the action takes the form

$$\boxed{\begin{aligned} S[x^\alpha, \tilde{p}_\alpha; N] &= \int d\tau (\tilde{p}_\alpha \dot{x}^\alpha - N\mathcal{H}) \\ \mathcal{H} &= \upsilon^\alpha \tilde{p}_\alpha + \frac{1}{2} g^{\alpha\beta} (\tilde{p}_\alpha - \tilde{A}_\alpha)(\tilde{p}_\beta - \tilde{A}_b) + \tilde{\phi}_v . \end{aligned} \quad (5.19)}$$

2.2 Special classes of observers

Since doing physics in preferred frames is always easier, it is worthwhile to investigate the privileged classes of observers. Up to now, we were dealing, almost tacitly, with the palatable Galilean observers which offered rigid non-rotating frames of reference. Another privileged observer would be one moving along a geodesic, i.e falling freely in space-time. Depending on the aforementioned properties, we can divide the observers in the following hierarchical classes

$$\text{Inertial} \begin{cases} \textit{Galilean} & \begin{cases} \upsilon^{(\alpha;\beta)} = 0 & \text{rigid} \\ \upsilon^{[\alpha;\beta]} = 0 & \text{nonrotating} \end{cases} \\ \textit{Gaussian} & \begin{cases} \upsilon^{[\alpha;\beta]} = 0 & \text{nonrotating} \\ \upsilon^\alpha{}_{;\beta} \upsilon^\beta = 0 & \text{freely falling} \end{cases} \end{cases} \tag{5.20}$$

Although inertial observers exist only in gravity free space-times, all the other types can be found in arbitrary Newton–Cartan space-times. Given an observer υ^α we can simplify the form of the action principle and the super-Hamiltonian by choosing specific fiducial co-ordinates and choosing a specific gauge.

2.3 The super-Hamiltonian in comoving coordinates

As in general relativity, we can introduce a Momentarily Comoving Reference Frame (MCRF) i.e. a frame of *co-moving coordinates*. To choose these co-ordinates, we pick up three independent scalar functions $Y^i(x^\alpha)$ with the property of being constant along the world line of the observer

$$\upsilon^\alpha Y^i{}_{,\alpha} = 0. \tag{5.21}$$

Then the set $\{T, Y^i\}$ *is* the set of the four comoving co-ordinates. In that case, the action (5.19) yields

$$\tilde{p}_\alpha \dot{x}^\alpha = p_\alpha \upsilon^\alpha \dot{T} + \tilde{p}_\alpha Y^\alpha{}_i \dot{Y}^i. \tag{5.22}$$

If we make the substitution

$$\begin{aligned} &\tilde{P}_T = p_\alpha \upsilon^\alpha, \quad \tilde{P}_i = \tilde{p}_\alpha Y^\alpha{}_i, \\ &Y^\alpha{}_i \equiv g^{\alpha\beta} Y^j{}_{,\beta} g_{ji}. \end{aligned} \tag{5.23}$$

the super-Hamiltonian reduces to

$$\mathcal{H} = \tilde{P}_T + \tilde{H}, \tag{5.24}$$

with

$$\tilde{H} = \frac{1}{2} g^{ij} \tilde{P}_i \tilde{P}_j - \tilde{A}^i \tilde{P}_j + \phi. \tag{5.25}$$

Note that A^α is a spacelike vector and thus completely characterized by the three projections $A^i \equiv A^\alpha Y^i{}_{,\alpha}$. In these co-ordinates the form reduces to a "deparametrised form" (Kuchař 1980)

$$S[Y^i, \tilde{P}_i] = \int dT (\tilde{P}_i \frac{dY^i}{dT} - \tilde{H}). \tag{5.26}$$

The physical interpretation is fairly straightforward: $\tilde{H} = -\tilde{P}_T$ is the energy of the particle under observation by an *arbitrary* observer in an *arbitrary* gauge.

VI The Quantum Challenge

Life, that dares send
A challenge to his end,
And when it comes, say 'Welcome, friend!'

