I. Kerr-Schild metric

The Kerr-Schild metric is given by,

$$g_{\mu\nu} = \eta_{\mu\nu} + 2H l_\mu l_\nu , \tag{1}$$

where $\eta_{\mu\nu}$ = diag(-1,1,1,1) is the Minkowski space-time metric, H is a scalar function of the coordinates and l_μ is a null vector with respect to both $\eta_{\mu\nu}$ and $g_{\mu\nu}$. This metric describes both the Schwarzschild and Kerr spacetimes in a slicing that is not pathological at the horizon. For the former space-time,Eqn. 1 leads to the Schwarzschild metric in Eddington-Finkelstein coordinates and for the latter it leads to the Kerr metric. In both cases the coordinates that I use are (t,x,y,z). For a Kerr space-time the choice of H and l_μ are

$$H = \frac{M r^3}{r^4 + a^2 z^2} \tag{2}$$

and

$$l_\mu = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right). \tag{3}$$

Where M is the mass of the Kerr black hole and $a = J/M$ is the angular momentum of the black hole. $r(x,y,z)$ is obtained from the relation,

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 \tag{4}$$

as

$$r^2 = \frac{1}{2}(\rho^2 - a^2) + \sqrt{\frac{1}{4}(\rho^2 - a^2) + a^2 z^2} , \tag{5}$$

with $\rho = \sqrt{x^2 + y^2 + z^2}$.

Taking the $a \to 0$ limit, the Schwarzschild space-time can be obtained. This leads to

$$H = \frac{M}{r} \tag{6}$$

and

$$l_\mu = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \tag{7}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and M is the mass as before. The resulting metric in Schwarzschild coordinates is the ingoing Eddington-Finkelstein form of the Schwarzschild metric,

$$ds^2 = -\left(1 - 2\frac{M}{r}\right)dt^2 + 4\frac{M}{r}drdt + \left(1 + 2\frac{M}{r}\right)dr^2 + r^2(d\theta^2 + (\sin\theta)^2 d\phi^2)\,. \qquad (8)$$

Another useful property of Eqn. 1 is its form-invariance under a Lorentz transformation. Under a Lorentz transformation, by definition the Minkowski space-time metric remains invariant, $\eta_{\mu\nu} \rightarrow \eta_{\mu\nu}$. $H \rightarrow \vec{H}$ goes to a new scalar function and $l_\mu \rightarrow \hat{l}_\mu$ remains a null vector. Hence, the form of the metric is left invariant with new definitions for H and l_μ. This property is computationally advantageous in that one may code the general form of the metric and then specify H, l_μ and their spatial derivatives for the problem of interest. The algebraic form of Eqn. 1 can be imposed during the course of a numerical evolution of boosted black hole space-times. There is a relationship between the lapse, shift and the 3-metric in the 3+1 form of Eqn. 1 that essentially amounts to a gauge choice that leads to a form of the space-time metric Eqn. 1 . But the Kerr-Schild metric is a restricted subset of solutions to Einstein's equations, so not all metrics can be written in the formEqn. 1 .

The null cone structure of this metric[boyer.1,MTW] is very well known for the Schwarzschild and Kerr spacetimes. In particular we know the outgoing null vector l_μ everywhere in the space-time. This makes this metric a good candidate for testing apparent horizon location algorithms as well as for studying differencing near the horizon in the black hole evolution problem. As stated earlier, this metric has no pathological behavior at the horizon. In the 3+1 splitting the lapse, shift, 3-metric and extrinsic curvature are all regular across the horizon and the slices of constant coordinate time cover the region $0 < r < \infty$. Because of the null cone structure of this metric and the non-pathological behavior at the horizon it is expected this coordinate system is a good one to use in numerical evolutions of black hole spacetimes.

A. 3+1 Splitting of the Kerr-Schild metric

The decomposition of the Kerr-Schild metric into a 3+1 form in Cartesian coordinates is straightforward and was carried out using a tensor package for Maple[choptuik.3]. Comparing Eqn. 1 to the standard 3+1 form of the space-time metric,

$$ds^2 = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i dt) \qquad (9)$$

and using the property that $\eta^{\mu\nu} l_\mu l_\nu = 0 \rightarrow l_t^2 = l_x^2 + l_y^2 + l_z^2$ it is possible to show that the lapse is

$$\alpha = \frac{1}{\sqrt{1 + 2H(l_x^2 + l_y^2 + l_z^2)}} \qquad (10)$$

$$= \frac{1}{\sqrt{1 + 2Hl_t^2}}\,, \qquad (11)$$

the shift is

$$\beta_i = 2Hl_t l_i \qquad (12)$$

and the 3-metric is

$$g_{ij} = \eta_{ij} + 2Hl_i l_j . \tag{13}$$

Using the definition of the extrinsic curvature, K_{ij}, we have,

$$K_{\mu\nu} = -(\perp \nabla_\mu \eta_\nu)$$

where $n_\nu = (-\alpha,0,0,0)$. The results for K_{ij} for general H and l_μ are lengthy and have not been included in this discussion.

