1 Equations

The code solves the evolution equations for the 3-metric, g_{ij} and extrinsic curvature, K_{ij},

$$\partial_t g_{ij} - \pounds_\beta g_{ij} = -2\alpha K_{ij}, \tag{1}$$
$$\partial_t K_{ij} - \pounds_\beta K_{ij} = -D_i D_j \alpha + \alpha \left(R_{ij} + K K_{ij} - 2K_{li} K^l_j\right). \tag{2}$$
$$\tag{3}$$

α is the lapse, β^i is the shift vector and $\pounds_\beta K_{ij} = \beta^k \partial_k K_{ij} + K_{kj} \partial_i \beta^k + K_{ik} \partial_j \beta^k$. Discussion of Causal DIFFERENCING HERE

2 Model Problem (Kerr-Schild metric)

The Kerr-Schild metric is given by,

$$g_{\mu\nu} = \eta_{\mu\nu} + 2H l_\mu l_\nu, \tag{4}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski spacetime metric, H is a scalar function of the coordinates and l^μ is a null vector with respect to both $\eta_{\mu\nu}$ and $g_{\mu\nu}$. This metric describes both the Schwarzschild and Kerr spacetimes in a slicing that is not pathological at the horizon. For the former spacetime, (4) leads to the Schwarzschild metric in Eddington-Finkelstein coordinates and for the latter it leads to the Kerr metric. In both cases the coordinates that I use are (t, x, y, z). For a Kerr spacetime the choice of H and l_μ are

$$H = \frac{M r^3}{r^4 + a^2 z^2} \tag{5}$$

and

$$l_\mu = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right). \tag{6}$$

Where M is the mass of the Kerr black hole and $a = J/M$ is the angular momentum of the black hole. $r(x,y,z)$ is obtained from the relation,

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 \tag{7}$$

as

$$r^2 = \frac{1}{2}\left(\rho^2 - a^2\right) + \sqrt{\frac{1}{4}\left(\rho^2 - a^2\right) + a^2 z^2}, \tag{8}$$

with $\rho = \sqrt{x^2 + y^2 + z^2}$.

Taking the $a \to 0$ limit, the Schwarzschild spacetime can be obtained. This leads to

$$H = \frac{M}{r} \tag{9}$$

and

$$l_\mu = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \tag{10}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and M is the mass as before. The resulting metric in Schwarschild coordinates is the ingoing Eddington-Finkelstein form of the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{4M}{r} dr dt + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{11}$$

Another useful property of (4) is its form-invariance under a Lorentz transformation. Under a Lorentz transformation, by definition the Minkowski spacetime metric remains invariant, $\eta_{\mu\nu} \to \eta_{\mu\nu}$. $H \to \hat{H}$ goes to a new scalar function and $l_\mu \to \hat{l}_\mu$ remains a null vector. Hence, the form of the metric is left invariant with new definitions for H and l_μ. This property is computationally advantageous in that one may code the general form of the metric and then specify H, l_μ and their spatial derivatives for the problem of interest. The algebraic form of (4) can be imposed during the course of a numerical evolution of boosted black hole spacetimes. There is a relationship between the lapse, shift and the 3-metric in the 3+1 form of (4) that essentially amounts to a gauge choice that leads to a form of the spacetime metric (4). But the Kerr-Schild metric is a restricted subset of solutions to Einstein's equations, so not all metrics can be written in the form(4).

2.1 3+1 Splitting of the Kerr-Schild metric

The decomposition of the Kerr-Schild metric into a 3+1 form in Cartesian coordinates is straight-forward and was carried out using a tensor package for Maple[**?**]. Comparing (4) to the standard 3+1 form of the spacetime metric,

$$ds^2 = -\alpha^2 dt^2 + g_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right) \tag{12}$$

and using the property that $\eta^{\mu\nu} l_\mu l_\nu = 0 \to l_t^2 = l_x^2 + l_y^2 + l_z^2$ it is possible to show that the lapse is

$$\alpha = \frac{1}{\sqrt{1 + 2H(l_x^2 + l_y^2 + l_z^2)}} \tag{13}$$

$$= \frac{1}{\sqrt{1 + 2H l_t^2}}, \tag{14}$$

the shift is

$$\beta_i = 2Hl_t l_i \tag{15}$$

and the 3-metric is

$$g_{ij} = \eta_{ij} + 2Hl_i l_j. \tag{16}$$

Using the definition of the extrinsic curvature, K_{ij}, we have,

$$K_{\mu\nu} = - \perp \nabla_\mu n_\nu,$$

where $n_\nu = (-\alpha, 0, 0, 0)$. The results for K_{ij} for general H and l_μ are lengthy and have not been included in this discussion.