Richard Crashaw 1613-1649

The passion to unravel the Laws of Nature, has led physicists to two well formulated and experimentally successful theories: general Relativity and Quantum Mechanics. Unfortunately, it seemed at the time that there was no connection between them and any attempt to reconcile the two will lead to weird results. Behind these attempts, was the ancient philosophical belief of economy and unity in Nature; trying to unify gravity and quantum mechanics would be a big step towards a more complete understanding of her Laws.

But alas! the attempts for unification, recovered an unsuspected number of subtle conceptual difficulties. Everything led to the quantisation of geometry that is space-time itself. Physicists tried to study quantum fields in classical curved background, but did not know *how* to interpret the resulting fields. The conflict was between the local character of Einstein's General Relativity (Principle of equivalence) and the global nature of certain Laws in quantum mechanics. But then, why are these problems not present in the case of Newtonian Gravity which is certainly in accordance with Einstein's equivalence principle?

1. Dimensional Pyramid[1]

If we wanted to order existing physical theories in a single diagram we would look at the role played by the physical constants c, G, h characterizing each theory. From the diagram we can distinguish the limiting positions of theories with respect to others.

[1] Kuchař 1980.

We start from Kuchař's *Classical Vertex*, the beginning of classical physics; quantum effects are negligible ($h \to 0$), gravitational effects causing the curvature of space-time are not present ($G \to 0$), and all interactions are instantaneous ($c \to \infty$ or $c^{-1} \to 0$). Here, as described in chapter III, global inertial frames, connected by the Galilei group are filled with classical particles moving and interacting at a distance.

Fig. 1. Dimensional Pyramid

Staying on a plane of $h = 0$, the *classical plane* and moving along the axis of G we experience the existence of Gravity. Since interactions are instantaneous we are dealing with Newton–Cartan theory of Gravitation (see Chapter III). If we move along the c^{-1} axis we enter the area of special relativity and Minkowski's flat space-time. Here, interactions propagate with finite speed, described by Poincaré-invariant field equations. Still, global inertial frames are available for experimenting. Leaving the last two axes to the classical plane, we enter General Relativity. No more global inertial frames; these are substituted by *local* inertial frames. The gravitational field is identified with the geometry of space-time and possesses its own dynamical degrees of freedom. If we move towards the G-axis, interactions become almost instantaneous and the dynamical degrees of freedom are lost; here space-time is still curved as described by the Newton–Cartan theory. Let us leave the classical plane from the classical vertex on the h-axis thus entering the quantum area. On the h-axis we expect to find classical non-relativistic quantum mechanics. If we move towards the c^{-1}–h plane we encounter relativistic quantum field theory in a flat space-time. whereas towards the G–h plane non-relativistic quantum theory in Newton–Cartan space-time. This limit is not well studied. One of the reasons might be the fact that since the dynamical degrees of freedom of the gravitational field remain frozen, no hint can be extracted for quantum gravity.

The greatest region –the interior of the schematic pyramid– still remains *terra incognita* and must correspond according to the diagram, to generally relativistic quantum field theory and quantum Gravity. This is yet to be studied and enlightened by current research.

The "Dimensional Pyramid" diagram, is helpful, since it illustrates the various limits from yet unknown theories as generally relativistic quantum field theory is.

2. Quantisation of a freely falling particle

In this section, we will try to describe the motion of a freely falling *quantum* particle in Newton–Cartan space-time. We will formulate laws in a manifestly covariant way. Later on we will retrieve Scrödinger's Equation and thus be lead to the result that Quantum Mechanics is actually compatible with the Principle of Equivalence.