$$\begin{aligned} K_{xx} = \Big(& -4H^2 l_z^2 l_x \left(\frac{\partial l_x}{\partial t}\right) - \left(\frac{\partial H}{\partial t}\right) l_x^2 + 4H^2 l_z^2 l_x \left(\frac{\partial l_t}{\partial x}\right) \\ & - 2Hl_z^2 \left(\frac{\partial H}{\partial t}\right) l_x^2 + 2Hl_z l_t \left(\frac{\partial H}{\partial z}\right) l_x^2 + 4H^2 l_z l_t l_x \left(\frac{\partial l_x}{\partial z}\right) \\ & - 4H^2 l_z l_t l_x \left(\frac{\partial l_z}{\partial x}\right) + 4H^2 l_y l_t l_x \left(\frac{\partial l_x}{\partial y}\right) - 4H^2 l_y l_t l_x \left(\frac{\partial l_y}{\partial x}\right) + \\ & 2Hl_y l_t \left(\frac{\partial H}{\partial y}\right) lx^2 + 4H^2 l_y^2 l_x \left(\frac{\partial l_t}{\partial x}\right) + 2Hl_x^3 \left(\frac{\partial H}{\partial x}\right) l_t \\ & - 2Hl_y^2 \left(\frac{\partial H}{\partial t}\right) l_x^2 - 4H^2 l_y^2 l_x \left(\frac{\partial l_x}{\partial t}\right) - 2Hl_x \left(\frac{\partial l_x}{\partial t}\right) \\ & + 2Hl_x \left(\frac{\partial l_t}{\partial x}\right) + 2H \left(\frac{\partial l_x}{\partial x}\right) l_t + 2 \left(\frac{\partial H}{\partial x}\right) l_x l_t - 4H^2 l_x^3 \left(\frac{\partial l_x}{\partial t}\right) \\ & - 2Hl_x^4 \left(\frac{\partial H}{\partial t}\right) + 4H^2 l_x^3 \left(\frac{\partial l_t}{\partial x}\right) \Big) \Big/ ((2HL_z^2) \sqrt{1 + 2Hl_t^2}) \end{aligned} \tag{14}$$

B. Eddington-Finkelstein metric

Substituting Eqn. 6 ,and Eqn. 7 in the expressions for the lapse, shift, 3-metric and extrinsic curvature we get the following results. The lapse is

$$\alpha = \frac{1}{\sqrt{1 + 2\frac{M}{r}}} . \tag{15}$$

The shift is

$$\beta_i = \frac{2M}{r^2} x_i . \tag{16}$$

The 3-metric is

$$g_{ij} = \eta_{ij} + \frac{2M}{r^3} x_i x_j \tag{17}$$

where $x_i = (x,y,z)$. The extrinsic curvature is

$$K_{ij} = \frac{2M}{r^4}\frac{1}{\sqrt{1+\frac{2M}{r}}}\left[r^2\eta_{ij} - \left(2+\frac{M}{r}\right)x_i x_j\right]. \tag{18}$$

The event horizon in this space-time is at $r=2M$. Since this is a static space-time the apparent horizon also will be located at $r=2M$. We shall use this in later chapters.

C. Kerr Metric

Using Eqn. 2 and Eqn. 3 the 3+1 form of the Kerr metric can be obtained. The lapse is

$$\alpha = \sqrt{\frac{(r^2+a^2)(r^4+a^2z^2)}{r^6+a^2r^4+2r^3Mx^2+2r^3Mz^2+2Mr^3y^2+a^2z^2r^2+2Mz^2a^2r+a^4z^2}} \tag{19}$$

The components of the shift are

$$\beta_x = 2\sqrt{\frac{M^2r^6(rx+ay)^2}{(r^4+a^2z^2)^2(r^2+a^2)^2}} \tag{20}$$

$$\beta_y = 2\sqrt{\frac{M^2r^6(-ry+ax)^2}{(r^4+a^2z^2)^2(r^2+a^2)^2}} \tag{21}$$

$$\beta_z = 2\sqrt{\frac{M^2r^4z^2}{(r^4+a^2z^2)^2}} \tag{22}$$

and the components of the 3-metric are

$$^{(3)}g_{xx} = \frac{(r^8+2r^6a^2+r^4a^4+a^2z^2r^4+2a^4z^2r^2+a^6z^2+2r^5Mx^2+4r^4Mxay+2r^3Ma^2y^2)}{(r^4+a^2z^2)(r^2+a^2)^2}, \tag{23}$$

$$^{(3)}g_{yy} = \frac{(r^8+2r^6a^2+r^4a^4+a^2z^2r^4+2a^4z^2r^2+a^6z^2+2r^5My^2+-4r^4Mxay+2r^3Ma^2x^2)}{(r^4+a^2z^2)(r^2+a^2)^2}, \tag{24}$$

$$^{(3)}g_{zz} = \frac{2Mrz^2+r^4+a^2z^2}{r^4+a^2z^2}, \tag{25}$$

$$^{(3)}g_{xy} = -\frac{2Mr^3(rx+ay)(-ry+ax)}{(r^4+a^2z^2)(r^2+a^2)^2}, \tag{26}$$