2.2 Algebraic Condition for the Lapse

Using the condition that $l^\mu l_\mu = 0$ an algebraic relationship may be set up between the lapse and the covariant components of the shift. By substituting the expressions for the lapse and shift vectors into $l_t^2 = lx^2 + ly^2 + lz^2$ the expression for the lapse is

$$\alpha = 1/\left(1 + \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2}\right)^{1/2}. \tag{17}$$

In our evolution code we store the contravariant components of the shift. During the course of an evolution we use the numerical 3-metric to lower the shift and then use those components to evaluate the lapse. We find that compared to using the analytically specified lapse that this helps in the evolution.

2.3 Eddington-Finkelstein metric

Substituting (9) and (10) in the expressions for the lapse, shift, 3-metric and extrinsic curvature we get the following results. The lapse is

$$\alpha = \frac{1}{\sqrt{1 + 2M/r}}. \tag{18}$$

The shift is

$$\beta_i = \frac{2M}{r^2} x_i. \tag{19}$$

The 3-metric is

$$g_{ij} = \eta_{ij} + \frac{2M}{r^3} x_i x_j, \tag{20}$$

where $x_i = (x, y, z)$. The extrinsic curvature is

$$K_{ij} = \frac{2M}{r^4} \frac{1}{\sqrt{1 + 2M/r}} \left[r^2 \eta_{ij} - (2 + \frac{M}{r}) x_i x_j \right]. \tag{21}$$

The event horizon in this spacetime is at $r = 2M$. Since this is a static spacetime the apparent horizon also will be located at $r = 2M$. We shall use this in later chapters.

2.4 Kerr Metric

Using (5) and (6) the 3+1 form of the Kerr metric can be obtained. The lapse is

$$\alpha = \sqrt{\frac{(r^2+a^2)(r^4+a^2z^2)}{r^6+a^2r^4+2r^3Mx^2+2r^3Mz^2+2Mr^3y^2+a^2z^2r^2+2Mz^2a^2r+a^4z^2}} \tag{22}$$

The components of the shift are

$$\beta_x = 2\sqrt{\frac{M^2r^6(rx+ay)^2}{(r^4+a^2z^2)^2(r^2+a^2)^2}} \tag{23}$$

$$\beta_y = 2\sqrt{\frac{M^2r^6(-ry+ax)^2}{(r^4+a^2z^2)^2(r^2+a^2)^2}} \tag{24}$$

$$\beta_z = 2\sqrt{\frac{M^2r^4z^2}{(r^4+a^2z^2)^2}}, \tag{25}$$

and the components of the 3-metric are

$$\begin{aligned} {}^{(3)}g_{xx} = {} & \left(r^8+2r^6a^2+r^4a^4+a^2z^2r^4+2a^4z^2r^2+a^6z^2+2r^5Mx^2+\right. \\ & \left. 4r^4Mxay+2r^3Ma^2y^2\right)/\left(\left(r^4+a^2z^2\right)\left(r^2+a^2\right)^2\right), \end{aligned} \tag{26}$$

$${}^{(3)}g_{xy} = -\frac{2Mr^3(rx+ay)(-ry+ax)}{(r^4+a^2z^2)(r^2+a^2)^2}, \tag{27}$$

$${}^{(3)}g_{xz} = \frac{2Mr^2(rx+ay)z}{(r^4+a^2z^2)(r^2+a^2)}, \tag{28}$$

$$\begin{aligned} {}^{(3)}g_{yy} = {} & \left(r^8+2r^6a^2+r^4a^4+a^2z^2r^4+2a^4z^2r^2+a^6z^2+2r^5My^2\right. \\ & \left. -4r^4Mxay+2r^3Ma^2x^2\right)/\left[\left(r^4+a^2z^2\right)\left(r^2+a^2\right)^2\right], \end{aligned} \tag{29}$$

$${}^{(3)}g_{yz} = -\frac{2Mr^2(-ry+ax)z}{(r^4+a^2z^2)(r^2+a^2)}, \tag{30}$$

$${}^{(3)}g_{zz} = \frac{2Mrz^2+r^4+a^2z^2}{r^4+a^2z^2}. \tag{31}$$

The Kerr spacetime has a number of interesting properties[**?**, **?**]. The first is the presence of a region of the spacetime known as the ergosphere. In this region any timelike observer is dragged in the direction of the rotation of the black hole. This effect is known as frame-dragging. The next property is that the event horizon is an oblate spheroid located at