2.1 Constraint Quantisation

Probably the easier way to acquire a common base with Quantum mechanics, is to try to quantize the Hamiltonian of a freely falling particle in Newton–Cartan space-time derived in the previous chapter. In order for that to come true, we will use Dirac's constraint quantisation method[2]. According to this method, we first turn the co-ordinate functions into operators

$$\underline{x}^\alpha = x^\alpha \quad , \quad \underline{\tilde p}_\alpha = -i\nabla_\alpha. \tag{6.1}$$

We then write down the classical Hamiltonian and treat it as an operator

$$\begin{aligned}\underline{\tilde{\mathcal{H}}} &= -\frac{1}{2}i[\upsilon^\alpha(x)\nabla_\alpha + \nabla_\alpha\upsilon^\alpha(x)] + \underline{\tilde H}_\upsilon \\ &= -i(\upsilon^\alpha\nabla_\alpha + \frac{1}{2}\upsilon^\alpha{}_{;\alpha}) + \underline{\tilde H}_\upsilon\end{aligned} \tag{6.2}$$

[2] Dirac 1951.

where

$$\underline{\tilde H}_\upsilon = -\frac{1}{2}g^\alpha\beta\nabla_\alpha\nabla_\beta + i(\tilde A^\alpha\nabla_\alpha + \frac{1}{2}\tilde A^\alpha{}_{;\alpha}) + \phi \tag{6.3}$$

is simply the Hamiltonian operator of the test particle as observed from $\{\upsilon^\alpha\}$. The final step is to impose the super-Hamiltonian constraint as a restriction of the state function:

$$\underline{\tilde{\mathcal{H}}}\tilde\psi(x) = 0 \tag{6.4}$$

Since we are dealing with operators and the commutators do not necessarily vanish, we require that the ordering is such that terms quadratic to the momenta are replaced by a covariant Laplacian (with respect to the degenerate metric $g^{\alpha\beta}$) and terms linear to the momenta are symmetrically ordered through anticommutators.

2.2 Working in the Hilbert space of state functions

The state functions

$$\tilde\psi(x) : \mathbb{M} \longrightarrow \mathbb{C} \tag{6.5}$$

($\mathbb{M}$ being the Newton–Cartan space-time) belong in a Hilbert space, equipped with an inner product on any leaf of constant time. The mathematical construction is actually a vector bundle E over $\mathbb{M}$ with typical fiber $\mathcal{H} = L^2(\mathbb{R}^3, d^3\mathbf{q})$, the quantum mechanical space for a single particle. If $\tilde\psi_1$ and $\tilde\psi_2$ both belong to the hilbert space of states (thus subject to the quantisation constraint 6.4), the inner product is defined with the help of the invariant volume element dV_T as

$$\langle\tilde\psi_1|\tilde\psi_2\rangle \equiv \int_T dV_T\tilde\psi_1^*(x)\tilde\psi_2(x). \tag{6.6}$$

It can be proved, that the inner product introduced in (6.6), does not depend on the choice of the time lead T. To prove that, we first have to find and explicit expression for the invariant volume element dV_T. We therefore introduce the four-dimensional Levi–Civita pseudo-tensor $\epsilon^{\alpha\beta\gamma\delta}$. In the case of Galilean coordinates $\{T, X^i\}$ this tensor is given by the relation

$$\epsilon^{\alpha\beta\gamma\delta}T_{,a}X^1{}_{,\beta}X^2{}_{,\gamma}X^3{}_{,\delta} = 1. \tag{6.7}$$

In the leaf T, we form a parallepiped with edges $d_1x^\alpha, d_2x^\alpha, d_3x^\alpha$:

$$T_{,\alpha}d_ix^\alpha = 0 \tag{6.8}$$

Using the contavariant metric $g^{\alpha\beta}$, we calculate the covectors corresponding to $d_i x^\alpha$ as

$$d_i x^\alpha = g^{\alpha\beta} d_i x_\beta. \tag{6.9}$$

Note that the covectors $d_i x_\alpha$ are arbitrary up to terms proportional to $T_{,\alpha}$. With the help of the space metric we calculate the volume of the formed parallepiped

$$dV_T = T_{,\alpha}\epsilon^{\alpha\beta\gamma\delta} d_1 x_\beta d_2 x_\gamma d_3 x_\delta. \tag{6.10}$$

Despite the fact that a certain arbitrariness exists for the covectors $d_i x_\alpha$, this does not contribute to the volume element. The volume element changes along time leafs lying on a geodesic of v^α by[3]

$$\mathcal{L}_v \int_{V_T} dV_T = \int_{V_T} dV_T v^\alpha{}_{;\alpha}. \tag{6.11}$$