$$^{(3)}g_{xz} = \frac{2Mr^2(rx+ay)z}{(r^4+a^2z^2)(r^2+a^2)^2}, \tag{27}$$

$$^{(3)}g_{yz} = -\frac{2Mr^2(-ry+ax)z}{(r^4+a^2z^2)(r^2+a^2)^2}. \tag{28}$$

(The Kerr space-time has a number of interesting properties[boyer.1,MTW]. The first is the presence of a region of the space-time known as the ergosphere. In this region any timelike observer is dragged in the direction of the rotation of the black hole. This effect is known as frame-dragging. The next property is that the event horizon is an oblate spheroid located at

$$r_+ = M + \sqrt{M^2 - a^2}\,. \tag{29}$$

Once again, since this metric is stationary, we would expect the apparent horizon to be located at the same location as the event horizon. In later chapters we show that this is the case for this form of the Kerr space-time metric.

IV. Boosted Kerr-Schild metric

We noted that the Kerr-Schild metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Hl_\mu l_\nu, \tag{30}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Hl_\mu l_\nu$$

was form-invariant under a Lorentz transformation. In this section we take advantage of this in our analysis of boosted Kerr black hole apparent horizons. The 3+1 splitting of the Kerr-Schild metric results in a 3-metric and extrinsic curvature that depend on l_μ, $\partial_\mu l_\nu$, H and $\partial_\mu H$, each of which can be a function of (t,x,y,z). Let $\bar{O}$ be the rest frame of the black hole. Then $(\bar{t}, \bar{x}^4)$ are the coordinates in the rest frame of the black hole. Here l_μ, $\partial_\mu \bar{l}_\nu$, $\bar{H}$and $\partial_\mu H$ are functions of $\bar{x}^i$. Note there is no time dependence in the unboosted case. In our approach to carrying out the 3+1 decomposition of the Kerr-Schild metric the form of the 3-metric and extrinsic curvature in terms of l_μ, H and their space-time derivatives remains fixed. That is, we can take the $\bar{O}$ frame and carry out a boost to the O frame in the direction $\bar{v}_i = (\bar{v}_x, \bar{v}_y, \bar{v}_z)$ using the transformation matrix,

$$\Lambda_t^{\bar{t}} = \gamma \tag{31}$$

$$\Lambda_i^{\bar{t}} = -v\gamma\bar{v}_i \tag{32}$$

$$\Lambda_j^{\bar{i}} = -(\gamma - 1)\bar{v}_j\bar{v}_i + \delta_{ij}\,. \tag{33}$$

Here $\delta_{ij}\bar{v}_i\bar{v}_j = 1$, v is the boost velocity and $\gamma = 1/(\sqrt{1 - v^2})$. This transformation leads to a new null vector, l_μ and a new function $H = \bar{H}(x_\mu)$.

$$l_\mu = \Lambda_{\bar{\mu}}^{\nu}\bar{l}_\nu \tag{34}$$

$$H = \overline{H}(\Lambda_{\mu}^{\bar{\nu}}\bar{x}_{\nu}). \tag{35}$$

These preserve the form of the space-time metric and its 3+1 decomposition in terms of l_{μ}, H and their space-time derivatives. We take advantage of this form invariance in constructing our model problem. The code that evaluates the 3-metric and extrinsic curvature remains fixed due to this invariance. All that we need to specify are the space-time derivatives of l_{μ} and H.

V. Boosted Eddington-Finkelstein case

For the Schwarzschild space-time we use $\overline{H}$ and $\bar{l}_{\mu}$ given by Eqn. 6 and Eqn. 7 . To boost this space-time we carry out the coordinate transformation prescribed earlier on $\overline{H}$ and $\bar{l}_{\mu}$ given by Eqn. 34 and Eqn. 35 .

If we consider boosts only in the z-direction then the coordinates transform as

$$\bar{t} = \gamma(t - vz) \tag{36}$$

$$\bar{x} = x \tag{37}$$

$$\bar{y} = y \tag{38}$$

$$\bar{z} = \gamma(z - vt), \tag{39}$$

and the radial coordinate becomes, $r = \sqrt{x^2 + y^2 + \gamma^2(z - vt)}^{2}$. Then $\overline{H} \rightarrow H = M/r$ with r as just defined. The components of the null vector in terms of the old are now

$$l_t = \gamma(\bar{l}_t - v\bar{l}_z) \tag{40}$$

$$l_x = \bar{l}_x \tag{41}$$

$$l_y = \bar{l}_y \tag{42}$$

$$l_z = \gamma(\bar{l}_z - v\bar{l}_t). \tag{43}$$

Where

$$\bar{l}_t = 1 \tag{44}$$

$$\bar{l}_x = \frac{x}{r} \tag{45}$$

$$\bar{l}_y = \frac{y}{r} \tag{46}$$

$$\bar{l}_z = \frac{\gamma(z - vt)}{r} \tag{47}$$

with r as defined above.

From Eqn. 42 , H and their space-time derivatives, the metric, extrinsic curvature, lapse and shift may be constructed.