$$r_+ = M + \sqrt{M^2-a^2}. \tag{32}$$

3 Boosted Kerr-Schild metric

The Kerr-Schild metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + 2Hl_\mu l_\nu, \tag{33}$$

is form-invariant under a Lorentz transformation. In this section we take advantage of this in our analysis of boosted Kerr black hole apparent horizons. The 3+1 splitting of the Kerr-Schild metric results in a 3-metric and extrinsic curvature that depend on l_μ, $\partial_\mu l_\nu$, H and $\partial_\mu H$, each of which can be a function of (t, x, y, z). Let $\bar{\mathcal{O}}$ be the rest frame of the black hole. Then $(\bar{t}, \bar{x^i})$ are the coordinates in the rest frame of the black hole. Here $\bar{l}_\mu$, $\partial_\mu \bar{l}_\nu$, $\bar{H}$ and $\partial_\mu \bar{H}$ are functions of $\bar{x^i}$. Note there is no time dependence in the unboosted case. In our approach to carrying out the 3+1 decomposition of the Kerr-Schild metric the form of the 3-metric and extrinsic curvature in terms of l_μ, H and their space-time derivatives remains fixed. That is, we can take the $\bar{\mathcal{O}}$ frame and carry out a boost to the $\mathcal{O}$ frame in the direction $\hat{v}_i = (\hat{v}_x, \hat{v}_y, \hat{v}_z)$ using the transformation matrix,

$$\begin{aligned}
\Lambda^{\bar{t}}_{t} &= \gamma && (34)\\
\Lambda^{\bar{t}}_{i} &= -v\gamma\hat{v}_i && (35)\\
\Lambda^{\bar{i}}_{j} &= (\gamma - 1)\hat{v}_i\hat{v}_j + \delta_{ij}. && (36)
\end{aligned}$$

Here $\delta_{ij}\hat{v}^i\hat{v}^j = 1$, v is the boost velocity and $\gamma = 1/\sqrt{1-v^2}$. This transformation leads to a new null vector, l_μ and a new function $H = \bar{H}(x^\mu)$.

$$\begin{aligned}
l_\mu &= \Lambda^{\nu}_{\bar{\mu}}\bar{l}_\nu && (37)\\
H &= \bar{H}(\Lambda^{\bar{\nu}}_{\mu}\bar{x}_\nu). && (38)
\end{aligned}$$

These preserve the form of the spacetime metric and its 3+1 decomposition in terms of l_μ, H and their spacetime derivatives. We take advantage of this form invariance in constructing our model problem. The code that evaluates the 3-metric and extrinsic curvature remains fixed due to this invariance. All that we need to specify are the spacetime derivatives of l_μ and H.

3.1 Boosted Eddington-Finkelstein case

For the Schwarzschild spacetime we use $\bar{H}$ and $\bar{l}_\mu$ given by equations (9) and (10). To boost this spacetime we carry out the coordinate transformation prescribed earlier on $\bar{H}$ and $\bar{l}_\mu$ given by equations (37) and (37).

If we consider boosts only in the z-direction then the coordinates transform as

$$\begin{aligned}
\bar{t} &= \gamma(t - vz) && (39)\\
\bar{x} &= x && (40)\\
\bar{y} &= y && (41)\\
\bar{z} &= \gamma(z - vt), && (42)
\end{aligned}$$

and the radial coordinate becomes, $r = \sqrt{x^2 + y^2 + \gamma^2(z - vt)^2}$. Then $\bar{H} \to H = M/r$ with r as just defined. The components of the null vector in terms of the old are now

$$
\begin{aligned}
l_t &= \gamma(\bar{l}_t - v\bar{l}_z) && (43)\\
l_x &= \bar{l}_x && (44)\\
l_y &= \bar{l}_y && (45)\\
l_z &= \gamma(\bar{l}_z - v\bar{l}_t). && (46)
\end{aligned}
$$

Where

$$
\begin{aligned}
\bar{l}_t &= 1 && (47)\\
\bar{l}_x &= x/r && (48)\\
\bar{l}_y &= y/r && (49)\\
\bar{l}_z &= \gamma(z - vt)/r && (50)
\end{aligned}
$$

with r as defined above.

¿From (46), H and their spacetime derivatives, the metric, extrinsic curvature, lapse and shift may be constructed.