The inner product of the Hilbert space as defined, has other good properties: the Hamiltonian $\underline{\tilde{H}}$ defined in (6.3) turns out to be Hermitian under the particular inner product. This is based on a version of Gauss Law for the Newton–Cartan space-time. This version is not exactly a four-dimensional generalization since we will still be dealing with spacelike vectors V^α lying on leaves of absolute time: $V^\alpha T_{,\alpha} = 0$. In Newton–Cartan space-time, we can not define a four-dimensional volume element and we are thus confined in three-dimensional leaves of absolute time.

Taking a volume V_T on the leaf T we take as n^α the unit spacelike vector normal to the boundary ∂V_T and put $n^\alpha = g^{\alpha\beta} n_\alpha$. On the boundary surface ∂V_T we form the parallelogram with edges $d_1 x^\alpha, d_2 x^\alpha$ and area

$$d(\partial V_T) = \epsilon^{\alpha\beta\gamma\delta} T_{,\alpha} n_\beta d_1 x_\gamma d_2 x_\delta. \tag{6.12}$$

The Gauss Law can then be written as

$$\int_{V_T} dV_T \nabla_\alpha V^\alpha = \int_{\partial V_T} d(\partial V_T) V^\alpha n_\alpha. \tag{6.13}$$

[3] To prove the following equation, we can use coming coordinates of the observer.

2.3 Properties of the Hamiltonian

The Hamiltonian, is –as mentioned in the previous section– Hermitian with respect to the scalar product of the Hilbert space introduced. The commutator can be calculated as

$$\begin{aligned}\langle\tilde{\psi}_1|\underline{\tilde{H}}_v\tilde{\psi}_2\rangle - \langle\underline{\tilde{H}}_v\tilde{\psi}_1|\tilde{\psi}_2\rangle &= \int_T dV_T \nabla_\alpha \tilde{j}^\alpha_{12} \\ &= \int_{\partial T} d(\partial T)\tilde{j}^\alpha_{12} n_\alpha.\end{aligned} \tag{6.14}$$

with

$$\tilde{j}^\alpha_{12} \equiv \frac{1}{2}[\tilde{\psi}^*_1(\nabla^\alpha\tilde{\psi}_2) - \nabla^\alpha\tilde{\psi}^*_1)\tilde{\psi}_2] + i\tilde{A}^\alpha\tilde{\psi}^*_1\tilde{\psi}_2 \tag{6.15}$$

being a spacelike vector, therefore allowing the Gauss Law to act. In a Hilbert space, where the state functions $\tilde{\psi}_1, \tilde{\psi}_2$ vanish sufficiently fast at the spatial infinity ∂T, the integral (6.15) goes to zero proving that *the Hamiltonian $\underline{\tilde{H}}_v$ is Hermitian.*

2.4 Interpretation of results

Limiting ourselves in a proper unit volume on the leaf T, we will identify

$$\tilde{\omega} \equiv \tilde{\psi}^*\tilde{\psi} \tag{6.16}$$

as the *probability density* and

$$\tilde{j}^\alpha = \frac{1}{2} i\tilde{\psi}^* \overleftrightarrow{\nabla}^\alpha \tilde{\psi} - \tilde{A}^\alpha\tilde{\psi}^*\tilde{\psi} \tag{6.17}$$

as the *probability current.*

The theory as formulated is manifestly covariant with respect to different observers, gauges and co-ordinates. The probability density is scalar whereas the inner product is an invariant under transformations of co-ordinates. The probability density and the probability current are both gauge independent

$$\tilde{\omega} \equiv \tilde{\psi}^*\tilde{\psi} = \psi^*\psi \equiv \omega \tag{6.18}$$

and

$$\tilde{j}^\alpha \equiv \frac{1}{2} i\tilde{\psi}^* \overleftrightarrow{\nabla}^\alpha \tilde{\psi} - \tilde{A}^\alpha\tilde{\psi}^*\tilde{\psi} = \frac{1}{2} i\psi^* \overleftrightarrow{\nabla}^\alpha \psi - A^\alpha\psi^*\psi \equiv j^\alpha. \tag{6.19}$$

The Hamiltonian constraint (6.4) must be the Scrödinger equation. In the next section, we will choose a specific preferred co-ordinate system and try to reveal the Schrödinger equation in a more familiar form.

3. Revealing the Schrödinger equation

Let us use the comoving co-ordinate system $\{T, Y^i\}$ to express the previously gained results. The Hamiltonian $\underline{\mathcal{H}}$ contains four-divergences of the vector potentials v^α and A^α. Thus, in order to calculate the Hamiltonian, we will need to know, how to express divergences of vector fields.

3.1 Vector divergence in comoving co-ordinates

Take V^α, an arbitrary vector field. Then

$$\begin{aligned} V^\alpha{}_{;\alpha} &= V^\alpha{}_{;\beta}(T_{,\alpha}v^\beta + Y^i_\alpha Y^\beta_i) \\ &= (V^\alpha T_{,\alpha})_{,\beta}v^\beta - (V^\gamma T_{,\gamma})v^\alpha Y^i_{\alpha;\beta}Y^\beta_i + V^i{}_{,i} - V^j Y^\alpha_j Y^i_{\alpha;\beta}Y^\beta_i \end{aligned} \tag{6.20}$$

This equation can be simplified using the christoffell symbol of the metric g^{ij}

$$\Gamma^i{}_{jk} = Y^\alpha_j Y^i_{\alpha;\beta} Y^\beta_k \tag{6.21}$$

where Y^α_i was defined in equation (5.23). Introducing the determinant of the metric $g \equiv \det g^{ij}$ we calculate

$$V^\alpha{}_{;\alpha} = (V^\alpha T_\alpha)_{,\beta}v^\beta + \frac{1}{2}(g_{,T}/g)V^\alpha T_\alpha + V^i{}_{|i} \tag{6.22}$$

where by "$_{|i}$" we denote covariant differentiation with respect to the metric g^{ij}.

3.2 Calculating the Hamiltonian Operator

The divergences of the vector fields v^α and A^α, will be

$$\begin{aligned} v^\alpha{}_{;\alpha} &= \frac{1}{2}(g_{,T}/g) \\ \tilde{A}^\alpha{}_{;\alpha} &= \tilde{A}^i{}_{|i} \end{aligned} \tag{6.23}$$

The term $-\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ can be shown to reduce to to the Laplacian of the instantaneous metric

$$\big(-\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\nabla_\beta\big)\tilde{\psi} = g^{ij}\tilde{\psi}_{|ij} \equiv \nabla^2\tilde{\psi}. \tag{6.24}$$

Putting all these back to the Hamiltonian

$$\underline{\tilde{\mathcal{H}}} = -i\big(v^\alpha\nabla_\alpha + \frac{1}{2}v^\alpha{}_{;\alpha}\big) - \frac{1}{2}g^\alpha\beta\nabla_\alpha\nabla_\beta + i\big(\tilde{A}^\alpha\nabla_\alpha + \frac{1}{2}\tilde{A}^\alpha{}_{;\alpha}\big) + \phi \tag{6.25}$$

and taking the constraint equation

$$\underline{\tilde{\mathcal{H}}}\tilde{\psi}(x) = 0 \tag{6.4}$$

we are lead to the familiar result

$$i(g^{1/4}\tilde{\psi})_{,T} = \big[-\frac{1}{2}\nabla^2 + i\big(\tilde{A}^i\nabla_i + \frac{1}{2}\tilde{A}^i{}_{|i}\big) + \phi\big]g^{1/4}\tilde{\psi}. \tag{6.26}$$

3.3 Schrödinger Equation in sight

Equation (6.26) is nothing but the Schrödinger equation of a non-relativistic particle moving in a space characterized by a time dependent metric $g_{ij}(T, Y^k)$ under the influence of a vector potential $\tilde{A}^i(T, Y^k)$ and a scalar potential $\phi(T, Y^k)$. From here, the equation of probability conservation for the inner product defined in the previous section is

$$\langle\tilde{\psi}_1|\tilde{\psi}_2\rangle = \int_T d^3Y (g^{1/4}\tilde{\psi})^*(g^{1/4}\tilde{\psi}). \tag{6.27}$$

As in all gauge theories, we can simplify our equations by choosing a particularly helpful gauge: e.g. $\tilde{A}^i$ being divergence free, or the scalar potential $\tilde{\phi}_v$ to vanish. The results for particular gauges produced in Kuchař (1980) are summarized in the table

Observer	**Schrödinger's Equation**
General	$i(g^{1/4}\tilde{\psi})_{,T} = [-\frac{1}{2}\nabla^2 + i\tilde{A}^i\nabla_i + \phi]g^{1/4}\tilde{\psi}$
Nonrotating	$i(g^{1/4}\tilde{\psi})_{,T} = [-\frac{1}{2}\nabla^2 + \tilde{\phi}_v]g^{1/4}\tilde{\psi}$
Galilean	$i\psi_{,T} = [-\frac{1}{2}\delta^{ij}\partial_i\partial_j + \phi]\psi$
Gaussian	$i(g^{1/4}\tilde{\psi})_{,T} = -\frac{1}{2}\nabla^2(g^{1/4}\tilde{\psi})$
Inertial	$i\tilde{\psi}_{,T} = -\frac{1}{2}\delta^{ij}\partial_i\partial_j\psi$

Table 6.I: Schrödinger's Equation for different observers

VII Epilogue

So the principles which are set forth in this treatise will, when taken up by thoughtful minds, lead to many another more remarkable result; and it is to be believed that it will be so on account of the nobility of the subject, which is superior to any other in nature.

Galileo Galilei (1638)

In the past few pages, we have reviewed the story of an unwanted child of Physics: the Newton–Cartan space time. From its birth, the theory was in parallel with Einstein's General Relativity; nowadays, the theory revives in a more modern language as a preliminary study for the quest of Physics: the unification of all laws in one principal Natural Law. As a background space where motion or quantum effects take place –the *substratum* of philosophers– Newton–Cartan space-time offers the possibility of a deep study in Quantum theory in classical gravitational potentials.

In such a limited space and time of this essay, we merely reviewed the main features. The spectrum of applications of the theory presented herein is actually a lot wider; recent research work offers even path integral formulations in Newton–Cartan space-times (Bièvre 1989), Bargmann structures, and research goes on.

The important result appearing in this review work, is the possibility to derive the correct quantum dynamics for a test particle in a gravitational field from the application of the equivalence principle. The whole theory is written in a four-dimensional covariant way Newtonian gravity appears as a geometrical theory. Note that none of the formulations developed adds anything to the physical predictions of the theory.

It is amazing to realize, how the same physical ideas can be expressed in so many different languages of mathematics; how many different frameworks can include the same physical reality. Already in the three-dimensional formulations we can

replace the fundamental laws by variation principles; the action at a distance by field description governed by the Poisson equation. In the four-dimensional formulation, we can either use the produced laws of geodesic motion, or alternatively use the action principle. In the words of Havas (1964):
"The variety of ways, in part based on entirely different sets of concepts, in which we can express the fundamental laws for gravitating matter, all leading to identical physical predictions, should caution us not to put undue stress on the supposed applications of a particular formulation of a theory, even if other formulations might not be available at a given time."

All these different formulations can help us build a deep inside knowledge of the object under consideration, and hope that *this* knowledge will help us to answer the questions about the world surrounding us. The resemblance of the gravity theory in Newton–Cartan space-time to the General theory of Relativity might help in the understanding of some features of Einstein's theory which are intimately connected with its four-dimensional structure and until now appeared to have no counterpart in Newtonian theory. Also, quantum theory expressed in a gravitating Newton–Cartan space-time might offer a better understanding to the interpretation of Quantum field Theory in Curved space-times.

He is not eternity or infinity, but eternal and infinite; He is not duration of space, but He endures and is present. He endures forever, and is everywhere present; and by existing always and everywhere, He constitutes duration and space ... And thus much concerning God; to discourse of whom from the appearances of things, does certainly belong to natural philosophy.

Isaac Newton from the General Scholium, Principia 1687

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1. Notation and conventions foldout

Notion	**Symbol**
Covariant Derivative	∇_α or $_{;\alpha}$
Partial Deivative	∂_α or $_{,\alpha}$
Time Derivative	$\dot{x}$
Affine connections	$\Gamma^\alpha{}_{\beta\gamma}$
Christoffel symbols	$\left\{ {}^{\ \alpha}_{\beta\ \gamma} \right\}$
Riemmann Tensor	$\mathcal{R}^\alpha{}_{\beta\gamma\delta}$
Ricci Tensor	$\mathcal{R}_{\alpha\beta}$
Ricci Scalar	$\mathcal{R}$
Space-time events	$\mathcal{P}, \mathcal{Q}, \mathcal{S}, \ldots$
Operators	$\underline{A}, \underline{B}, \ldots$
Real Numbers	$\mathbb{R}$
Galilei Group	$\mathbb{G}$
Space metric	$g^{\alpha\beta}$
Time metric	$h_{\alpha\beta}$
Action	$S[x]$
Hamiltonian	$\mathcal{H}$
Lie derivative	$\mathcal{L}$

2. Newton–Cartan Space-time foldout

Structure: $\{g^{\alpha\beta}, h_{\alpha\beta}, \nabla_\gamma\}$

with:
$g^{\alpha\beta}$: Space metric with signature $(0;1,1,1)$
$h_{\alpha\beta}$: time metric with signature $(1;0,0,0)$

satisfying the conditions:

$$g^{\alpha\beta}h_{\beta\gamma}$$
$$\nabla_\gamma g^{\alpha\beta} = 0 = \nabla_\gamma h_{\alpha\beta}$$
$$h_{\alpha[\beta}\mathcal{R}^\gamma{}_{\delta]\mu\nu} = 0$$

Space-time fields

T : absolute time
X^i : Galilean co-ordinates (non-rotating cartesians)
u^α : four-velocity of Galilean observer (rigid)
ϕ : scalar gravitational potential
ϕ^i : gravitational field strength

obeying the conditions:

$$h_{\alpha\beta} = T_\alpha T_\beta \text{ with } T_\alpha \equiv T_{,\alpha}$$
$$X^{i\alpha}{}_{;\beta} = 0 \text{ with } X^{i\alpha} \equiv g^{\alpha\beta}X^i{}_{,\beta}$$
$$g^{\alpha\beta} = \delta_{ij}X^{i\alpha}X^{j\beta}$$
$$u^\alpha T_\alpha = 1$$
$$u^\alpha X^i{}_{,\alpha} = 0$$
$$u^\alpha{}_{;\beta} = \phi^\alpha T_\beta$$
$$\phi^\alpha = g^{\alpha\beta}\phi_{,\beta}$$
$$\mathcal{R}^\alpha{}_{\beta\gamma\delta} = T_\beta \phi^\alpha{}_{;[\gamma}T_{\delta]}$$

Quasi-Galilei gauge group

$$\tilde{X}^i = (\alpha)^i{}_j X^j \quad \textit{and} \quad \tilde{T} \equiv \tilde{X}^0 = T - T_0$$
$$\delta^{mn}(\alpha)^i{}_m(\alpha)^j{}_n = \delta^{ij}$$
$$\tilde{X}^i = X^i - R^i(T)$$
$$\tilde{u}^\alpha = u^\alpha + U_i(T)X^i{}_{,\beta}$$
$$\tilde{\phi} = \phi + \mathcal{Q}_i(T)\tilde{X}^i + \varphi(T)$$
$$U^i \equiv \dot{R}^i \equiv dR^i/dT, \qquad \mathcal{Q}^i \equiv \dot{U}^i \equiv \ddot{R}^i$$